ON TOPOLOGIES ON THE GROUP $(\mathbb{Z}_p)^{\mathbb{N}}$

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ABSTRACT. It is proved that, on any Abelian group of infinite cardinality $\mathbf{m}$, there exist precisely $2^{2^{\mathbf{m}}}$ nonequivalent bounded Hausdorff group topologies. Under the continuum hypothesis, the number of nonequivalent compact and locally compact Hausdorff group topologies on the group $(\mathbb{Z}_p)^{\mathbb{N}}$ is determined.

The systematic study of Abelian topological groups was initiated in Pontryagin's paper [18] and van Kampen's paper [9] (see also [1]). In these papers, an exceptionally important relationship between an Abelian topological group and the topological group of its characters was revealed. It is not an exaggeration to say that this relationship is one of the most important tools for studying Abelian topological groups. This relationship manifests itself most pronouncedly for compact and locally compact groups.

Topological groups close to compact and locally compact ones are, respectively, totally bounded and locally bounded Abelian topological groups. This type of topologies on Abelian groups was studied by A. Weil [21], who showed that an Abelian topological group is totally bounded if and only if it is embedded in a compact Abelian group as a dense subgroup and that such a group is locally bounded if and only if it is embedded in a locally compact group as a dense subgroup. Another deep characteristic property of locally bounded and totally bounded topological, not necessarily Abelian, groups were found by A.D. Alexandroff [2]. He showed in particular that a topological group G is locally bounded if and only if G admits a left invariant generalized measure, see [2] for more details.

Among all bounded group topologies on a given Abelian group there is a maximal topology; the Weil completion of a group with this topology coincides with the Bohr compactification of this group and can be continuously mapped onto any compact group extension.

The next important step in the study of Abelian topological groups was made by Kakutani [7], who showed that the cardinality of an infinite compact Abelian group G and the cardinality of the discrete group G^* of its characters are related by

$$|G| = 2^{|G^*|}.$$

In the 1950s, interest in the study of general properties of Abelian topological groups motivated the interest in the study of the relationship between the algebraic structure of a given group and the possible topologies consistent with its group structure. This topic includes, in particular, the problem of describing Abelian groups admitting a compact topology, which was posed by Kaplansky in [8]. This problem was completely solved by Hulanicki in [14, 15].

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We mention two results concerning the possible compact topologies on a given Abelian group. In the very early 1950s, Kulikov [11] showed that the cardinality of the family of all compact Abelian groups of cardinality $2^{\mathbf{m}}$, where $\mathbf{m}$ is an infinite cardinal, equals $2^{\mathbf{m}}$. In the light of this result the following example of Fuchs [12] looks quite surprising. He constructed an Abelian group G of cardinality $2^{\mathbf{m}}$ admitting infinitely many different compact topologies. Moreover, the cardinality of the set of pairwise non-homeomorphic compact invariant topologies on G equals $2^{\mathbf{m}}$.

It is well known that, on any Abelian group G, there are as many nonequivalent Hausdorff group topologies as possible. To be more precise, the cardinality of the set of nonequivalent Hausdorff group topologies on G equals $2^{2^{|G|}}$. This result was obtained by Podewski [17], who studied the problem in its most general setting. Namely, given any set with a finite or countable system of algebraic structures, Podewski found necessary and sufficient conditions on these algebraic structures ensuring the existence of a nondiscrete Hausdorff topology consistent with all of them [17]. Moreover, Podewski showed that these conditions ensure the existence of the maximal possible number of pairwise non-equivalent Hausdorff topologies which agree with all algebraic structures. For Abelian groups and fields, Podewski's conditions always hold; details can be found in [17]. In 1970, Arnautov constructed an example of an infinite commutative ring such that any Hausdorff ring topology on this ring is discrete [4]. In the 1979s–1980s, examples of infinite non-Abelian groups admitting only discrete Hausdorff group topologies were given [13], [16], [20].

Fuchs' example shows that properties of various topologies that an abstract Abelian group G carries directly depend on the algebraic properties of the group G.

In our view, the influence of the algebraic structure of an Abelian group on the structure of the set of Hausdorff group topologies on this group has been studied insufficiently. All topologies considered in the following are assumed to be Hausdorff.

This paper studies the structure of the set of possible topologies on the well known Abelian group

$$G = (\mathbb{Z}_p)^{\mathbb{N}} , \tag{1}$$

where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime p. Henceforth for two sets X and Y the set X^Y is understood in usual sense of set theory. As customary in topology, we denote the prime field with p elements by the same symbol $\mathbb{Z}_p$.

Note that there are four canonical, in a certain sense, topologies on G. The group G has the natural compact topology determined by the structure of the direct product (1); we denote this topology by τ_c. The second topology is the discrete topology τ_d on G. The character group of (G, τ_c), which we denote by $(G, \tau_c)^* = \oplus_{\aleph_0}\mathbb{Z}_p$, is countable and discrete. Thus, we obtain two topological groups, $(G, \tau_c) \times (G, \tau_c)^*$ and $(G, \tau_c) \times (G, \tau_d)$. Since these two groups are algebraically isomorphic to G, