ON THE HOMOGENIZATION OF THE STOKES PROBLEM IN A PERFORATED DOMAIN

M. HILLAIRET

Abstract. In this paper we consider the Stokes equations on a bounded perforated domain completed with non-zero constant boundary conditions on the holes. We investigate configurations for which the holes are identical spheres and their number N goes to infinity while their radius $1/N$ tends to zero. We prove that, under the assumption that the holes do not concentrate in any box of size $1/N^{1/3}$, the solution is well approximated asymptotically by solving a Stokes-Brinkman problem.

1. Introduction

In this paper we focus on the homogenization of the Stokes equations in a perforated domain with non-zero constant boundary conditions on the holes. Precisely, we assume that Ω is a smooth bounded domain in $\mathbb{R}^3$. Given $N \in \mathbb{N}$, we introduce N centers $h_1^N, \dots, h_N^N$ in Ω such that the $B_i^N = B(h_i^N, 1/N)$ satisfy

$$B_i^N \Subset \Omega\,, \qquad \overline{B_i^N} \cap \overline{B_j^N} = \emptyset\,, \qquad \text{for } i \neq j \text{ in } \{1,\dots,N\}\,. \tag{A0}$$

Given a N-uplet $(v_i^N)_{i=1,\dots,N} \in (\mathbb{R}^3)^N$, it is classical that there exists a unique solution to

$$\left\{ \begin{array}{rcl} -\Delta u + \nabla p &=& 0\,, \\ \operatorname{div} u &=& 0\,, \end{array} \right. \qquad \text{on } \mathcal{F}^N := \Omega \setminus \bigcup_{i=1}^N B_i^N\,, \tag{1}$$

completed with boundary conditions

$$\left\{ \begin{array}{rcll} u &=& v_i^N\,, & \text{on } \partial B_i^N\,, \\ u &=& 0\,, & \text{on } \partial\Omega\,. \end{array} \right. \tag{2}$$

We are interested here in the behavior of this solution when N goes to infinity and the asymptotics of the data $(h_i^N, v_i^N)_{i=1,\dots,N}$ are given.

When the holes are distributed periodically, a major contribution in the analysis of this problem is due to G. Allaire [1]. In this reference, the author proves that there exists a critical value of the ratio between the size of the holes and their mutual distance for which there is a transition from the Stokes equations towards the Stokes-Brinkman equations. If the holes are "denser" the transition holds with a Darcy law while if the holes are "more dilute" we obtain again a Stokes problem asymptotically. This former result is an adaptation to the Stokes equations of a previous analysis on the Laplace equation in [3]. We refer the reader to [2, 5] for a review of equivalent results for other fluid models.

Date: April 14, 2016.

In [1], the Stokes equations are completed with vanishing boundary conditions while a volumic source term is added in the bulk. The problem with non-zero constant boundary conditions is introduced in [4] for the modeling of a thin spray in a highly viscous fluid. In this case, the holes represent droplets of another phase called "dispersed phase". This phase can be made of another fluid or small rigid spheres. The Stokes equations should then be completed with evolution equations for this dispersed phase yielding a time-evolution problem with moving holes. Computing the asymptotics of the stationary Stokes problem is then a tool for understanding the instantaneous response of the dispersed phase to the drag forces exerted by the flow on the droplets/spheres. We refer the reader to [4, 11] for more details on the modeling. In [4], the authors adapt the result of [1] on the derivation of the Stokes-Brinkman system. A comparable analysis with another purpose is provided in [10] when the dispersed phase is sufficiently dilute so that one recovers asymptotically the Stokes system. We emphasize that there is a significative new difficulty in introducing non-vanishing boundary conditions. Indeed, the boundary conditions on the holes may be highly oscillating (when jumping from one hole to another). Hence, if one were trying to solve this new system by lifting the boundary conditions, he would introduce a highly oscillating source term in the Stokes equations that is out of the scope of the analysis in [1].

The result in [4] is obtained under the assumption that the distance between two centers h_i^N and h_j^N is larger than $2/N^{1/3}$. This assumption is quite restrictive and prevents from extension to a time-dependant problem or a random model (in the spirit of [12]). Furthermore, the proof in [4] relies heavily on explicit formulas for solutions to the Stokes equations in annuli and exterior domains. Considering holes with non-spherical shapes thus requires new tools. Our main motivation in this paper is to provide another approach that may help to overcome these two difficulties.

In order to consider the limit $N \to \infty$, we make now precise the different assumptions on the data of our Stokes problem (1)-(2). This includes:

- the positions of the centers $(h_i^N)_{i=1,\dots,N}$,
- the velocities prescribed on the holes $(v_i^N)_{i=1,\dots,N}$.

First, similarly to [4], we assume that:

$$\text{(A1)} \qquad \frac{1}{N}\sum_{i=1}^{N} |v_i^N|^2 \text{ is uniformly bounded}\,.$$

Second, we make precise the assumptions on the separation between the holes. This is the main point where our assumptions differ from [4]. It is nowadays classical that the properties of Stokes flows in domains with obstacles change drastically when the distance between obstacles decreases becoming comparable to their diameters (see [8]). We want to avoid this phenomenon in the pairwise as in the global interactions between holes through the flow. We quantify this by introducing:

$$\begin{aligned} d^N_{min} &= \min_{i=1,\dots,N} \left\{ \mathrm{dist}(h_i^N,\partial\Omega), \min_{j\neq i} |h_i^N - h_j^N| \right\}, \\ M^N &= \sup_{x\in\Omega} \left\{ \# \left\{ i \in \{1,\dots,N\} \text{ s.t. } h_i^N \in \overline{B(x,1/N^{1/3})} \right\} \right\}. \end{aligned}$$

We assume below that

$$\lim_{N\to\infty} N d^N_{min} = +\infty\,, \tag{A2}$$

$$(M^N)_{N\in\mathbb{N}} \text{ is bounded.} \tag{A3}$$

We note that under these assumptions, we have that for N sufficiently large the $(B_i^N)_{i=1,\dots,N}$ are disjoint and do not intersect $\partial\Omega$. For N large enough, assumption (A0) only fixes that the holes are inside Ω. There exists then a unique pair $(u^N,p^N) \in H^1(\mathcal{F}^N)\times L^2(\mathcal{F}^N)$ solution to (1)-(2) (see next section for more details). The pressure is unique up to an additive constant that we may fix by requiring that p^N has mean 0. It can be seen as the Lagrange multiplier of the divergence-free condition in (1). Hence, we focus on the convergence of the sequence $(u^N)_{N\in\mathbb{N}}$ and will not go into details on what happens to the pressure (in contrast with [1]). The u^N are defined on different domains. In order to compute a limit for this sequence of vector-fields, we unify their domain of definition by extending u^N with the values v_i^N on B_i^N for any $i=1,\dots,N$. We still denote u^N the extension for simplicity. This is now a sequence in $H^1_0(\Omega)$.

The final assumptions prescribe the asymptotics of the distribution $(h_i^N,v_i^N)_{i=1,\dots,N}$. We introduce the empiric measure

$$S_N = \frac{1}{N}\sum_{i=1}^N \delta_{h_i^N,v_i^N} \in \mathbb{P}(\mathbb{R}^3\times\mathbb{R}^3),$$

and we assume:

$$\int_{\mathbb{R}^3} S_N(\mathrm{d}v) \rightharpoonup \rho(x)\mathrm{d}x \text{ weakly in the sense of measures on } \mathbb{R}^3\,, \tag{A4}$$

$$\int_{\mathbb{R}^3} vS_N(\mathrm{d}v) \rightharpoonup j(x)\mathrm{d}x \text{ weakly in the sense of (vectorial-)measures on } \mathbb{R}^3\,. \tag{A5}$$

We recall that, by assumption (A0), the measure S_N is supported in $\Omega\times\mathbb{R}^3$ so that, in the weak limit, $\rho \geqslant 0$ and ρ and j have support included in Ω.

Our main result reads:

Theorem 1. *Let* $(v_i^N,h_i^N)_{i=1,\dots,N}$ *be a sequence of data satisfying* (A0) *for arbitrary* $N\in\mathbb{N}$. *Assume furthermore that* (A1)–(A5) *hold true with*

$$j\in L^2(\Omega)\,,\quad \rho\in L^\infty(\Omega)\,.$$

Then, the associated sequence of extended velocity-fields $(u^N)_{N\in\mathbb{N}}$ *converges in* $H^1_0(\Omega)-w$ *to the unique velocity-field* $\bar{u}\in H^1(\Omega)$ *such that there exists a pressure* $\bar{p}\in L^2(\Omega)$ *for which* $(\bar{u},\bar{p})$ *solves:*

$$\left\{\begin{array}{rcl} -\Delta\bar{u}+\nabla\bar{p} & = & 6\pi(j-\rho\bar{u})\,, \\ \mathrm{div}\,\bar{u} & = & 0\,, \end{array}\right. \qquad \text{on } \Omega, \tag{3}$$

completed with boundary conditions

$$\bar{u}=0\,,\quad \text{on } \partial\Omega\,. \tag{4}$$

We remark that, with the assumptions (A1) and (A3), we may extract a subsequence such that the first momentums of S_N in v converge to some $(\rho,j)\in L^\infty(\Omega)\times L^2(\Omega)$. Hence, assumptions (A4) and (A5) only fix that the whole sequence converges to the same density ρ and momentum distribution j. For simplicity, we do not include a source term in (1) even if our result extends in a straightfoward way to this case (due to the linearity of the Stokes equations).

We emphasize that our result is not in contradiction with [1]. Indeed, in the periodic case, if the distance between holes is smaller than $1/N^{1/3}$ we are in a case where [1] shows that the asymptotic problem is a Darcy law. But, in our case, if the holes were arranging periodically with distance constantly less than $1/N^{1/3}$ in some part of the domain, there would exist one cube of size $1/N^{1/3}$ in which the number of holes goes to infinity. This possibility is ruled out by assumption (A3). On the opposite, if the distance between holes is too often much larger than $1/N^{1/3}$ in our framework, the asymptotic ρ and j vanish and we obtain again the Stokes equations in the limit as in [1]. We also remark that the results herein extend the previous results in [1] and [4]. Indeed, on the one hand, the assumptions in [4] are clearly more restrictive than (A2)-(A3). On the other hand, our assumptions allow the holes to be distributed periodically. In this case, we would have, with the terminology of [1], that "a_ε" $=1/N$ and "ε" $=(\sigma_\varepsilon^2/N)^{1/3}$ and "M_0" $=6\pi\mathbb{I}_3$. The case in which [1] derives the Stokes-Brinkman problem corresponds then to $j=0$ and $\sigma_\varepsilon\to\bar{\sigma}\in(0,\infty)$ when $N\to\infty$ (or equivalently $\varepsilon\to 0$). In that case, straightforward computations show that $\rho=\bar{\sigma}^{-2}$ and system (3)-(4) corresponds to the one derived in [1].

To conclude, one novelty of this paper is that we expect the two assumptions (A2)-(A3) are sufficiently general to tackle the time-evolution problem. Another novelty of the paper stems from the method of proof. We shall apply arguments that are not highly sensitive to the explicit value of solutions to the Stokes problem. The two main ingredients of the proof are the decrease of stokeslets (see (14)) and conservation arguments (see next subsection). In particular, we plan to consider more general shapes of holes and boundary conditions on holes in future works.

1.1. **Outline of the proof.** Our proof is based on a classical compactness argument: we first obtain uniform bounds on the sequence u^N and then extract a subsequence such that, passing to the weak limit in the weak formulation for the N-problem, we obtain the weak formulation of the limit problem.

We first prove that the sequence $(u^N)_{N\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)$. This part is obtained by applying a variational characterization of solutions to Stokes problems and relies upon assumptions (A1) and (A2) only. We may then extract a subsequence (that we do not relabel) converging to some $\bar{u}$ in $H^1_0(\Omega)$ (and strongly in any $L^q(\Omega)$ for $q \in [1,6[$). In order to identify a system satisfied by $\bar{u}$ all that remains is devoted to the proof that:

$$I_w := \int_\Omega \nabla \bar{u} : \nabla w \,,$$

satisfies:

$$I_w = 6\pi \int_\Omega (j(x) - \rho(x)\bar{u}(x)) \cdot w(x) \mathrm{d}x \,,$$

for arbitrary divergence-free $w \in C^\infty_c(\Omega)$. So, we fix a divergence-free $w \in C^\infty_c(\Omega)$ and we note that by construction, we have

$$I_w = \lim_{N\to\infty} I^N_w \qquad \text{with} \qquad I^N_w = \int_\Omega \nabla u^N : \nabla w \,, \quad \forall\, N \in \mathbb{N}.$$

We compute then I^N_w by applying that u^N is a solution to the Stokes problem (1)-(2). As the support of all the integrals I^N_w is Ω and the support of w is not adapted to the Stokes problem (1)-(2), this requires a special care. So, we introduce a covering $(T^N_\kappa)_{\kappa\in\mathcal{K}^N}$ of $Supp(w)$ with cubes of width $1/N^{1/3}$ and we split

$$I^N_w = \sum_{\kappa\in\mathcal{K}^N} \int_{T^N_\kappa} \nabla u^N : \nabla w.$$

Given N and κ, we apply that there are not too many particles in T^N_κ because of assumption (A5). This enables to localize the method of reflections [9, 10] and replace w with

$$\sum_{i\in\mathcal{I}^N_\kappa} U^N[w(h^N_i)](x - h^N_i) \,,$$

in the integral on T^N_κ. We denote here

- $\mathcal{I}^N_\kappa$ the subset of indices $i \in \{1, \dots, N\}$ for which $h^N_i \in T^N_\kappa$,
- $(U^N[v](y), P^N[v](y))$ the solution to the Stokes problem outside $B(0, 1/N)$ with boundary condition $U[v](y) = v$ on $\partial B(0, 1/N)$ and vanishing condition at infinity.

We obtain that

$$\int_{T^N_\kappa} \nabla u^N : \nabla w \sim \sum_{i\in\mathcal{I}^N_\kappa} \int_{T^N_\kappa} \nabla u^N : \nabla [U^N[w(h^N_i)]](x - h^N_i) \,.$$

Then, we observe that the pair

$$\big(U^N[w(h^N_i)](x - h^N_i), P^N[w(h^N_i)](x - h^N_i)\big)$$

is a solution to the Stokes problem outside B_i^N. Hence, we apply that u^N is divergence-free, introduce the pressure and integrate by parts to obtain that:

$$\int_{T_\kappa^N} \nabla u^N : \nabla w \sim \sum_{i \in \mathcal{I}_\kappa^N} \int_{\partial B_i^N} (\partial_n U^N[w(h_i^N)] - P^N[w(h_i^N)]n) \cdot v_i^N \mathrm{d}\sigma$$
$$+ \int_{\partial T_\kappa^N} \sum_{i \in \mathcal{I}_\kappa^N} (\partial_n U^N[w(h_i^N)] - P^N[w(h_i^N)]n) \cdot u^N \mathrm{d}\sigma\,.$$

We skip for conciseness that (U^N, P^N) depend on $(x - h_i^N)$ in these last identities. It is classical by the Stokes law that:

$$\int_{\partial B_i^N} (\partial_n U^N[w(h_i^N)] - P^N[w(h_i^N)]n) \mathrm{d}\sigma = \frac{6\pi}{N} w(h_i^N)\,,$$

and, by interpreting the Stokes system as the conservation of normal stress, that:

$$\int_{\partial T_\kappa^N} (\partial_n U[w(h_i^N)] - P^N[w(h_i^N)]n) \mathrm{d}\sigma = -\frac{6\pi}{N} w(h_i^N)\,.$$

To take advantage of this last identity, we use that T_κ^N has small width to replace u^N by some mean value $\bar{u}_\kappa^N$ on ∂T_κ^N. Say for simplicity that:

$$\bar{u}_\kappa^N = \frac{1}{|T_\kappa^N|} \int_{T_\kappa^N} u^N(x) \mathrm{d}x\,, \tag{5}$$

and assume that replacing u^N by $\bar{u}_\kappa^N$ induces a small error in the boundary integral. We obtain then that:

$$\int_{T_\kappa^N} \nabla u^N : \nabla w \sim \sum_{i \in \mathcal{I}_\kappa^N} \frac{6\pi}{N} w(h_i^N) \cdot v_i^N - \sum_{i \in \mathcal{I}_\kappa^N} \frac{6\pi}{N} w(h_i^N) \cdot \bar{u}_\kappa^N.$$

Summing over κ yields:

$$I_w^N \sim \sum_{i=1}^N \frac{6\pi}{N} w(h_i^N) \cdot v_i^N - \sum_{\kappa \in \mathcal{K}^N} \left[\sum_{i \in \mathcal{I}_\kappa^N} \frac{6\pi}{N} w(h_i^N) \right] \cdot \bar{u}_\kappa^N.$$

The first term on the right-hand side converges by assumption (A3) to :

$$6\pi \int_\Omega j(x) \cdot w(x) \mathrm{d}x.$$

To compute the limit of the second term, we introduce:

$$\sigma^N = 6\pi \sum_{\kappa \in \mathcal{K}^N} \left[\sum_{i \in \mathcal{I}_\kappa^N} w(h_i^N) \right] \mathbf{1}_{T_\kappa^N}\,,$$

so that:

$$\sum_{\kappa \in \mathcal{K}^N} \left[\sum_{i \in \mathcal{I}_\kappa^N} \frac{6\pi}{N} w(h_i^N) \right] \cdot \bar{u}_\kappa^N = \int_\Omega \sigma^N \cdot u^N(x) \mathrm{d}x.$$

For $w \in C_c^\infty(\Omega)$, we have that σ^N is bounded in $L^1(\Omega)$ and, under assumption (A4), it converges to $\sigma(x) = \rho(x)w(x)$ in $\mathcal{D}'(\Omega)$. However, this is not sufficient to compute the limit of this last term. Indeed we have strong convergence of the sequence u^N in $L^q(\Omega)$ for $q < 6$ only. Consequently, we need the supplementary assumption (A5) which entails that σ^N is bounded in $L^\infty(\Omega)$. Now, σ^N converges in $L^q(\Omega) - w$ for arbitrary $q \in (1, \infty)$ (up to the extraction of a subsequence) and combining this fact with the strong convergence of u^N we obtain that:

$$\lim_{N \to \infty} \sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N} \frac{6\pi}{N} w(h_i^N) \cdot \bar{u}_\kappa^N = \int_\Omega \rho(x) w(x) \cdot \bar{u}(x) \mathrm{d}x.$$

This would end the proof if we could actually define $\bar{u}_\kappa^N$ as in (5) and prove that it induces a small error by replacing u^N with the mean $\bar{u}_\kappa^N$ on ∂T_κ^N. Unfortunately, for this, we need that the combination of stokeslets to which u^N is multiplied is a solution to the Stokes equations on the set where the mean is taken (in particular we cannot choose T_κ^N here contrary to what we have written in (5)). We also remark that, in order that the sum:

$$\sum_{\kappa \in \mathcal{K}^N} \sum_{i \in \mathcal{I}_\kappa^N} \frac{6\pi}{N} w(h_i^N) \cdot \bar{u}_\kappa^N$$

corresponds to the multiplication with an L^1-bounded sequence $(\sigma^N)_{N \in \mathbb{N}}$, we not only need (A5) but also that the mean $\bar{u}_\kappa^N$ is taken on a set whose volume is of magnitude $1/N$. To handle both technical difficulties simultaneously, we introduce a parameter δ (which will be large), we "delete" the particles in a $1/(\delta N^{1/3})-$neighborhood of ∂T_κ^N and we construct $\bar{u}_\kappa^N$ as the mean of u^N on the $1/(2\delta N^{1/3})$-neighborhood of ∂T_κ^N (inside T_κ^N). By a suitable choice of the covering T_κ^N we prove that the cost of this deletion process is $O(1/\sqrt{\delta})$. Hence, we obtain that:

$$\left| I_w - 6\pi \int_\Omega (j(x) - \rho(x)\bar{u}(x)) \cdot w(x) \mathrm{d}x \right| \lesssim \frac{1}{\sqrt{\delta}}$$

for arbitrary large δ.

1.2. **Notations.** In the whole paper, for arbitrary $x \in \mathbb{R}^3$ and $r > 0$, we denote $B_\infty(x, r)$ the open ball with center x and radius r for the ℓ^∞ norm. The classical euclidean balls are denoted $B(x, r)$. For $x \in \mathbb{R}^3$ and $0 < \lambda_1 < \lambda_2$ we also denote:

$$A(x, \lambda_1, \lambda_2) := B_\infty(x, \lambda_2) \setminus \overline{B_\infty(x, \lambda_1)}\,.$$

The operator distance (between sets) is always computed with the ℓ^∞ norm. We will constantly use a truncation function associated to the parameter N. This truncation function is constructed in a classical way. We introduce $\chi \in C_c^\infty(\mathbb{R}^3)$ a truncation function such that $\chi = 1$ on $[-1,1]^3$ and $\chi = 0$ outside $[-2,2]^3$. We denote $\chi^N = \chi(N\cdot)$ its rescaled versions. This truncation function satisfies :

- $\chi^N = 1$ on $B_\infty(0,1/N)$ and $\chi^N = 0$ outside $B_\infty(0,2/N)$,
- $\nabla\chi^N$ has support in $A(0,1/N,2/N)$ and size $O(N)$.

When we truncate vector-fields with χ^N we shall create *a priori* non divergence-free vector-fields. To lift the divergence of these vector-fields, we use extensively the Bogovskii operator $\mathfrak{B}_{x,\lambda_1,\lambda_2}$ on the "cubic" annulus $A(x,\lambda_1,\lambda_2)$ (again $x \in \mathbb{R}^3$ and $0 < \lambda_1 < \lambda_2$). We recall that $w = \mathfrak{B}_{x,\lambda_1,\lambda_2}[f]$ is defined for arbitrary $f \in L^2(A(x,\lambda_1,\lambda_2))$, whose mean vanishes, and yields an $H^1_0(A(x,\lambda_1,\lambda_2))$ vector-field such that $\mathrm{div}\, w = f$. As the returned vector-field vanishes on $\partial A(x,\lambda_1,\lambda_2)$ we extend it tacitly by 0 to obtain an $H^1(\mathbb{R}^3)$ function.

For legibility we also make precise a few conventions. We have the following generic notations:

- u is a velocity-field solution to a Stokes problem, with associated pressure p,
- w is a test-function,
- I is an integral while $\mathcal{I}$ is a set of indices,
- T is a cube, depending on the width we shall use different exponants,
- n denotes the outward normal to the open set under consideration.

We shall also use extensively the symbol $\lesssim$ to denote that we have an inequality with a non-significant constant. We mean that we denote $a \lesssim b$ when there exists a constant C, which is not relevant to our problem, such that $a \leqslant Cb$. In most cases "not relevant" will mean that it does not depend on the parameters N and/or δ. If a more precise statement of this "non-relevance" is required we shall make it precise.

2. Analysis of the Stokes problem

In this section, we recall how is solved the Stokes problem:

$$\left\{ \begin{array}{rcl} -\Delta u + \nabla p & = & 0\,, \\ \mathrm{div}\, u & = & 0\,, \end{array} \right. \qquad \text{on } \mathcal{F}\,, \tag{6}$$

completed with boundary conditions

$$u(x) = u_*\,, \qquad \text{on } \partial\mathcal{F}\,, \tag{7}$$

for a lipschitz domain $\mathcal{F}$ and boundary condition $u_* \in H^{\frac{1}{2}}(\partial\mathcal{F})$. We consider the different cases: $\mathcal{F}$ is a bounded set, an exterior domain, or a perforated cube. In the second case, we complement the system with a vanishing condition at infinity.

2.1. Reminders on the Stokes problem in a bounded or an exterior domain.

We first assume that $\mathcal{F}$ is a bounded domain with a lipschitz boundary $\partial\mathcal{F}$. In this setting, a standard way to solve the Stokes problem (6)-(7) is to work with a generalized formulation (see [7, Section 4]). For this, we introduce:

$$D(\mathcal{F}) := \left\{ u \in H^1(\mathcal{F}) \text{ s.t. } \mathrm{div}\, u = 0 \right\}\,, \quad D_0(\mathcal{F}) := \left\{ u \in H^1_0(\mathcal{F}) \text{ s.t. } \mathrm{div}\, u = 0 \right\}\,.$$

By [7, Theorem III.4.1], we have that $D_0(\mathcal{F})$ is the closure for the $H^1_0(\Omega)-$norm of

$$\mathcal{D}_0(\mathcal{F}) = \{w \in C^\infty_c(\mathcal{F}) \text{ s.t. } \mathrm{div}\, w = 0\} \,.$$

We have then the following definition

Definition 2. *Given $u_* \in H^{\frac{1}{2}}(\partial\mathcal{F})$, a vector-field $u \in D(\mathcal{F})$ is called generalized solution to (6)-(7) if*

- *$u = u_*$ on $\partial\mathcal{F}$ in the sense of traces,*
- *for arbitrary $w \in D_0(\mathcal{F})$, there holds:*

$$\int_{\mathcal{F}} \nabla u : \nabla w = 0\,. \tag{8}$$

This generalized formulation is obtained assuming that we have a classical solution, multiplying (6) with arbitrary $w \in \mathcal{D}_0(\mathcal{F})$ and performing integration by parts. De Rham theory ensures that conversely, if one constructs a generalized solution then it is possible to find a pressure p such that (6) holds in the sense of distributions. Standard arguments yield:

Theorem 3. *Assume that the boundary of the fluid domain $\partial\mathcal{F}$ splits into $(N+1) \in \mathbb{N}$ lipschitz connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_N$. Given $u_* \in H^{\frac{1}{2}}(\partial\mathcal{F})$ satisfying*

$$\int_{\Gamma_i} u_* \cdot n \mathrm{d}\sigma = 0\,, \quad \forall\, i \in \{0, \dots, N\}, \tag{9}$$

then

- *there exists a unique generalized solution u to (17)-(18);*
- *this generalized solution realizes*

$$\inf \left\{ \int_{\mathcal{F}} |\nabla u|^2, u \in D(\mathcal{F}) \text{ s.t. } u_{|\partial\mathcal{F}} = u_* \right\} \,. \tag{10}$$

Proof. Existence and uniqueness of the generalized solution is a consequence of [7, Theorem IV.1.1]. A key argument in the proof of this reference is the property of traces that we state in the following lemma:

Lemma 4. *For arbitrary $u_* \in H^{\frac{1}{2}}(\partial\mathcal{F})$ satisfying (9) there holds:*

- *there exists $u_{bdy} \in D(\mathcal{F})$ having trace u_* on $\partial\mathcal{F}$,*
- *for arbitrary $u_{bdy} \in D(\mathcal{F})$ having trace u_* on $\partial\mathcal{F}$ there holds*

$$\{u \in D(\mathcal{F}) \text{ s.t. } u_{|\partial\mathcal{F}} = u_*\} = u_{bdy} + D_0(\mathcal{F})\,.$$

Then, given $u \in D(\mathcal{F})$ the generalized solution to (17)-(18) and $w \in D_0(\mathcal{F})$, the fundamental property (8) of u entails that:

$$\begin{aligned} \int_{\mathcal{F}} |\nabla(u+w)|^2 &= \int_{\mathcal{F}} |\nabla u|^2 + 2\int_{\mathcal{F}} \nabla u : \nabla w + \int_{\mathcal{F}} |\nabla w|^2\,, \\ &= \int_{\mathcal{F}} |\nabla u|^2 + \int_{\mathcal{F}} |\nabla w|^2\,. \end{aligned}$$

Consequently, the norm on the left-hand side is minimal if and only if $w = 0$. Combining this remark with the above lemma yields that the generalized solution to (17)-(18) is the unique minimizer of (10) in $\left\{v \in D(\mathcal{F}) \text{ s.t. } v_{|\partial\mathcal{F}} = u_*\right\}$. $\square$

As mentioned previously, once it is proven that there exists a unique generalized solution u to (6)-(7), it is possible to recover a pressure p so that (6)-(7) holds in the sense of distributions. If the data are smooth (*i.e.* $\mathcal{F}$ has smooth boundaries and u_* is smooth) one proves also that $(u,p) \in C^\infty(\overline{\mathcal{F}})$.

We turn to the exterior problem as developed in [7, Section 5]. We assume now that $\mathcal{F} = \mathbb{R}^3 \setminus \overline{B^a}$ where $B^a = B(0, 1/a)$ and we consider the Stokes problem (6) with boundary condition

$$u = u_* \text{ on } \partial B^a\,, \qquad \lim_{|x|\to\infty} u(x) = 0\,, \tag{11}$$

for some $u_* \in H^{\frac{1}{2}}(\partial B^a)$. For the exterior problem, we keep the definition of generalized solution up to change a little the function spaces. We denote in this case:

- $\mathcal{D}(\mathcal{F}) = \left\{w_{|\mathcal{F}}\,,\ w \in C^\infty_c(\mathbb{R}^3) \text{ s.t. } \operatorname{div} w = 0\right\}$,
- $D(\mathcal{F})$ is the closure of $\mathcal{D}(\mathcal{F})$ for the norm:

$$\|w\|_{D(\mathcal{F})} = \left(\int_{\mathcal{F}} |\nabla w|^2\right)^{\frac{1}{2}}.$$

We keep the definition of $\mathcal{D}_0(\mathcal{F})$ as in the bounded-domain case and we construct $D_0(\mathcal{F})$ as the closure of $\mathcal{D}_0(\mathcal{F})$ with respect to this latter homogeneous H^1-norm. We note that, in the exterior domain case, we still have that $D(\mathcal{F}) \subset W^{1,2}_{loc}(\mathcal{F})$ (see [7, Lemma II.6.1]) so that we have a trace operator on ∂B^a and an equivalent to Lemma 4.

As in the case of bounded domains, the Stokes problem (6)-(11) with boundary conditions u_* prescribing no flux through ∂B^a has a unique generalized solution (see [7, Theorem V.2.1], actually the no-flux assumption is not necessary for the exterior problem). Thus, this solution satisfies:

- $\nabla u \in L^2(\mathbb{R}^3 \setminus \overline{B^a})$,
- for any $w \in D_0(\mathbb{R}^3 \setminus \overline{B^a})$ there holds:

$$\int_{\mathbb{R}^3\setminus\overline{B^a}} \nabla u : \nabla w = 0.$$

Explicit formulas are provided when the boundary condition $u_* = v$ with $v \in \mathbb{R}^3$ constant (see [4, Section 6.2] for instance):

$$u(x) = U^a[v](x) \ := \ \frac{1}{4a}\left(\frac{3}{|x|} + \frac{1}{a^2|x|^3}\right)v + \frac{3}{4a}\left(\frac{1}{|x|} - \frac{1}{a^2|x|^3}\right)\frac{v\cdot x}{|x|^2}x\,, \tag{12}$$

$$p(x) = P^a[v](x) \ := \ \frac{3}{2a}\frac{v\cdot x}{|x|^4}x\,. \tag{13}$$

We call this classical solution stokeslet in what follows. With these explicit formulas, we remark that:

$$|U^a[v](x)| \lesssim \frac{|v|}{a|x|}\,, \quad |\nabla U^a[v](x)| + |P^a[v](x)| \lesssim \frac{|v|}{a|x|^2}\,, \quad \forall\, x \in \mathbb{R}^3 \setminus \overline{B_a}\,, \tag{14}$$

and we recall that the force exerted by the flow on ∂B^a reads:

$$\int_{\partial B_a} (\partial_n U^a[v] - P^a[v]n)\mathrm{d}\sigma = \frac{6\pi}{a} v. \tag{15}$$

For convenience, the stokeslet $U^a[v]$ is extended by $U^a[v] = v$ on B^a in what follows.

2.2. **Stokes problem in a perforated cube.** In this last subsection, we fix $(N, M) \in (\mathbb{N} \setminus \{0\})^2$, and a divergence-free $w \in C_c^\infty(\mathbb{R}^3)$. We denote T^N an open cube of width $1/N^{1/3}$ and $B_i^N = B(x_i, 1/N) \subset T^N$ for $i = 1, \dots, M$. We assume further that there exists d_m satisfying

$$\min_{i=1,\dots,M} \left\{ \mathrm{dist}(x_i, \partial T^N), \min_{j\neq i} (|x_i - x_j|) \right\} \geqslant d_m > \frac{4}{N}\,. \tag{16}$$

We consider the Stokes problem:

$$\left\{ \begin{array}{rcl} -\Delta u + \nabla p &=& 0\,, \\ \mathrm{div}\, u &=& 0\,, \end{array} \right. \qquad \text{on } \mathcal{F} = T^N \setminus \bigcup_{i=1}^M \overline{B_i^N}\,, \tag{17}$$

completed with boundary conditions

$$\left\{ \begin{array}{rcll} u(x) &=& w(x)\,, & \text{on } B_i^N\,, \forall\, i = 1, \dots, M\,, \\ u(x) &=& 0\,, & \text{on } \partial T^N\,. \end{array} \right. \tag{18}$$

Assumption (16) entails that the B_i^N do not intersect and do not meet the boundary ∂T^N. So, the set $T^N \setminus \bigcup_{i=1}^M \overline{B_i^N}$ has a lipschitz boundary satisfying:

$$\partial \left[T^N \setminus \bigcup_{i=1}^M \overline{B_i^N} \right] = \partial T^N \cup \bigcup_{i=1}^M \partial B_i^N.$$

For any $i = 1, \dots, M$, direct computations show that:

$$\int_{\partial B_i^N} w \cdot n \mathrm{d}\sigma = \int_{B_i^N} \mathrm{div}\, w = 0.$$

Hence, the problem (17)-(18) is solved by applying Theorem 3 and it admits a unique generalized solution $u \in H^1(\mathcal{F})$. We want to compare this solution with:

$$u_s(x) = \sum_{i=1}^M U^N[w(x_i)](x - x_i),$$

where U^N is the stokeslet as defined in (12). The main result of this subsection is:

Proposition 5. *There exists a constant K independent of (N,M,d_m,w) for which:*

$$\|(u-u_s)\|_{L^6(\mathcal{F})} + \|\nabla(u-u_s)\|_{L^2(\mathcal{F})} \leqslant K\|w\|_{W^{1,\infty}(\mathbb{R}^3)}\sqrt{\frac{M}{N}}\left[\frac{1}{N} + \sqrt{\frac{M}{Nd_m}}\right].$$

Proof. We split the error term into two pieces. First, we reduce the boundary conditions of the Stokes problem (17)-(18) to constant boundary conditions. Then, we compare the solution to the Stokes problem with constant boundary conditions to the combination of stokeslets u_s. In the whole proof, the symbol $\lesssim$ is used when the implicit constant in our inequality does not depend on N, M, d_m and w.

So, we introduce u_c the unique generalized solution to the Stokes problem on $\mathcal{F}$ with boundary conditions:

$$\left\{\begin{array}{rcll} u_c &=& w(x_i)\,, & \text{on } B_i^N\,,\,\forall\, i=1,\dots,M\,,\\ u_c &=& 0\,, & \text{on } \partial T^N\,. \end{array}\right. \tag{19}$$

Again, existence and uniqueness of this velocity-field holds by applying Theorem 3. We split then:

$$\begin{array}{rcl} \|(u-u_s)\|_{L^6(\mathcal{F})} &\leqslant& \|(u-u_c)\|_{L^6(\mathcal{F})} + \|(u_c-u_s)\|_{L^6(\mathcal{F})}\,,\\ \|\nabla(u-u_s)\|_{L^2(\mathcal{F})} &\leqslant& \|\nabla(u-u_c)\|_{L^2(\mathcal{F})} + \|\nabla(u_c-u_s)\|_{L^2(\mathcal{F})}. \end{array}$$

To control the first term on the right-hand sides, we note that $(u-u_c)$ is the unique generalized solution to the Stokes problem on $\mathcal{F}$ with boundary conditions:

$$\left\{\begin{array}{rcll} (u-u_c)(x) &=& w(x)-w(x_i)\,, & \text{on } B_i^N\,,\,\forall\, i=1,\dots,M\,,\\ (u-u_c)(x) &=& 0\,, & \text{on } \partial T^N\,. \end{array}\right.$$

Hence, by the variational characterization of Theorem 3, $\|\nabla(u-u_c)\|_{L^2(\mathcal{F})}$ realizes the minimum of $\|\nabla\tilde{w}\|_{L^2(\mathcal{F})}$ amongst

$$\left\{\tilde{w}\in H^1(\mathcal{F}) \text{ s.t. } \operatorname{div}\tilde{w}=0\,,\ \tilde{w}_{|_{\partial T^N}}=0\,,\ \tilde{w}_{|_{\partial B_i^N}} = w(\cdot)-w(x_i)\,,\ \forall\, i=1,\dots,M\right\}.$$

We construct thus a suitable $\tilde{w}$ in this space. We set:

$$\tilde{w} = \sum_{i=1}^M \tilde{w}_i$$

with, for $i=1,\dots,M$:

$$\tilde{w}_i = \left(\chi^N(\cdot-x_i)(w(\cdot)-w(x_i)) - \mathfrak{B}_{x_i,1/N,2/N}\left[x\mapsto (w(x)-w(x_i))\cdot\nabla\chi^N(x-x_i)\right]\right).$$

We recall that χ^N is a chosen function that truncates between $B_\infty(0,1/N)$ and $B_\infty(0,2/N)$ and that we denote $\mathfrak{B}_{x_i,1/N,2/N}$ the Bogovskii operator on the annulus $A(x_i,1/N,2/N)$. The properties of this operator are analyzed in Appendix A. The above vector-field $\tilde{w}_i$ is well-defined as, for $i = 1, \dots, M$, there holds:

$$\begin{aligned}\int_{A(x_i,1/N,2/N)} (w(x) - w(x_i)) \cdot \nabla \chi^N(x - x_i) \mathrm{d}x \\ &= \int_{B_\infty(x_i,2/N)\setminus B_\infty(x_i,1/N)} \mathrm{div}(\chi^N(\cdot - x_i)(w(\cdot) - w(x_i)))\,, \\ &= \int_{\partial B_\infty(x_i,1/N)} (w(x) - w(x_i)) \cdot n \mathrm{d}\sigma\,, \\ &= \int_{B_\infty(x_i,1/N)} \mathrm{div}(w) = 0\,,\end{aligned}$$

and we can apply the Bogovskii operator to $x \mapsto (w(\cdot) - w(x_i)) \cdot \nabla \chi^N(\cdot - x_i)$ on the annulus $A(x_i, 1/N, 2/N)$. We note that $\tilde{w}_i$ has support in $B_\infty(x_i, 2/N)$ so that, as $d_m > 4/N$, the $\tilde{w}_i$ have disjoint supports inside T^N. This yields that $\tilde{w}$ is indeed divergence-free and fits the required boudary conditions. Furthermore, there holds:

$$\|\nabla \tilde{w}\|_{L^2(\mathcal{F})} \leqslant \left[\sum_{i=1}^{M} \|\nabla \tilde{w}_i\|^2_{L^2(B_\infty(x_i,2/N))} \right]^{\frac{1}{2}}.$$

For $i \in \{1, \dots, M\}$ we have by direct computations:

$$\begin{aligned}\|\nabla \chi^N(\cdot - x_i)(w(\cdot) - w(x_i))\|^2_{L^2(B_\infty(x_i,2/N))} &\lesssim \frac{\|w\|^2_{W^{1,\infty}}}{N^3}\,, \\ \|\chi^N(\cdot - x_i)\nabla(w(\cdot) - w(x_i))\|^2_{L^2(B_\infty(x_i,2/N))} &\lesssim \frac{\|w\|^2_{W^{1,\infty}}}{N^3}\,,\end{aligned}$$

and, by applying Lemma 15:

$$\begin{aligned}\|\nabla \mathfrak{B}_{x_i,1/N,2/N} \left[x \mapsto (w(x) - w(x_i)) \cdot \nabla \chi^N(x - x_i)\right] \|^2_{L^2(B_\infty(x_i,2/N))} \\ &\lesssim \|x \mapsto (w(x) - w(x_i)) \cdot \nabla \chi^N(x - x_i)\|^2_{L^2(B_\infty(x_i,2/N))} \\ &\lesssim \frac{\|w\|^2_{W^{1,\infty}}}{N^3}.\end{aligned}$$

Gathering all these inequalities in the computation of $\tilde{w}$ yields finally:

$$\|\nabla \tilde{w}\|_{L^2(\mathcal{F})} \lesssim \sqrt{M} \frac{\|w\|_{W^{1,\infty}}}{N^{\frac{3}{2}}}\,.$$

The variational characterization of generalized solutions to Stokes problems entails that we have the same bound for $(u - u_c)$. At this point, we argue that the straightforward extension of u and u_c (by w and $w(x_i)$ on the B_i^N respectively) satisfy $(u - u_c) \in H^1_0(T^N) \subset L^6(T^N)$

so that

$$\begin{aligned}\|u-u_c\|_{L^6(\mathcal{F})} &\leqslant \|u-u_c\|_{L^6(T^N)} \lesssim \|\nabla(u-u_c)\|_{L^2(T^N)}\\ &\lesssim \left(\|\nabla(u-u_c)\|^2_{L^2(\mathcal{F})} + M\frac{\|w\|^2_{W^{1,\infty}}}{N^3}\right)^{\frac{1}{2}}\\ &\lesssim \sqrt{M}\frac{\|w\|_{W^{1,\infty}}}{N^{\frac{3}{2}}}.\end{aligned}$$

We emphasize that, by a scaling argument, the constant deriving from the embedding $H^1_0(T^N) \subset L^6(T^N)$ does not depend on N so that it is not significant to our problem.

We turn to estimating $u_c - u_s$. Due to the linearity of the Stokes equations, we split

$$u_c = \sum_{i=1}^{M} u_{c,i},$$

where $u_{c,i}$ is the generalized solution to Stokes problem on $\mathcal{F}$ with boundary conditions:

$$\left\{\begin{array}{lll} u_{c,i} &=& w(x_i)\,, \quad \text{on } \partial B_i^N\,,\\ u_{c,i} &=& 0\,, \quad \text{on } \partial T^N \cup \bigcup_{j\neq i} \partial B_j^N\,.\end{array}\right.$$

We have then

$$\|\nabla(u_c-u_s)\|_{L^2(\mathcal{F})} \leqslant \sum_{i=1}^{M} \|\nabla(u_{c,i} - U^N[w(x_i)](\cdot - x_i))\|_{L^2(\mathcal{F})}. \tag{20}$$

Similarly, we expand :

$$u_s = \sum_{i=1}^{M} U_i\,, \text{ where } U_i(x) = U^N[w(x_i)](x-x_i)\,, \quad \forall\, x\in\mathbb{R}^3.$$

For $i \in \{1,\ldots,M\}$ we extend $u_{c,i}$ by 0 on $\mathbb{R}^3 \setminus T^N$ and B_j^N for $j \neq i$. The extension we still denote by $u_{c,i}$ satisfies $u_{c,i} \in H^1(\mathbb{R}^3 \setminus \overline{B_i^N})$ and is divergence-free. In particular, we have $u_{c,i} \in D(\mathbb{R}^3 \setminus \overline{B_i^N})$. Consequently, $u_{c,i} - U_i \in D(\mathbb{R}^3 \setminus \overline{B_i^N})$ and:

$$\begin{aligned}\|\nabla(u_{c,i} - U_i(\cdot - x_i))\|^2_{L^2(\mathcal{F})} &\leqslant \int_{\mathbb{R}^3\setminus\overline{B_i^N}} |\nabla u_{c,i}(x) - \nabla U_i(x)|^2 \mathrm{d}x\\ &\leqslant \int_{\mathbb{R}^3\setminus\overline{B_i^N}} |\nabla u_{c,i}|^2 - 2\int_{\mathbb{R}^3\setminus\overline{B_i^N}} \nabla u_{c,i} : \nabla U_i + \int_{\mathbb{R}^3\setminus\overline{B_i^N}} |\nabla U_i|^2\,.\end{aligned}$$

To compute the product term, we apply that $u_{c,i}$ and $U_i = U^N[w(x_i)](\cdot - x_i)$ have the same trace on ∂B_i^N and that U_i is a generalized solution to the Stokes problem on $\mathbb{R}^3 \setminus \overline{B_i^N}$. So, integrals of the form $\int_{\mathbb{R}^3\setminus\overline{B_i^N}} \nabla U_i : \nabla w$ (for $w \in D(\mathbb{R}^3 \setminus \overline{B_i^N})$) depend only on the trace of w on ∂B_i^N. This entails that:

$$\int_{\mathbb{R}^3\setminus\overline{B_i^N}} \nabla u_{c,i} : \nabla U_i = \int_{\mathbb{R}^3\setminus\overline{B_i^N}} |\nabla U_i|^2\,,$$

and we have:

$$\|\nabla(u_{c,i} - U^N(\cdot - x_i))\|^2_{L^2(\mathcal{F})} \leqslant \int_{\mathbb{R}^3\setminus\overline{B_i^N}} |\nabla u_{c,i}|^2 - \int_{\mathbb{R}^3\setminus\overline{B_i^N}} |\nabla U_i|^2 . \tag{21}$$

To conclude, we find a bound from above for

$$\int_{\mathbb{R}^3\setminus\overline{B_i^N}} |\nabla u_{c,i}(x)|^2 \mathrm{d}x = \int_{\mathcal{F}} |\nabla u_{c,i}(x)|^2 \mathrm{d}x.$$

As $u_{c,i}$ is a generalized solution to a Stokes problem on $\mathcal{F}$, this can be done by constructing a divergence-free $\bar{w}_i$ satisfying the same boundary condition as $u_{c,i}$. We define:

$$\bar{w}_i = \chi_{d_m/4}(\cdot - x_i)U_i - \mathfrak{B}_{x_i,d_m/4,d_m/2}\left[x \mapsto U_i(x)\cdot\nabla\chi_{d_m/4}(x-x_i)\right]$$

where $\chi_{d_m/4} := \chi^{4/d_m}$ (with the family of truncation functions of the introduction) As previously, we have here a divergence-free function which satisfies the right boundary conditions because $\chi_{d_m/4}(\cdot - x_i) = 1$ on B_i^N (since $d_m/4 > 1/N$) and vanishes on all the other boundaries of $\partial\mathcal{F}$ (since the distance between one hole center and the other holes or ∂T^N is larger than $d_m - 1/N > d_m/2$). Again, similarly as in the computation of $\tilde{w}_i$ we apply the properties of the Bogovskii operator $\mathfrak{B}_{x_i,d_m/4,d_m/2}$ and there exists an absolute constant K for which:

$$\|\nabla\bar{w}_i\|^2_{L^2(\mathcal{F})} \leqslant \int_{\mathbb{R}^3\setminus\overline{B_i^N}} |\chi_{d_m/4}(\cdot - x_i)\nabla U_i|^2 + K\left(\int_{A(x_i,d_m/4,d_m/2)} |\nabla U_i(x)|^2 + |\nabla\chi_{d_m/2}(x-x_i)\otimes U_i(x)|^2\mathrm{d}x\right)$$

As we have the same bound for $u_{c,i}$, we plug the right-hand side above in (21) and get:

$$\|\nabla(u_{c,i} - U_i)\|^2_{L^2(\mathcal{F})} \lesssim \int_{A(x_i,d_m/4,d_m/2)} |\nabla U_i(x)|^2 \mathrm{d}x + \int_{A(x_i,d_m/4,d_m/2)} |\nabla\chi_{d_m/4}(x-x_i)\otimes U_i(x)|^2\mathrm{d}x\,.$$

With the explicit decay properties for U_i (see (14)) and $\nabla\chi_{d_m/4}$ we derive:

$$\int_{A(x_i,d_m/4,d_m/2)} \left(|\nabla U_i(x)|^2 + |\nabla\chi_{d_m/4}(x-x_i)\otimes U_i(x)|^2\right)\mathrm{d}x \lesssim \frac{\|w\|^2_{W^{1,\infty}}}{N^2 d_m}\,.$$

Combining these bounds for $i = 1,\ldots,M$ in (20) we get:

$$\|\nabla(u_c - u_s)\|_{L^2(\mathcal{F})} \leqslant \frac{M\|w\|_{W^{1,\infty}}}{N\sqrt{d_m}}.$$

By similar arguments, we also have:

$$\|u_c - u_s\|_{L^6(\mathcal{F})} = \|u_c - u_s\|_{L^6(T^N)} \leqslant \sum_{i=1}^M \|u_{c,i} - U_i\|_{L^6(\mathbb{R}^3\setminus\overline{B_i^N})}.$$

As $u_{c,i}, U_i \in D(\mathbb{R}^3 \setminus \overline{B_i^N})$ and $u_{c,i}, U_i$ share the same value on ∂B_i^N, there holds $u_{c,i} - U_i \in D_0(\mathbb{R}^3 \setminus \overline{B_i^N})$ and we may use the classical inequality (see [7, (II.6.9)]):

$$\|u_{c,i} - U_i\|_{L^6(\mathbb{R}^3 \setminus \overline{B_i^N})} \lesssim \|\nabla u_{c,i} - \nabla U_i\|_{L^2(\mathbb{R}^3 \setminus \overline{B_i^N})}, \quad \forall\, i = 1, \ldots, M,$$

(again the constant arising from this embedding does not depend on N by a standard scaling argument). This yields again the bound:

$$\|(u_c - u_s)\|_{L^6(\mathcal{F})} \leqslant \frac{M\|w\|_{W^{1,\infty}}}{N\sqrt{d_m}},$$

and ends the proof of our proposition. □

3. Proof of Theorem 1 – Uniform estimates

From now on, we fix a sequence of data $(v_i^N, h_i^N)_{i=1,\ldots,N}$ associated to $(B_i^N)_{i=1,\ldots,N}$ that satisfy (A0) for arbitrary $N \in \mathbb{N}$ and such that (A1)–(A5) hold true with

$$j \in L^2(\Omega), \quad \rho \in L^\infty(\Omega).$$

Because of assumption (A0), the existence result of the previous section applies so that there exists a unique generalized solution $u^N \in H^1(\mathcal{F}^N)$ to (1)-(2). In what follows, we extend implicitly u^N by its boundary values on the ∂B_i^N:

$$u^N = \begin{cases} u^N, & \text{in } \mathcal{F}^N, \\ v_i^N, & \text{in } B_i^N, \text{ for } i = 1, \ldots, N. \end{cases}$$

As the B_i^N do not overlap and do not meet $\partial\Omega$, it is straightforward that these velocity-fields yield a sequence in $H^1_0(\Omega)$ of divergence-free vector-fields. Moreover, we have the property:

$$\|\nabla u^N\|_{L^2(\mathcal{F}^N)} = \|\nabla u^N\|_{L^2(\Omega)}.$$

Our target result reads:

Theorem 6. *The sequence of extended generalized solutions $(u^N)_{N\in\mathbb{N}}$ converges weakly in $H^1_0(\Omega)$ to $\bar{u}$ satisfying*

- (B1) $\bar{u} \in H^1_0(\Omega)$,
- (B2) $\operatorname{div} \bar{u} = 0$ *on* Ω,
- (B3) *for any divergence-free $w \in C^\infty_c(\Omega)$ we have:*

$$\int_\Omega \nabla\bar{u} : \nabla w = 6\pi \int_\Omega [j - \rho\bar{u}] \cdot w. \tag{22}$$

Theorem 1 is a corollary of this theorem as (B1)-(B2)-(B3) corresponds to the generalized formulation of the Stokes-Brinkman system (3)-(4). The remainder of the paper is devoted to the proof of this result. In this section, we prove that u^N is bounded in $H^1_0(\Omega)$, extract a subsequence weakly converging to some $\bar{u}$ and justify that it remains then to prove that $\bar{u}$ satisfies (B3). The two next sections are devoted to the computations of integrals:

$$I_w = \int_\Omega \nabla \bar{u} : \nabla w$$

for a fixed but arbitrary divergence-free $w \in C_c^\infty(\Omega)$.

So, we first compute uniform bounds on u^N by applying the variational characterization of solutions to the Stokes problem (10). Given $N \in \mathbb{N}$, we set:

$$v^N(x) = \sum_{i=1}^N \nabla \times \left(\frac{\chi^N(x - h_i^N)}{2}\, v_i^N \times (x - h_i^N) \right) =: \sum_{i=1}^N v_i(x).$$

Then, $v^N \in C_c^\infty(\mathbb{R}^3)$ is the curl of a smooth potential vector so that $\operatorname{div} v^N = 0$. Because of assumption (A2), there exists a $N_0 \in \mathbb{N}$ such that:

$$N d^N_{min} > 4\,, \quad \forall\, N > N_0.$$

Let $N > N_0$ from now on. Because χ^N has support in $B_\infty(0, 2/N)$ we have that $Supp\, v_i \subset B_\infty(h_i^N, 2/N)$ and the $(v_i)_{i=1,\dots,N}$ have disjoint supports. Because χ^N is 1 on $B(0,1/N) \subset B_\infty(0,1/N)$ we derive further that, for $i \in \{1,\dots,N\}$:

$$\begin{aligned} v_i(x) &= 0\,, \quad \text{on } \partial\mathcal{F}^N \cup \bigcup_{j \neq i} B_j^N\,, \\ v_i(x) &= \nabla \times \left(\frac{1}{2}\, v_i^N \times (x - h_i^N) \right) = v_i^N\,, \quad \text{on } B_i^N. \end{aligned}$$

By combination, we obtain:

$$\begin{aligned} v^N(x) &= v_i^N\,, \quad \text{on } B_i^N\,, \quad \forall\, i = 1,\dots,N\,, \\ v^N(x) &= 0\,, \quad \text{on } \partial\mathcal{F}^N\,. \end{aligned}$$

We have then by Theorem 3 that:

$$\|\nabla u^N\|_{L^2(\mathcal{F}^N)} \leqslant \|\nabla v^N\|_{L^2(\mathcal{F}^N)} = \left(\sum_{i=1}^N \|\nabla v_i\|^2_{L^2(\mathbb{R}^3)} \right)^{\frac{1}{2}}. \tag{23}$$

For arbitrary $N \in \mathbb{N}$ and $i \in \{1,\dots,N\}$, there holds:

$$\begin{aligned} |\nabla v_i(x)| &\lesssim |\nabla \chi^N(x - h_i^N)||v_i^N| + |\nabla^2 \chi^N(x - h_i^N)||v_i^N||x - h_i^N| \\ &\lesssim N \left(|\nabla \chi(N(x - h_i^N))| + |\nabla^2 \chi(N(x - h_i^N))| \right) |v_i^N|\,. \end{aligned}$$

Consequently, by a standard scaling argument:

$$\int_{\mathbb{R}^3} |\nabla v_i(x)|^2 \mathrm{d}x \lesssim \frac{1}{N} \left(\int_{\mathbb{R}^3} |\nabla \chi(|y|)|^2 + |\nabla^2 \chi(|y|)|^2 \mathrm{d}y \right) |v_i^N|^2\,.$$

Then, for $N > N_0$, we combine the previous computation into:

$$\|\nabla v^N\|^2_{L^2(\mathcal{F}^N)} \lesssim \frac{1}{N}\sum_{i=1}^N |v_i^N|^2 \,.$$

Note that χ is fixed a priori so that all constants depending on χ may be considered as non-significant. Assumption (A1) then yields that there exists $\mathcal{E}^\infty < \infty$ so that:

$$\left(\frac{1}{N}\sum_{i=1}^N |v_i^N|^2\right)^{\frac{1}{2}} \leqslant \mathcal{E}^\infty \,, \quad \forall\, N > N_0. \tag{24}$$

By (23) the norm of u^N in $H^1_0(\Omega)$ is also bounded by $\mathcal{E}^\infty$. We keep the symbol $\mathcal{E}^\infty$ to denote the above bound in what follows.

As u^N is bounded in $H^1_0(\Omega)$, it is weakly-compact and we denote by $\bar{u}$ a cluster-point for the weak topology. It is straightforward that $\bar{u}$ satisfies $\operatorname{div} \bar{u} = 0$ on Ω. So $\bar{u}$ satisfies (B1) and (B2) of our theorem. The remainder of the proof consists in showing that it satisfies (B3) also. Indeed, we remark that ρ is the density of a probability measure. Hence $\rho \geqslant 0$ on Ω. By a simple energy estimate one may then show that, given $j \in L^2(\Omega)$, there exists at most one $\bar{u} \in H^1_0(\Omega)$ that satisfies simultaneously (B1)-(B2)-(B3). A direct corollary of this remark is that, if we prove that (B3) is satisfied by $\bar{u}$ we have uniqueness of the possible cluster point to the sequence $(u^N)_{N\in\mathbb{N}}$ and the whole sequence converges to this $\bar{u}$ in $H^1_0(\Omega) - w$.

4. Proof of Theorem 1 – Computations for finite N

From now on, we assume that u^N converges weakly to $\bar{u}$ in $H^1_0(\Omega)$ (we do not relabel the subsequence for simplicity) and we fix a divergence-free $w \in C^\infty_c(\Omega)$. We aim to compute the scalar product:

$$\int_\Omega \nabla \bar{u} : \nabla w.$$

By definition, we have:

$$\int_\Omega \nabla \bar{u} : \nabla w = \lim_{N\to\infty} I^N \text{ with } I^N = \int_\Omega \nabla u^N : \nabla w\,, \quad \forall\, N \in \mathbb{N}\,.$$

As classical, we want to apply the equation satisfied by u^N in order to compute I^N in a way that makes possible to use the assumption on the convergence of the empiric measures S_N. To do this, we fix an integer $\delta \geqslant 4$, we construct, for fixed N, a suitable test-function w^s (depending actually on δ and N) so that

- we make an error of order $1/\sqrt{\delta}$ by replacing w with w^s in I^N,
- replacing w with w^s in I^N we prove that,

  $$\int_\Omega \nabla u^N : \nabla w^s \to 6\pi \int_\Omega (j - \rho\bar{u})\cdot w + error\,,$$

  when $N \to \infty$, with an *error* of size $1/\sqrt{\delta}$.

As δ can be taken arbitrary large, this will yield the expected result.

We explain now the construction of w^s. The integer $\delta \geqslant 4$ is fixed in the remainder of this section. For a given $N \in \mathbb{N}$, applying the construction in Appendix B, we obtain $(T_\kappa^N)_{\kappa\in\mathbb{Z}^3}$ a covering of $\mathbb{R}^3$ with cubes of width $1/N^{1/3}$ such that denoting:

$$\mathcal{Z}_\delta^N := \left\{ i \in \{1,\dots,N\} \text{ s.t. } \mathrm{dist}\left(h_i^N , \bigcup_{\kappa\in\mathbb{Z}^3} \partial T_\kappa^N \right) < \frac{1}{\delta N^{1/3}} \right\},$$

there holds:

$$\frac{1}{N}\sum_{i\in\mathcal{Z}_\delta^N}(1+|v_i^N|^2) \leqslant \frac{12}{\delta}\frac{1}{N}\sum_{i=1}^N(1+|v_i^N|^2) \leqslant \frac{12(1+|\mathcal{E}^\infty|^2)}{\delta}. \tag{25}$$

Moreover, for $N \geqslant N_w$, for a N_w depending only on w and Ω, keeping only the indices $\mathcal{K}^N$ such that T_κ^N intersect $Supp(w)$, we obtain a covering $(T_\kappa^N)_{\kappa\in\mathcal{K}^N}$ of $Supp(w)$ such that all the cubes are included in Ω (see the appendix for more details). We assume $N \geqslant N_w$ from now on. We emphasize that we do not make precise the set of indices $\mathcal{K}^N$. The only relevant property to our computations is that

$$\#\mathcal{K}^N \leqslant N|\Omega|. \tag{26}$$

This inequality is derived by remarking that the T_κ^N are disjoint cubes of volume $1/N$ that are all included in Ω. Associated to this covering, we introduce the following notations. For arbitrary $\kappa \in \mathcal{K}^N$, we set

$$\mathcal{I}_\kappa^N := \{ i \in \{1,\dots,N\} \text{ s.t. } h_i^N \in T_\kappa^N \}, \quad M_\kappa^N := \#\mathcal{I}_\kappa^N,$$

and $\mathcal{I}^N := \bigcup_{\kappa\in\mathcal{K}^N}\mathcal{I}_\kappa^N$. Because of assumption (A3), there exists $M^\infty \in \mathbb{N}$ such that:

$$M_\kappa^N \leqslant M^\infty, \quad \forall\, \kappa \in \mathcal{K}^N, \quad \forall\, N \in \mathbb{N}. \tag{27}$$

In brief, the set of indices $\{1,\dots,N\}$ contains the three important subsets:

- the subset $\mathcal{I}^N$ contains all the indices that are "activated" in our computations,
- the subset $\mathcal{Z}_\delta^N$ contains the indices that are close to boundaries of the partition,
- the subset $\{1,\dots,N\}\setminus\mathcal{I}^N$ contains indices that are not activated.

We emphasize that $\mathcal{Z}_\delta^N$ contains indices that can be in both $\mathcal{I}^N$ and its complement.

We construct then w^s piecewisely on the covering of $Supp(w)$. Given $\kappa \in \mathcal{K}^N$, we set:

$$w_\kappa^s(x) = \sum_{i\in\mathcal{I}_\kappa^N\setminus\mathcal{Z}_\delta^N} U^N[w(h_i^N)](x-h_i^N), \quad \forall\, x \in \mathbb{R}^3, \tag{28}$$

and

$$w^s = \sum_{\kappa\in\mathcal{K}^N} w_\kappa^s \mathbf{1}_{T_\kappa^N}$$

We note that $w^s \notin H_0^1(\mathcal{F}^N)$ because of jumps at interfaces ∂T_κ^N. It will be sufficient for our purpose that $w^s \in H^1(\mathring{T}_\kappa^N)$ for arbitrary $\kappa \in \mathcal{K}^N$. In a cube $\mathring{T}_\kappa^N$ the test function w^s is thus a combination of stokeslets centered in the h_i^N that are contained in the cell. We delete from this combination the centers that are too close to ∂T_κ^N (namely $1/(\delta N^{1/3})$ close to ∂T_κ^N). We proceed by proving that we make a small error by replacing w with w^s in I^N :

Proposition 7. *There exists $K \in (0,\infty)$ and $N_\delta \in \mathbb{N}$ depending only on δ and w for which, given $N > N_\delta$, there holds:*

$$\left| \int_\Omega \nabla u^N : \nabla w - \sum_{\kappa \in \mathcal{K}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w^s \right| \leqslant K \left(\frac{1}{\sqrt{\delta}} + \frac{1}{N^{\frac{1}{6}}} + \frac{1}{\sqrt{N d_{min}}} \right) . \tag{29}$$

Proof. We proceed in several steps by introducing different intermediate test-functions. In this proof, we use symbol $\lesssim$ to denote inequalities with constants that do not depend on N and δ.

First step: Construction of auxiliary test-functions. For arbitrary $\kappa \in \mathcal{K}^N$, we consider the Stokes problem on $\mathring{T}_\kappa^N \setminus \bigcup_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \overline{B_i^N}$ with boundary conditions:

$$\left\{ \begin{array}{rcll} u(x) & = & w(x)\,, & \text{on } \partial B_i^N \text{ for } i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N\,, \\ u(x) & = & 0\,, & \text{on } \partial T_\kappa^N\,. \end{array} \right. \tag{30}$$

We note that this problem enters the framework of Section 2.2. Indeed, let denote:

$$d_m^\kappa := \min_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \left\{ \mathrm{dist}(h_i^N, \partial T_\kappa^N)\,, \min_{j \neq i} |h_i^N - h_j^N| \right\}$$

Because we deleted the indices of $\mathcal{Z}_\delta^N$, we have that:

$$d_m^\kappa \geqslant \min \left(d_{min}^N, \frac{1}{\delta N^{\frac{1}{3}}} \right) .$$

In particular, for N sufficiently large depending only on δ ($N > N_\delta = \max(N_0, (4\delta)^{3/2})$) the d_m^κ satisfy assumption (16) uniformy in $\kappa \in \mathcal{K}^N$. We remark that making N_δ larger than δ^3, we have also that, for $N > N_\delta$, there holds:

$$\frac{1}{\delta N^{\frac{1}{3}}} \geqslant \frac{1}{N^{\frac{2}{3}}} .$$

Consequently, we apply below that not only assumption (16) is satisfied, but also:

$$\frac{1}{N d_m^\kappa} \leqslant \max \left(\frac{1}{N d_{min}^N}, \frac{1}{N^{\frac{1}{3}}} \right) , \quad \forall\, N > N_\delta\,. \tag{31}$$

So, for $N > N_\delta$ the arguments developed in Section 2.2 entail that there exists a unique generalized solution to the Stokes problem on $\mathring{T}_\kappa^N \setminus \bigcup_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \overline{B_i^N}$ with boundary condition (30). We denote this solution by $\bar{w}_\kappa$. We keep notation $\bar{w}_\kappa$ to denote its extension to Ω (by w on the holes and by 0 outside $\mathring{T}_\kappa^N$). As $\mathring{T}_\kappa^N \subset \Omega$, we obtain a divergence-free $\bar{w}_\kappa \in H_0^1(\Omega)$. We then add the $\bar{w}_\kappa$ into:

$$\bar{w} = \sum_{\kappa \in \mathcal{K}^N} \bar{w}_\kappa\,.$$

To summarize, this vector field satisfies:

- $\bar{w} \in H^1_0(\Omega)$,
- $\mathrm{div}\, \bar{w} = 0$ on Ω,
- $\bar{w} = w(x)$ on B_i^N for all $i \in \mathcal{I}^N \setminus \mathcal{Z}^N_\delta$.

We correct now the value of $\bar{w}$ on the B_i^N when $i \in \mathcal{Z}^N_\delta \cup (\{1,\ldots,N\} \setminus \mathcal{I}^N)$ in order that it fits the same boundary conditions as w on $\mathcal{F}^N$. We set:

$$\begin{aligned}\tilde{w} \;&=\; \sum_{i \in \mathcal{Z}^N_\delta} \Big[\chi^N(\cdot - h_i^N) w - \mathfrak{B}_{h_i^N,1/N,2/N}[x \mapsto w(x)\cdot \nabla\chi^N(x-h_i^N)]\Big] \\ &\quad + \prod_{i \in \mathcal{Z}^N_\delta} (1-\chi^N(\cdot - h_i^N))\bar{w} + \sum_{i\in\mathcal{Z}^N_\delta} \mathfrak{B}_{h_i^N,1/N,2/N}[x \mapsto \bar{w}(x)\cdot\nabla\chi^N(x-h_i^N)]\,.\end{aligned}$$

Briefly, one may interpret the construction of $\tilde{w}$ as follows. The sum on the first line creates a divergence-free lifiting of the boundary conditions prescribed by w on the ∂B_i^N for $i \in \mathcal{Z}^N_\delta$. On the second line is a divergence-free truncation of $\bar{w}$ that creates a vector-field vanishing on $\cup_{i\in\mathcal{Z}^N_\delta} B_i^N$. We remark that this vector-field is well defined because, by similar computations as we did in the proof of Proposition 5, we have:

$$\int_{A(h_i^N,1/N,2/N)} \bar{w}(x)\cdot\nabla\chi^N(x-h_i^N)\mathrm{d}x = \int_{A(h_i^N,1/N,2/N)} w(x)\cdot\nabla\chi^N(x-h_i^N)\mathrm{d}x = 0\,, \quad \forall\, i \in \mathcal{Z}^N_\delta.$$

Hence, we may apply the Bogovskii operator which lifts the divergence term in the brackets with a vector-field vanishing on the boundaries of $A(h_i^N,1/N,2/N)$ that we extend by 0 on $\mathbb{R}^3 \setminus A(h_i^N,1/N,2/N)$. When $i \in \{1,\ldots,N\}\setminus(\mathcal{I}^N \cup \mathcal{Z}^N_\delta)$ we have that h_i^N is in the $1/\delta N^{1/3}$ core of the cube T^N_κ that contains him. Moreover the index of this cube is not in $\mathcal{K}^N$. As $N > N_\delta$, we have then $B(h_i^N,1/N) \subset T^N_\kappa$ where there already holds that $\bar{w}(x) = w(x) = 0$. Consequently, there is nothing to correct on these holes. This is the reason why they do not appear in the above construction.

Direct computations show that $\mathrm{div}\,\tilde{w} = 0$ on Ω. On the other hand, because $N > N_0$, the family of balls $(B_\infty(h_i^N,2/N))_{i=1,\ldots,N}$ are disjoint and included in Ω. Hence, the truncations that we perform in $\tilde{w}$ do not perturb the value of $\bar{w}$ neither on the B_i^N for $i \in \{1,\ldots,N\}\setminus\mathcal{Z}^N_\delta$ nor on $\partial\Omega$. This remark entails that

- for $i \in \mathcal{I}^N \setminus \mathcal{Z}^N_\delta$:
$$\tilde{w}(x) = \bar{w}(x) = w(x)\,, \quad \text{on } B_i^N\,,$$
- for $i \in \mathcal{Z}^N_\delta$:
$$\tilde{w}(x) = \chi^N(x-h_i^N)w(x) = w(x)\,, \quad \text{on } B_i^N\,,$$
- for $i \in \{1,\ldots,N\}\setminus(\mathcal{I}^N \cup \mathcal{Z}^N_\delta)$:
$$\tilde{w}(x) = \bar{w}(x) = 0 = w(x)\,, \quad \text{on } B_i^N\,,$$
- $w(x) = 0\,,$ on $\partial\Omega$.

Consequently, by restriction, there holds that $w - \tilde{w} \in H^1_0(\mathcal{F}^N)$ is divergence-free. As u^N is a generalized solution to a Stokes problem on $\mathcal{F}^N$ we have thus:

$$\int_{\mathcal{F}^N} \nabla u^N : \nabla (w - \tilde{w}) = 0.$$

We rewrite this identity as follows:

$$\int_{\Omega} \nabla u^N : \nabla w = \sum_{\kappa \in \mathcal{K}^N} \int_{T^N_\kappa} \nabla u^N : \nabla w^s - E_1 - E_2 \,, \tag{32}$$

with :

$$\begin{aligned} E_1 &= \sum_{\kappa \in \mathcal{K}^N} \int_{T^N_\kappa} \nabla u^N : \nabla (w^s_\kappa - \bar{w}_\kappa)\,, \\ E_2 &= \int_{\Omega} \nabla u^N : \nabla (\bar{w} - \tilde{w})\,. \end{aligned}$$

Second step: Control of error term E_1. For arbitrary $\kappa \in \mathcal{K}^N$, we apply Proposition 5 to $\bar{w}_\kappa$ and its corresponding combination of stokeslets (namely, the restriction w^s_κ of w^s to $\mathring{T}^N_\kappa$). By construction, d^κ_m satisfies the requirement $d^\kappa_m > 4/N$ for $N > N_\delta$. We have thus:

$$\|\nabla (w^s_\kappa - \bar{w}_\kappa)\|_{L^2(T^N_\kappa)} \lesssim \sqrt{\frac{M^N_\kappa}{N}} \left(\frac{1}{N} + \sqrt{\frac{M^N_\kappa}{N d^\kappa_m}} \right) \|w\|_{W^{1,\infty}}\,.$$

Note here that $\#(\mathcal{I}^N_\kappa \setminus \mathcal{Z}^N_\delta) \leqslant \#\mathcal{I}^N_\kappa = M^N_\kappa$. Consequently, introducing this last bound in the computation of E_1 and applying a standard Cauchy-Schwarz inequality together with (26)-(27) yields:

$$\begin{aligned} |E_1| &\lesssim \sum_{\kappa \in \mathcal{K}^N} \|\nabla u^N\|_{L^2(T^N_\kappa)} \sqrt{\frac{M^N_\kappa}{N}} \left(\frac{1}{N} + \sqrt{\frac{M^N_\kappa}{N d^\kappa_m}} \right) \|w\|_{W^{1,\infty}}\,, \\ &\lesssim \left(\sum_{\kappa \in \mathcal{K}^N} \|\nabla u^N\|^2_{L^2(T^N_\kappa)} \right)^{\frac{1}{2}} \sqrt{M^\infty} \left(\frac{1}{N} + \sqrt{\frac{M^\infty}{N d^\kappa_m}} \right) \|w\|_{W^{1,\infty}}\,. \end{aligned}$$

Here, we note again that, by construction, the T^N_κ are disjoint and included in Ω so that

$$\left(\sum_{\kappa \in \mathcal{K}^N} \|\nabla u^N\|^2_{L^2(T^N_\kappa)} \right)^{\frac{1}{2}} \leqslant \|\nabla u^N\|_{L^2(\Omega)}.$$

Applying the uniform bound for u^N in $H^1_0(\Omega)$ we conclude then that:

$$|E_1| \lesssim \mathcal{E}^\infty \left(\frac{\sqrt{M^\infty}}{N} + \frac{M^\infty}{\sqrt{N d^\kappa_m}} \right) \|w\|_{W^{1,\infty}}\,,$$

and, introducing (31), that:

$$|E_1| \lesssim \mathcal{E}^\infty M^\infty \left(\frac{1}{N^{\frac{1}{6}}} + \frac{1}{\sqrt{N d^N_{min}}} \right) \|w\|_{W^{1,\infty}} . \tag{33}$$

Second step: Control of error term E_2. As for the second term, we replace $\tilde{w}$ by its explicit construction. We remark that because the supports of the $(\chi^N(\cdot - h^N_i))_{i\in\{1,\dots,N\}}$ are disjoint (as $d^N_{min} > 4/N$ for $N > N_0$) we have:

$$1 - \prod_{i\in\mathcal{Z}^N_\delta} (1 - \chi^N(x - h^N_i)) = \sum_{i\in\mathcal{Z}^N_\delta} \chi^N(x - h^N_i)\,, \quad \forall\, x \in \Omega.$$

Consequently, we split:

$$\begin{aligned} \bar{w} - \tilde{w} \ &= \sum_{i\in\mathcal{Z}^N_\delta} \Big[\chi^N(\cdot - h^N_i)\bar{w} - \mathfrak{B}_{h^N_i,1/N,2/N}[x \mapsto \bar{w}(x)\cdot\nabla\chi^N(x - h^N_i)]\Big] \\ &\quad - \sum_{i\in\mathcal{Z}^N_\delta} \Big[\chi^N(\cdot - h^N_i)w - \mathfrak{B}_{h^N_i,1/N,2/N}[x \mapsto w(x)\cdot\nabla\chi^N(x - h^N_i)]\Big] . \end{aligned}$$

By direct computations and application of Lemma 15 to the Bogovskii operator $\mathfrak{B}_{h^N_i,1/N,2/N}$, we find $E^i_2 \in L^2(B_\infty(h^N_i, 2/N))$, $i \in \mathcal{Z}^N_\delta$, for which:

$$\nabla(\bar{w} - \tilde{w}) = \sum_{i\in\mathcal{Z}^N_\delta} E^i_2 \mathbf{1}_{B_\infty(h^N_i,2/N)}\,,$$

and such that:

$$\|E^i_2\|^2_{L^2(B_\infty(h^N_i,2/N))} \lesssim \frac{1}{N}\|w\|^2_{W^{1,\infty}} + N^2\|\bar{w}\|^2_{L^2(B_\infty(h^N_i,2/N))} + \|\nabla\bar{w}\|^2_{L^2(B_\infty(h^N_i,2/N))} .$$

Introducing these bounds in the computation of E_2, and reproducing similar computations as for E_1, we derive:

$$\begin{aligned} |E_2| \ &\leqslant \sum_{i\in\mathcal{Z}^N_\delta} \int_{B_\infty(h^N_i,2/N)} |\nabla u^N||E^i_2|\,, \\ &\leqslant \left(\sum_{i\in\mathcal{Z}^N_\delta} \|\nabla u^N\|^2_{L^2(B_\infty(h^N_i,2/N))} \right)^{\frac{1}{2}} \left(\sum_{i\in\mathcal{Z}^N_\delta} \|E^i_2\|^2_{L^2(B_\infty(h^N_i,2/N)))} \right)^{\frac{1}{2}}, \\ &\lesssim \mathcal{E}^\infty \left(\sum_{i\in\mathcal{Z}^N_\delta} \|E^i_2\|^2_{L^2(B_\infty(h^N_i,2/N)))} \right)^{\frac{1}{2}}, \end{aligned}$$

where we applied again that the $(B_\infty(h_i^N,2/N))_{i\in\mathcal{Z}_\delta^N}$ are disjoint and cover a subset of Ω. To complete the proof, it remains to compute:

$$\sum_{i\in\mathcal{Z}_\delta^N}\|E_2^i\|^2_{L^2(B_\infty(h_i^N,2/N))} \leqslant \sum_{i\in\mathcal{Z}_\delta^N}\frac{1}{N}\|w\|^2_{W^{1,\infty}}+N^2\|\bar{w}\|^2_{L^2(B_\infty(h_i^N,2/N))}+\|\nabla\bar{w}\|^2_{L^2(B_\infty(h_i^N,2/N))}\,.$$

We recall that, by choice of the covering (see (25)), we have:

$$\sum_{i\in\mathcal{Z}_\delta^N}\frac{1}{N}\lesssim\frac{1}{\delta}(1+|\mathcal{E}^\infty|^2). \tag{34}$$

Consequently, there holds:

$$\sum_{i\in\mathcal{Z}_\delta^N}\frac{1}{N}\|w\|^2_{W^{1,\infty}}\lesssim\frac{\|w\|^2_{W^{1,\infty}}}{\delta}(1+|\mathcal{E}^\infty|^2)\,.$$

To compute the terms depending on $\bar{w}$ we apply again Proposition 5 in order to write $\bar{w}=w^s+l.o.t.$ To this end, we remark that given $i\in\mathcal{Z}_\delta^N$ we have that h_i^N is in the $1/(\delta N^{1/3})$-neighborhood of some ∂T_κ^N then either $B_\infty(h_i^N,2/N)\subset T_\kappa^N$ or $B_\infty(h_i^N,2/N)$ intersects other cubes. In that second case, as the diameter of $B_\infty(h_i^N,2/N)$ is much smaller than the one of a cube T_κ^N, we have that $B_\infty(h_i^N,2/N)$ may intersect at most 2 "successive" cubes in the covering of $\mathbb{R}^3$, in all directions, and thus at most 8 cubes. Consequently:

$$\begin{aligned}\|\bar{w}\|^2_{L^2(B_\infty(h_i^N,2/N))} &\leqslant 8\sup_{\kappa\in\mathcal{K}^N}\|\bar{w}_\kappa\|^2_{L^2(B_\infty(h_i^N,2/N)\cap T_\kappa^N)}\,,\\ \|\nabla\bar{w}\|^2_{L^2(B_\infty(h_i^N,2/N))} &\leqslant 8\sup_{\kappa\in\mathcal{K}^N}\|\nabla\bar{w}_\kappa\|^2_{L^2(B_\infty(h_i^N,2/N)\cap T_\kappa^N)}\,.\end{aligned}$$

As $\bar{w}=0$ on any cube T_κ^N whose index $\kappa\notin\mathcal{K}^N$, the corresponding indices κ do not appear in these latter supremums. If $B_\infty(h_i^N,2/N)\cap T_\kappa^N\neq\emptyset$ then we may introduce w_κ^s

$$\begin{aligned}\|\nabla\bar{w}_\kappa\|^2_{L^2(B_\infty(h_i^N,2/N)\cap T_\kappa^N)} &\lesssim \|\nabla(\bar{w}_\kappa-w_\kappa^s)\|^2_{L^2(T_\kappa^N)}+\|\nabla w_\kappa^s\|^2_{L^2(B_\infty(h_i^N,2/N))}\,,\\ N^2\|\bar{w}_\kappa\|^2_{L^2(B_\infty(h_i^N,2/N)\cap T_\kappa^N)} &\lesssim N^2\|(\bar{w}_\kappa-w_\kappa^s)\|^2_{L^2(B_\infty(h_i^N,2/N)\cap T_\kappa^N)}+N^2\|w_\kappa^s\|^2_{L^2(B_\infty(h_i^N,2/N))}\,.\end{aligned}$$

We compute the terms involving w_κ^s by using the explicit formula (28) and the expansion of stokeslet (14). Remarking that for $N>N_0$ the distance between h_j^N and $B_\infty(h_i^N,2/N)$ is larger than $d_{min}^N/2$ (for arbitrary $j\in\mathcal{I}_\kappa^N\setminus\mathcal{Z}_\delta^N$) and recalling that there are at most M_κ^N indices in $\mathcal{I}_\kappa^N\setminus\mathcal{Z}_\delta^N$ we derive the bound:

$$|w_\kappa^s(x)|\lesssim\frac{M_\kappa^N}{Nd_{min}^N}\|w\|_{W^{1,\infty}}\,,\quad |\nabla w_\kappa^s(x)|\lesssim\frac{M_\kappa^N}{N|d_{min}^N|^2}\|w\|_{W^{1,\infty}}\,,\quad \forall\, x\in B_\infty(h_i^N,2/N)\,,$$

and consequently:

$$\begin{aligned}
\|\nabla w^s_\kappa\|^2_{L^2(B_\infty(h^N_i,2/N))} &\lesssim \frac{|M^N_\kappa|^2}{N^5|d^N_{min}|^4}\|w\|^2_{W^{1,\infty}} \lesssim \frac{|M^\infty|^2}{N}\|w\|^2_{W^{1,\infty}}\,,\\
N^2\|w^s_\kappa\|_{L^2(B_\infty(h^N_i,2/N))} &\lesssim \frac{|M^N_\kappa|^2}{N^3|d^N_{min}|^2}\|w\|^2_{W^{1,\infty}} \lesssim \frac{|M^\infty|^2}{N}\|w\|^2_{W^{1,\infty}}\,.
\end{aligned}$$

where we applied $Nd^N_{min} \geqslant 4$ and (27). For the remainder terms, we apply again Proposition 5 yielding:

$$\|\nabla(\bar{w}_\kappa - w^s_\kappa)\|^2_{L^2(T^N_\kappa)} \lesssim \frac{M^N_\kappa}{N}\left(\frac{1}{N} + \sqrt{\frac{M^N_\kappa}{Nd^\kappa_m}}\right)^2 \|w\|^2_{W^{1,\infty}} \lesssim \frac{|M^\infty|^2}{N}\|w\|^2_{W^{1,\infty}}\,,$$

and, by combining Hölder inequality together with the bound in $L^6(T^N_\kappa)$ obtained in Proposition 5:

$$\begin{aligned}
N^2\|(\bar{w}_\kappa - w^s_\kappa)\|^2_{L^2(B_\infty(h^N_i,2/N)\cap T^N_\kappa)} &\leqslant N^2|B_\infty(h^N_i,2/N)|^{\frac{2}{3}}\|(\bar{w}_\kappa - w^s_\kappa)\|^2_{L^6(B_\infty(h^N_i,2/N)\cap T^N_\kappa)}\,,\\
&\lesssim \|(\bar{w}_\kappa - w^s_\kappa)\|^2_{L^6(\mathring{T}^N_\kappa \setminus \bigcup_{j\in\mathcal{I}^N_\kappa\setminus\mathcal{Z}^N_\delta} B^N_j)}\,,\\
&\lesssim \frac{M^N_\kappa}{N}\left(\frac{1}{N} + \sqrt{\frac{M^N_\kappa}{Nd^\kappa_m}}\right)^2 \|w\|^2_{W^{1,\infty}} \lesssim \frac{|M^\infty|^2}{N}\|w\|^2_{W^{1,\infty}}\,.
\end{aligned}$$

Gathering these computations and applying (34), we obtain that for $N > N_\delta$:

$$\sum_{i\in\mathcal{Z}^N_\delta} \frac{1}{N}\|w\|^2_{W^{1,\infty}} + N^2\|\bar{w}\|^2_{L^2(B_\infty(h^N_i,2/N))} + \|\nabla\bar{w}\|^2_{L^2(B_\infty(h^N_i,2/N))} \lesssim \frac{1}{\delta}\|w\|^2_{W^{1,\infty}}|M^\infty|^2(1+|\mathcal{E}^\infty|^2),$$

so that:

$$|E_2| \lesssim \frac{M^\infty}{\sqrt{\delta}}\|w\|_{W^{1,\infty}}(1+|\mathcal{E}^\infty|^2)^{\frac{3}{2}}\,. \tag{35}$$

Combining (33) and (35) in (32), we obtain the expected result. □

5. Proof of Theorem 1 – Asymptotics $N \to \infty$

In this section, we end the proof of Theorem 1 keeping the notations introduced in the previous section. A straightforward corollary of Proposition 7 reads :

Corollary 8. *There exists $K \in (0,\infty)$ such that for arbitrary $\delta \geqslant 4$, there holds:*

$$\limsup_{N\to\infty}\left|\int_\Omega \nabla u^N : \nabla w - \sum_{\kappa\in\mathcal{I}^N}\int_{T^N_\kappa}\nabla u^N : \nabla w^s\right| \leqslant \frac{K}{\sqrt{\delta}}.$$

So in this section, we prove the following proposition:

Proposition 9. *There exists $K \in (0,\infty)$ such that, for arbitrary $\delta \geqslant 4$, there holds:*

$$\limsup_{N\to\infty} \left| \sum_{\kappa \in \mathcal{I}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w^s - 6\pi \int_\Omega (j - \rho \bar{u}) \cdot w \right| \leqslant \frac{K}{\sqrt{\delta}}\,.$$

This will end the proof of Theorem 1. Indeed, combining the above corollary and proposition, we obtain that there exists K which does not depend on δ such that, for arbitrary $\delta \geqslant 4$:

$$\limsup_{N\to\infty} \left| \int_\Omega \nabla u^N : \nabla w - 6\pi \int_\Omega (j - \rho\bar{u}) \cdot w \right| \leqslant \frac{K}{\sqrt{\delta}}\,.$$

As

$$\lim_{N\to\infty} \int_\Omega \nabla u^N : \nabla w = \int_\Omega \nabla \bar{u} : \nabla w\,,$$

and δ can be made arbitrary large, this entails that

$$\int_\Omega \nabla \bar{u} : \nabla w = 6\pi \int_\Omega (j - \rho\bar{u}) \cdot w\,,$$

and we obtain that $\bar{u}$ satisfies (B3).

We give now a proof of Proposition 9. From now on δ is fixed larger than 4 and we assume, with the conventions of the previous section, that:

$$N \geqslant \max(N_0, N_w, N_\delta)\,.$$

For such a N, we denote:

$$\tilde{I}^N = \sum_{\kappa\in\mathcal{K}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w^s = \sum_{\kappa\in\mathcal{K}^N} \int_{T_\kappa^N} \nabla u^N : \nabla w^s_\kappa.$$

First, let fix $\kappa \in \mathcal{K}^N$ and simplify

$$\tilde{I}^N_\kappa := \int_{T_\kappa^N} \nabla u^N : \nabla w^s_\kappa\,.$$

By definition, we have that:

$$w^s_\kappa(x) = \sum_{i \in \mathcal{I}^N_\kappa \setminus \mathcal{Z}^N_\delta} U^N[w(h_i^N)](x - h_i^N)\,, \quad \forall\, x \in \mathbb{R}^3\,,$$

so that, introducing the associated pressures $x \mapsto P^N[w(h_i^N)](x-h_i^N)$, we obtain (recall that u^N is divergence-free and constant on the B_i^N):

$$
\begin{aligned}
\tilde{I}_\kappa^N &= \int_{T_\kappa^N} \nabla u^N : \nabla w_\kappa^s \,, \\
&= \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \int_{T_\kappa^N \setminus B_i^N} \nabla u^N(x) : [\nabla U^N[w(h_i^N)](x-h_i^N) - P^N[w(h_i^N)](x-h_i^N)\mathbb{I}_3] \mathrm{d}x \,, \\
&= \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \left(\int_{\partial B_i^N} \left\{ \partial_n U^N[w(h_i^N)](x-h_i^N) - P^N[w(h_i^N)](x-h_i^N) n \right\} \cdot v_i^N \mathrm{d}\sigma \right. \\
&\qquad \left. + \int_{\partial T_\kappa^N} \left\{ \partial_n U^N[w(h_i^N)](x-h_i^N) - P^N[w(h_i^N)](x-h_i^N) n \right\} \cdot u^N(x) \mathrm{d}\sigma \right) , \\
&= \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} I_{i,int}^N + I_{i,ext}^N \,,
\end{aligned}
$$

where, we denoted:

$$
\begin{aligned}
I_{i,int}^N &= \int_{\partial B_i^N} \left\{ \partial_n U^N[w(h_i^N)](x-h_i^N) - P^N[w(h_i^N)](x-h_i^N) n \right\} \cdot v_i^N \mathrm{d}\sigma \,, \\
I_{i,ext}^N &:= \int_{\partial T_\kappa^N} \left\{ \partial_n U^N[w(h_i^N)](x-h_i^N) - P^N[w(h_i^N)](x-h_i^N) n \right\} \cdot u^N(x) \mathrm{d}\sigma \,.
\end{aligned}
$$

Recalling that (U^N, P^N) is the solution to the Stokes problem in the exterior of a ball of radius $1/N$, and that v_i^N is constant on ∂B_i^N, we have an explicit value for the interior integral whatever the value of the index i (see (15)):

$$
I_{i,int}^N = \frac{6\pi}{N} w(h_i^N) \cdot v_i^N \,.
$$

For the other term, we apply that the diameter of T_κ^N is small so that we may approximate u^N on ∂T_κ^N by a constant. Namely, we choose:

$$
\bar{u}_\kappa^N = \frac{1}{|[T_\kappa^N]_{2\delta}|} \int_{[T_\kappa^N]_{2\delta}} u^N(x) \mathrm{d}x,
$$

where $[T_\kappa^N]_{2\delta}$ is the $1/(2\delta N^{1/3})$ neighborhood of ∂T_κ^N inside $\mathring{T}_\kappa^N$. At this point, we remark that we have actually two notations for the same quantity. Indeed, a simple draw shows that introducing x_κ^N the center of T_κ^N, we have:

$$
\mathring{T}_\kappa^N = B_\infty\left(x_\kappa^N, \frac{1}{2N^{1/3}}\right) \quad \text{while} \quad [T_\kappa^N]_{2\delta} = A\left(x_\kappa^N, \frac{1-1/\delta}{2N^{1/3}}, \frac{1}{2N^{1/3}}\right) .
$$

So, we replace:

$$
\begin{aligned}
I_{i,ext}^N &= \int_{\partial T_\kappa^N} \left\{ \partial_n U^N[w(h_i^N)](x-h_i^N) - P^N[w(h_i^N)](x-h_i^N)n \right\} \cdot \bar{u}_\kappa^N \mathrm{d}\sigma \Big) \\
&\quad + \int_{\partial T_\kappa^N} \left\{ \partial_n U^N[w(h_i^N)](x-h_i^N) - P^N[w(h_i^N)](x-h_i^N)n \right\} \cdot (u^N(x) - \bar{u}_\kappa^N) \mathrm{d}\sigma\,.
\end{aligned}
$$

For the first term on the right-hand side of this last identity, we apply that the flux through hypersurfaces of the normal stress tensor is conserved by solutions to the Stokes problem so that, applying (15), we have:

$$
\begin{aligned}
&\int_{\partial T_\kappa^N} \left\{ \partial_n U^N[w(h_i^N)](x-h_i^N) - P^N[w(h_i^N)](x-h_i^N)n \right\} \mathrm{d}\sigma \Big) \\
&\qquad = -\int_{\partial B_i^N} \left\{ \partial_n U^N[w(h_i^N)](x-h_i^N) - P^N[w(h_i^N)](x-h_i^N)n \right\} \mathrm{d}\sigma \\
&\qquad = -\frac{6\pi}{N} w(h_i^N).
\end{aligned}
$$

Finally, we obtain:

$$
\tilde{I}_\kappa^N = \frac{6\pi}{N} \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} (w(h_i^N) \cdot v_i^N - w(h_i^N) \cdot \bar{u}_\kappa^N) + Err_\kappa \tag{36}
$$

with:

$$
Err_\kappa = \int_{\partial T_\kappa^N} \left\{ \sum_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \partial_n U^N[w(h_i^N)](\cdot - h_i^N) - P^N[w(h_i^N)](\cdot - h_i^N)n \right\} \cdot (u^N - \bar{u}_\kappa^N) \mathrm{d}\sigma\,.
$$

We control this error term with the following lemma:

Lemma 10. *There exists a constant C_δ depending only on δ such that,*

$$
|Err_\kappa| \leqslant \frac{C_\delta}{N^{\frac{5}{6}}} \|\nabla u^N\|_{L^2(T_\kappa^N)}\,, \quad \forall\, \kappa \in \mathcal{K}^N\,.
$$

Proof. For $N > \max(N_0, N_w, N_\delta)$, we have that

$$
[T_N^\kappa]_{2\delta} \subset T_\kappa^N \setminus \bigcup_{i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N} \overline{B_i^N}.
$$

Indeed, $B_i^N = B(h_i^N, 1/N)$ and, for $i \in \mathcal{I}_\kappa^N \setminus \mathcal{Z}_\delta^N$ we have that h_i^N is $1/(\delta N^{1/3})$ far from ∂T_κ^N. These centers are thus $1/(2\delta N^{1/3})$ far from $[T_N^\kappa]_{2\delta}$ which is larger than $1/N$ since $N > (4\delta)^{3/2}$. In particular all the stokeslets in w_κ^s satisfy:

$$
\left\{
\begin{aligned}
\Delta U^N[w(h_i^N)](x-h_i^N) - \nabla P^N[w(h_i^N)](x-h_i^N) &= 0\,, \\
\operatorname{div} U^N[w(h_i^N)](x-h_i^N) &= 0\,,
\end{aligned}
\right.
\qquad \text{on } [T_N^\kappa]_{2\delta}\,. \tag{37}
$$

Consequently, we split

$$\partial[T^{\kappa}_N]_{2\delta} = \partial T^N_\kappa \cup \partial T^N_{\kappa,\delta}$$

where

$$\partial T^N_{\kappa,\delta} = \{x \in T^N_\kappa \text{ s.t. } \mathrm{dist}(x, \partial T^N_\kappa) = 1/(2\delta N^{1/3})\}\,.$$

We remark then that for any divergence-free $v \in H^1([T^N_\kappa]_{2\delta})$ satisfying

$$\left\{\begin{array}{rcll} v &=& u^N - \bar{u}^N_\kappa\,, & \text{on } \partial T^N_\kappa\,,\\ v &=& 0\,, & \text{on } \partial T^N_{\kappa,\delta}\,,\end{array}\right.$$

integrating by parts Err_κ and applying (37), we have:

$$Err_\kappa = \int_{[T^N_\kappa]_{2\delta}} \left\{ \sum_{i\in\mathcal{I}^N_\kappa\setminus\mathcal{Z}^N_\delta} \nabla U^N[w(h^N_i)](\cdot - h^N_i)\right\} : \nabla v\,,$$

so that:

$$|Err_\kappa| \leqslant \left\{ \sum_{i\in\mathcal{I}^N_\kappa\setminus\mathcal{Z}^N_\delta} \|\nabla U^N[w(h^N_i)](\cdot - h^N_i)\|_{L^2([T^N_\kappa]_{2\delta})}\right\} \|\nabla v\|_{L^2([T^N_\kappa]_{2\delta})}\,. \tag{38}$$

Let choose a suitable v in order to apply this estimate. We recall that we introduced x^N_κ the center of T^N_κ and that we remarked that

$$T^N_\kappa = B_\infty\left(x^N_\kappa, \frac{1}{2N^{1/3}}\right)\,, \quad [T^N_\kappa]_{2\delta} = A\left(x^N_\kappa, \frac{(1-1/\delta)}{2N^{1/3}}, \frac{1}{2N^{1/3}}\right)\,.$$

So, we introduce $\zeta_\delta \in C^\infty(\mathbb{R}^3)$ such that

$$\zeta_\delta(x) = 0 \text{ in } B_\infty\left(0, \frac{1-1/\delta}{2}\right) \quad \text{and} \quad \zeta_\delta(x) = 1 \text{ outside } B_\infty\left(0,\frac{1}{2}\right)$$

and we set

$$\begin{aligned} v(x) &= \zeta_\delta(N^{1/3}(x - x^N_\kappa))(u^N(x) - \bar{u}^N_\kappa)\\ &\qquad - \mathfrak{B}_{x^N_\kappa,(1-1/\delta)/(2N^{1/3}),1/(2N^{1/3})}[x \mapsto (u^N(x) - \bar{u}^N_\kappa)\cdot\nabla[\zeta_\delta(N^{1/3}(x-x^N_\kappa))]]\,.\end{aligned}$$

Again v is well-defined as one shows by direct computations that the argument of the Bogovskii operator has mean zero on $A(x^N_\kappa, (1-1/\delta)/(2N^{1/3}), 1/(2N^{1/3}))$. Applying Lemma 15, we have then that there exists a constant C_δ depending only on δ for which:

$$\|\nabla v\|_{L^2([T^N_\kappa]_{2\delta})} \leqslant C_\delta\left[\|\nabla u^N\|_{L^2([T^N_\kappa]_{2\delta})} + N^{\frac{1}{3}}\|u^N(x) - \bar{u}^N_\kappa\|_{L^2([T^N_\kappa]_{2\delta})}\right]\,.$$

Here we note that the $\bar{u}^N_\kappa$ is exactly the mean of u^N on $[T^N_\kappa]_{2\delta}$. Consequently, applying the Poincaré-Wirtinger inequality in the annulus $[T^N_\kappa]_{2\delta}$ with the remark on the best constant as in Lemma 13 we obtain finally that:

$$\|\nabla v\|_{L^2([T^N_\kappa]_{2\delta})} \leqslant C_\delta \|\nabla u^N\|_{L^2([T^N_\kappa]_{2\delta})}\,. \tag{39}$$

As for the stokeslet, we make again the remark that for any $i \in \mathcal{I}^N_\kappa \setminus \mathcal{Z}^N_\delta$ the minimum distance between h_i^N and $[T^N_\kappa]_{2\delta}$ is larger than $1/(2\delta N^{1/3})$. Hence, applying the expansion (14) of the stokeslet $U^N[w(h_i^N)]$ we obtain that

$$\begin{aligned}\|\nabla U^N[w(h_i^N)](\cdot - h_i^N)\|_{L^2([T^N_\kappa]_{2\delta})} &\leqslant \left(\int_{1/(2\delta N^{1/3})}^{\infty} \frac{\mathrm{d}r}{N^2 r^2}\right)^{\frac12} |w(h_i^N)| \\ &\leqslant \frac{\sqrt{2\delta}}{N^{\frac56}} |w(h_i^N)|\,.\end{aligned}$$

This entails that:

$$\sum_{i\in\mathcal{I}^N_\kappa\setminus\mathcal{Z}^N_\delta} \|\nabla U^N[w(h_i^N)](\cdot - h_i^N)\|_{L^2([T^N_\kappa]_{2\delta})} \leqslant \frac{\sqrt{2\delta}M^N_\kappa}{N^{\frac56}} \|w\|_{L^\infty}\,. \tag{40}$$

Combining (39) and (40) in (38) yields the expected result. □

Summing (36) over κ, we obtain that:

$$\begin{aligned}\tilde{I}^N &= \frac{6\pi}{N} \sum_{\kappa\in\mathcal{K}^N} \sum_{i\in\mathcal{I}^N_\kappa\setminus\mathcal{Z}^N_\delta} (w(h_i^N)\cdot v_i^N - w(h_i^N)\cdot \bar{u}^N_\kappa) + Err \\ &= \frac{6\pi}{N} \sum_{i\in\mathcal{I}^N\setminus\mathcal{Z}^N_\delta} w(h_i^N)\cdot v_i^N - \frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}^N} \sum_{i\in\mathcal{I}^N_\kappa\setminus\mathcal{Z}^N_\delta} w(h_i^N)\cdot \bar{u}^N_\kappa + Err.\end{aligned} \tag{41}$$

where

$$Err = \sum_{\kappa\in\mathcal{K}^N} Err_\kappa.$$

Hence, applying Lemma 10, a Cauchy-Schwarz inequality and remarking again that the $(T^N_\kappa)_{\kappa\in\mathcal{K}^N}$ form a partition of a subset of Ω with a number of elements satisfying (26), we have:

$$|Err| \leqslant C_\delta \sum_{\kappa\in\mathcal{K}^N} \frac{\|\nabla u^N\|_{L^2(T^N_\kappa)}}{N^{5/6}} \leqslant \frac{C_\delta}{N^{\frac13}} \|\nabla u^N\|_{L^2(\Omega)} \leqslant \frac{C_\delta \mathcal{E}^\infty}{N^{\frac13}}\,. \tag{42}$$

So the asymptotics of $\tilde{I}^N$ is given by the two first terms on the right-hand side of (41). We make precise these asymptotics in the two following lemmas:

Lemma 11. *There exists a constant K independent of δ for which:*

$$\limsup_{N\to\infty} \left| \frac{6\pi}{N} \sum_{i\in\mathcal{I}^N\setminus\mathcal{Z}^N_\delta} w(h_i^N)\cdot v_i^N - 6\pi\int_\Omega j(x)\cdot w(x)\mathrm{d}x \right| \leqslant \frac{K}{\delta}\,.$$

Proof. As $w \in C^\infty_c(\Omega)$ and $(T^N_\kappa)_{\kappa\in\mathcal{K}^N}$ is a covering of $Supp(w)$ we have by assumption (A5) that:

$$\int_\Omega j(x)\cdot w(x)\mathrm{d}x = \lim_{N\to\infty} \sum_{i=1}^N w(h_i^N)\cdot v_i^N = \lim_{N\to\infty} \sum_{i\in\mathcal{I}^N} w(h_i^N)\cdot v_i^N\,.$$

Hence, our proof reduces to find a uniform bound on

$$\sum_{i\in\mathcal{I}^N} w(h_i^N)\cdot v_i^N - \sum_{i\in\mathcal{I}^N\setminus\mathcal{Z}_\delta^N} w(h_i^N)\cdot v_i^N = \sum_{i\in\mathcal{Z}_\delta^N} w(h_i^N)\cdot v_i^N\,.$$

However, for large N, there holds:

$$\left|\frac{1}{N}\sum_{i\in\mathcal{Z}_\delta^N} w(h_i^N)\cdot v_i^N\right| \leqslant \left(\frac{1}{N}\sum_{i\in\mathcal{Z}_\delta^N}|v_i^N|^2\right)^{\frac{1}{2}}\left(\frac{1}{N}\sum_{i\in\mathcal{Z}_\delta^N}|w(h_i^N)|^2\right)^{\frac{1}{2}}\,.$$

Here, we apply (25) that has guided our choice for the covering $(T_\kappa^N)_{\kappa\in\mathcal{K}^N}$:

$$\begin{aligned}\left(\frac{1}{N}\sum_{i\in\mathcal{Z}_\delta^N}|v_i^N|^2\right) &\leqslant \frac{12}{\delta}\left(1+|\mathcal{E}^\infty|^2\right)\,,\\ \left(\frac{1}{N}\sum_{i\in\mathcal{Z}_\delta^N}|w(h_i^N)|^2\right) &\leqslant \frac{12}{\delta}\left(1+|\mathcal{E}^\infty|^2\right)\|w\|_{L^\infty}^2\,.\end{aligned}$$

Combining these two estimates, we obtain:

$$\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{i\in\mathcal{Z}_\delta^N} w(h_i^N)\cdot v_i^N\right| \leqslant \frac{12}{\delta}\left(1+|\mathcal{E}^\infty|^2\right)\|w\|_{L^\infty}\,.$$

□

Lemma 12. *There exists a constant K independent of δ for which:*

$$\limsup_{N\to\infty}\left|\frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}^N}\sum_{i\in\mathcal{I}_\kappa^N\setminus\mathcal{Z}_\delta^N} w(h_i^N)\cdot\bar{u}_\kappa^N - 6\pi\int_\Omega \rho(x)\bar{u}(x)\cdot w(x)\mathrm{d}x\right| \leqslant \frac{K}{\sqrt{\delta}}\|w\|_{L^\infty}\,.$$

Proof. As in the previous proof, let first complete the sum by reintroducing the $\mathcal{Z}_\delta^N$ indices:

$$\frac{1}{N}\sum_{\kappa\in\mathcal{K}^N}\sum_{i\in\mathcal{I}_\kappa^N\setminus\mathcal{Z}_\delta^N} w(h_i^N)\cdot\bar{u}_\kappa^N = \frac{1}{N}\sum_{\kappa\in\mathcal{K}^N}\sum_{i\in\mathcal{I}_\kappa^N} w(h_i^N)\cdot\bar{u}_\kappa^N + \tilde{E}rr^N \tag{43}$$

where:

$$\tilde{E}rr^N = \frac{1}{N}\sum_{\kappa\in\mathcal{K}^N}\sum_{i\in\mathcal{I}_\kappa^N\cap\mathcal{Z}_\delta^N} w(h_i^N)\cdot\bar{u}_\kappa^N.$$

For the first term on the right-hand side of (43), we remark that:

$$\frac{1}{N}\sum_{\kappa\in\mathcal{K}^N}\sum_{i\in\mathcal{I}_\kappa^N} w(h_i^N)\cdot\bar{u}_\kappa^N = \left(1-\left(1-\frac{1}{\delta}\right)^3\right)^{-1}\sum_{\kappa\in\mathcal{K}^N}\int_{[T_\kappa^N]_{2\delta}}\left(\sum_{i\in\mathcal{I}_\kappa^N} w(h_i^N)\right)\cdot u^N\,.$$

So, we introduce:

$$\sigma^N = \left(1 - \left(1 - \frac{1}{\delta}\right)^3\right)^{-1} \sum_{\kappa \in \mathcal{K}^N} \left(\sum_{i \in \mathcal{I}_\kappa^N} w(h_i^N)\right) \mathbf{1}_{[T_\kappa^N]_{2\delta}}\,,$$

for which:

$$\frac{1}{N}\sum_{\kappa \in \mathcal{K}^N}\sum_{i \in \mathcal{I}_\kappa^N} w(h_i^N)\cdot \bar{u}_\kappa^N = \int_\Omega \sigma^N(x)\cdot u^N(x)\mathrm{d}x.$$

On the one-hand, we note that:

$$\|\sigma^N\|_{L^1(\Omega)} \leqslant \frac{1}{N}\sum_{\kappa\in\mathcal{K}^N} M_\kappa^N \|w\|_{L^\infty}\,,$$

where $\sum_{\kappa\in\mathcal{K}^N} M_\kappa^N \leqslant N$, so that:

$$\|\sigma^N\|_{L^1(\Omega)} \leqslant \|w\|_{L^\infty}\,.$$

Complementarily, because of assumption (A3), we also have :

$$\begin{aligned}\|\sigma^N\|_{L^\infty(\Omega)} &\leqslant \left(1 - \left(1 - \frac{1}{\delta}\right)^3\right)^{-1} \sup_{\kappa\in\mathcal{K}^N} M_\kappa^N \|w\|_{L^\infty}\\ &\leqslant \left(1 - \left(1 - \frac{1}{\delta}\right)^3\right)^{-1} M^\infty \|w\|_{L^\infty}\,,\end{aligned}$$

and σ^N is bounded in all L^q-spaces.

On the other hand, for any $v \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \sigma^N(x)\cdot v(x)\mathrm{d}x = \frac{1}{N}\sum_{\kappa\in\mathcal{K}^N}\sum_{i\in\mathcal{I}_\kappa^N} w(h_i^N)\cdot \bar{v}_\kappa^N$$

with

$$\bar{v}_\kappa^N = \frac{1}{|[T_N^\kappa]_{2\delta}|}\int_{[T_N^\kappa]_{2\delta}} v(x)\mathrm{d}x.$$

We remark that, for any $i \in \mathcal{I}_\kappa^N$, h_i^N is inside T_κ^N whose diameter is $1/N^{1/3}$. This entails:

$$\left|\bar{v}_\kappa^N - v(h_i^N)\right| \lesssim \frac{1}{N^{1/3}}\|\nabla v\|_{L^\infty}.$$

Gathering these identities for all indices i in all the cubes T_κ^N, we infer :

$$\left|\int_\Omega \sigma^N(x)\cdot v(x)\mathrm{d}x - \frac{1}{N}\sum_{i\in\mathcal{I}^N} w(h_i^N)\cdot v(h_i^N)\right| \lesssim \frac{1}{N^{1/3}}\|\nabla v\|_{L^\infty}\|w\|_{L^\infty}\,.$$

Consequently, assumption (A4) implies that:

$$\lim_{N\to\infty}\int_\Omega \sigma^N(x)\cdot v(x)\mathrm{d}x = \int_\Omega \rho(x)w(x)\cdot v(x)\mathrm{d}x\,,$$

and $\sigma^N \rightharpoonup \rho w$ weakly in $L^q(\Omega)$ for arbitrary $q \in (1,\infty)$. Combining then the weak convergence of σ^N in $L^2(\Omega)$ and the strong convergence of u^N in $L^2(\Omega)$ (up to the extraction of a subsequence), we have:

$$\lim_{N\to\infty}\int_\Omega \sigma^N \cdot u^N = \int_\Omega \rho w \cdot \bar{u}.$$

As for the remainder term, we introduce:

$$\tilde{\sigma}^N = \left(1-\left(1-\frac{1}{\delta}\right)^3\right)^{-1} \sum_{\kappa\in\mathcal{K}^N}\left(\sum_{i\in\mathcal{I}^N_\kappa\cap\mathcal{Z}^N_\delta}|w(h_i^N)|\right)\mathbf{1}_{[T^N_\kappa]_{2\delta}}\,.$$

so that:

$$|\tilde{E}rr^N| \leqslant \int_\Omega \tilde{\sigma}^N(x)|u^N(x)|\mathrm{d}x\,.$$

With similar arguments as in the previous computations, we have, applying (25):

$$\|\tilde{\sigma}^N\|_{L^1(\Omega)} \leqslant \frac{1}{N}\#\mathcal{Z}^N_\delta\|w\|_{L^\infty} \leqslant \frac{1}{\delta}\|w\|_{L^\infty}(1+|\mathcal{E}^\infty|^2).$$

Furthermore, we have:

$$\|\tilde{\sigma}^N\|_{L^\infty(\Omega)} \lesssim \delta M^\infty\|w\|_{L^\infty}.$$

Consequently, by interpolation, we obtain:

$$\|\tilde{\sigma}^N\|_{L^{4/3}(\Omega)} \lesssim \frac{|M^\infty|^{1/4}}{\sqrt{\delta}}\|w\|_{L^\infty}(1+|\mathcal{E}^\infty|^2)^{\frac{3}{4}}\,.$$

As u^N is bounded in $L^4(\Omega)$ by sobolev embedding, this yields that:

$$|\tilde{E}rr^N| \lesssim \frac{|M^\infty|^{1/4}}{\sqrt{\delta}}\|w\|_{L^\infty}(1+|\mathcal{E}^\infty|^2)^{\frac{7}{4}}\,,\quad \forall\, N\in\mathbb{N},$$

and there exists a constant K depending only on $\mathcal{E}^\infty$, M^∞ and $\|w\|_{L^\infty}$ for which:

$$\limsup_{N\to\infty}|\tilde{E}rr^N| \leqslant \frac{K}{\sqrt{\delta}}\,.$$

This ends the proof. □

Appendix A. Auxiliary technical lemmas

We recall here several standard lemmas that help in the above proofs.

First, we recall the Poincaré-Wirtinger inequality [7, Theorem II.5.4] which states that for arbitrary lipschitz domain $\mathcal{F}$, there holds:

$$\left\|u-\frac{1}{|\mathcal{F}|}\int_{\mathcal{F}}u(x)\mathrm{d}x\right\|_{L^2(\mathcal{F})} \leqslant C_{PW}\|\nabla u\|_{L^2(\mathcal{F})}\,.$$

We extensively use this inequality when $\mathcal{F}$ is an annulus. In this case, a standard scaling argument entails the following remark on the constant C_{PW}:

Lemma 13. *Given $(x,\lambda,a) \in \mathbb{R}^3 \times (0,\infty) \times (0,1)$ there exists a constant C_a depending only on a (and expecially not on (x,λ)) for which :*

$$\left\| u - \frac{1}{|A(x,a\lambda,\lambda)|} \int_{A(x,a\lambda,\lambda)} u(x)\mathrm{d}x \right\|_{L^2(A(x,a\lambda,\lambda))} \leqslant C_a \lambda \|\nabla u\|_{L^2(A(x,a\lambda,\lambda))}\,.$$

Second, we focus on the properties of the Bogovskii operators $\mathfrak{B}$. This means we are interested in solving the divergence problem:

$$\mathrm{div}\, v = f\,, \quad \text{on } \mathcal{F}\,, \tag{44}$$

whose data is f and unknown is v. We recall the result due to M.E. Bogovskii (see [7, Theorem III.3.1]):

Lemma 14. *Let $\mathcal{F}$ be a lipschitz bounded domain in $\mathbb{R}^3$. Given $f \in L^2(\mathcal{F})$ such that*

$$\int_{\mathcal{F}} f(x)\mathrm{d}x = 0\,,$$

there exists a solution $v := \mathfrak{B}_{\mathcal{F}}[f] \in H^1_0(\mathcal{F})$ to (44) such that

$$\|\nabla v\|_{L^2(\mathcal{F})} \leqslant C\|f\|_{L^2(\mathcal{F})}$$

with a constant C depending only on $\mathcal{F}$.

In the case of annuli, the above result yields the following lemma by a standard scaling argument:

Lemma 15. *Let $(x,\lambda,a) \in \mathbb{R}^3 \times (0,\infty) \times (0,1)$. Given $f \in L^2(A(x,a\lambda,\lambda))$ such that*

$$\int_{\mathcal{F}} f(x)\mathrm{d}x = 0\,,$$

there exists a solution $v := \mathfrak{B}_{x,a\lambda,\lambda}[f] \in H^1_0(A(x,a\lambda,\lambda))$ to (44) such that

$$\|\nabla v\|_{L^2(A(x,a\lambda,\lambda))} \leqslant C_a \|f\|_{L^2(A(x,a\lambda,\lambda))},$$

with a constant C_a depending only on a (and especially neither on f nor on (x,λ)) .

Appendix B. Proof of a covering lemma

This appendix is devoted to the construction of coverings that are adapted to the empiric measures S_N. We prove the following general lemma:

Lemma 16. *Let $(N,d) \in [\mathbb{N}^*]^2$, $d \geqslant 2$, and $\mu \in \mathcal{M}_+(\mathbb{R}^3)$ a positive bounded measure. There exists $(T^N_\kappa)_{\kappa\in\mathbb{Z}^3}$ a covering of $\mathbb{R}^3$ with disjoint cubes of width $1/N^{1/3}$ such that denoting*

$$\mathcal{C}^N_d := \left\{ x \in \mathbb{R}^3 \ s.t. \ dist\left(x, \bigcup_{\kappa\in\mathbb{Z}^3} \partial T^N_\kappa\right) < \frac{1}{(d+1)N^{1/3}} \right\}$$

there holds

$$\mu(\mathcal{C}_d^N) \leqslant \frac{6}{d}\mu(\mathbb{R}^3). \tag{45}$$

In Section 4, we apply the previous lemma with $N \in \mathbb{N}^*$, $d = \delta - 1$ and

$$\mu := \frac{1}{N}\sum_{i=1}^{N}(1 + |v_i^N|^2)\delta_{h_i^N},$$

to obtain a covering $(T_\kappa^N)_{\kappa\in\mathbb{Z}^3}$ satisfying (25). Assuming then $N \geqslant [\mathrm{dist}(Supp\, w, \mathbb{R}^3 \setminus \Omega)/4]^{-3}$, we obtain that the subcovering $(T_\kappa^N)_{\kappa\in\mathcal{K}^N}$ containing only the cubes that intersect $Supp(w)$ is made of cubes T_κ^N that are included in the $1/N^{1/3}$ neighborhood of $Supp(w)$ (as T_κ^N has diameter $1/N^{1/3}$). By direct computations, we obtain then that, for $\kappa \in \mathcal{K}^N$, the distance between T_κ^N and $\mathbb{R}^3 \setminus \Omega$ is strictly positive so that $T_\kappa^N \subset \Omega$.

Proof. By a standard scaling argument, it suffices to prove the result for $N = 1$. Let $d \geqslant 2$. First, for arbitrary $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ we set:

$$\tilde{T}_k = \left[\frac{k_1}{d}, \frac{k_1+1}{d}\right[\times \left[\frac{k_2}{d}, \frac{k_2+1}{d}\right[\times \left[\frac{k_3}{d}, \frac{k_3+1}{d}\right[$$

These cubes with tildas and index k are cubes of width $1/d$. We call them "small cubes." It is straightforward that $(\tilde{T}_k)_{k\in\mathbb{Z}^3}$ forms a partition of $\mathbb{R}^3$. For arbitrary

$$\kappa = (k_1, k_2, k_3) + \{0, \ldots, d-1\}^3,$$

we set then:

$$T_\kappa = \bigcup_{k\in\kappa} \tilde{T}_k = \left[\frac{k_1}{d}, \frac{k_1}{d} + 1\right[\times \left[\frac{k_2}{d}, \frac{k_2}{d} + 1\right[\times \left[\frac{k_3}{d}, \frac{k_3}{d} + 1\right[.$$

These cubes without tildas and with index κ are cubes of width 1. We call them "large cubes". We introduce then the $1/d$-neighborhood of the boundary of this large cube:

$$[T_\kappa]_d := \bigcup_{k\in\partial\kappa} \tilde{T}_k .$$

where

$$\begin{aligned}\partial\kappa &= \{k \in \{k_1, k_1 + d - 1\} \times \{k_2, \ldots, k_2 + d - 1\} \times \{k_3, \ldots, k_3 + d - 1\}\}\\ &\quad \cup \{k \in \{k_1, \ldots, k_1 + d - 1\} \times \{k_2, k_2 + d - 1\} \times \{k_3, \ldots, k_3 + d - 1\}\}\\ &\quad \cup \{k \in \{k_1, \ldots, k_1 + d - 1\} \times \{k_2, \ldots, k_2 + d - 1\} \times \{k_3, k_3 + d - 1\}\}\end{aligned}$$

(which means taking the small cubes whose indices are in the boundary of κ). We remark that we may split $[T_\kappa]_d$ into 6 subsets corresponding to the top, bottom, left, right, front and back faces of the cube T_κ. For instance, the bottom face of $[T_\kappa]_d$ reads:

$$\bigcup_{k\in\{k_1,\ldots,k_1+d-1\}\times\{k_2,\ldots,k_2+d-1\}\times\{k_3\}} \tilde{T}_k .$$

For arbitrary $k^\ell = \ell(1,1,1)$, with $\ell \in \{0,\dots,d-1\}$ we also denote

$$\mathcal{K}_\ell = \left\{ \kappa = (k^\ell + \pi + \{0,\dots,d-1\}^3)\,, \quad \pi \in d\mathbb{Z}^3 \right\}$$

We emphasize that $\mathcal{K}_\ell$ is a set made of sets (corresponding to large cubes). Any set $\mathcal{K}_\ell$ corresponds to a partition of $\mathbb{Z}^3$ and then to a covering of $\mathbb{R}^3$ with disjoint large cubes.

Given $\ell \in \{0,\dots,d-1\}$ we consider now

$$\mathcal{C}^\ell_d = \left\{ x \in \mathbb{R}^3 \text{ s.t. } \mathrm{dist}\left(x, \bigcup_{\kappa\in\mathcal{K}_\ell} \partial T_\kappa\right) < \frac{1}{(d+1)N^{1/3}} \right\}.$$

We remark that, for fixed ℓ there holds:

$$\mathcal{C}^\ell_d \subset \bigcup_{\kappa\in\mathcal{K}_\ell} [T_\kappa]_d.$$

We denote $\partial\mathcal{K}_l$ the set of indices k such that $\tilde{T}_k$ contributes to this $1/d$-neigborhood. Setting $\partial\mathcal{K}_l = \bigcup\{\partial\kappa,\ \kappa\in\mathcal{K}_\ell\}$, we have thus:

$$\mathcal{C}^\ell_d \subset \bigcup_{k\in\partial\mathcal{K}_l} \tilde{T}_k\,.$$

We can decompose this union of small cubes by regrouping together the cubes that belong to left / right / top /bottom / front / back faces of large cubes. For instance, the indices k of small cubes belonging to bottom faces of large cubes satisfy

$$k \in \mathbb{Z}^2 \times \{\ell + d\mathbb{Z}\}\,.$$

For two different ℓ and ℓ' in $\{0,\dots,d-1\}$ the same index k cannot belong to the bottom faces of two different cubes in the coverings $\mathcal{K}_\ell$ and $\mathcal{K}_{\ell'}$ of $\mathbb{R}^3$. We have the same properties for top / right / left / front / back faces. Consequently, in the family of coverings $(\mathcal{K}_\ell)_{\ell\in\{0,\dots,d-1\}}$ one small cube $\tilde{T}^N_k$ belongs at most once to a top / bottom / right / left / front / back face of a large cube so that:

$$\text{any } k\in\mathbb{Z}^3 \text{ belongs to at most 6 different } \partial\mathcal{K}_l\,. \tag{46}$$

Let now introduce the measure μ. For any $k\in\mathbb{Z}^3$, we denote:

$$\tilde{\mu}_k = \mu(\tilde{T}_k),$$

and we consider the sum:

$$Rem := \sum_{\ell\in\{0,\dots,d-1\}} \mu(\mathcal{C}^\ell_d).$$

With the previous definitions, we have:

$$Rem \leqslant \sum_{\ell\in\{0,\dots,d-1\}} \sum_{k\in\partial\mathcal{K}_\ell} \tilde{\mu}_k.$$

Because of (46), we have then that any $k \in \mathbb{Z}^3$ appears at most 6 times in this sum. Consequently:

$$Rem \leqslant 6 \sum_{k \in \mathbb{Z}^3} \tilde{\mu}_k \leqslant 6\mu(\mathbb{R}^3)\,.$$

The measure μ being positive, this implies that one of the terms in the sum defining Rem is less than Rem/d. In other words, there exists at least one $\ell^0 \in \{0, \dots, d-1\}$ such that:

$$\mu(\mathcal{C}_d^{\ell^0}) \leqslant \frac{6}{d}\mu(\mathbb{R}^3)\,.$$

The covering $(T_\kappa)_{\kappa \in \mathcal{K}_{\ell^0}}$ is then made of disjoint cubes of width 1 satisfying (45). We have obtained the required covering of $\mathbb{R}^3$. □

Acknowledgement. The author would like to thank Ayman Moussa, Franck Sueur, Laure Saint-Raymond and Mark Wilkinson for many stimulating discussions on the topic. The author is partially supported by the ANR-13-BS01-0003-01 project DYFICOLTI.

References

[1] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. *Arch. Rational Mech. Anal.*, 113(3):209–259, 1990.

[2] V. Bonnaillie-Noël, C. Lacave, and N. Masmoudi. Permeability through a perforated domain for the incompressible 2D Euler equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(1):159–182, 2015.

[3] D. Cioranescu and F. Murat. Un terme étrange venu d'ailleurs. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980)*, volume 60 of *Res. Notes in Math.*, pages 98–138, 389–390. Pitman, Boston, Mass.-London, 1982.

[4] L. Desvillettes, F. Golse, and V. Ricci. The mean-field limit for solid particles in a Navier-Stokes flow. *J. Stat. Phys.*, 131(5):941–967, 2008.

[5] E. Feireisl and Y. Lu. Homogenization of stationary Navier-Stokes equations in domains with tiny holes. *J. Math. Fluid Mech.*, 17(2):381–392, 2015.

[6] E. Guazzelli and J. F. Morris. *A physical introduction to suspension dynamics*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2012.

[7] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.

[8] J. Happel and H. Brenner. *Low Reynolds number hydrodynamics with special applications to particulate media*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1965.

[9] J. L. Höfer, R. and Velàzquez. The method of reflections, homogenization and screening for poisson and stokes equations in perforated domains. arXiv:1603.06750, March 2016.

[10] P.-E. Jabin and F. Otto. Identification of the dilute regime in particle sedimentation. *Comm. Math. Phys.*, 250(2):415–432, 2004.

[11] P.-E. Jabin and B. Perthame. Notes on mathematical problems on the dynamics of dispersed particles interacting through a fluid. In *Modeling in applied sciences*, Model. Simul. Sci. Eng. Technol., pages 111–147. Birkhäuser Boston, Boston, MA, 2000.

[12] J. Rubinstein. On the macroscopic description of slow viscous flow past a random array of spheres. *J. Statist. Phys.*, 44(5-6):849–863, 1986.

Institut Montpelliérain Alexander Grothendieck, Université de Montpellier, Place Eugène Bataillon, 34095 Montpellier Cedex 5 France
E-mail address: matthieu.hillairet@umontpellier.fr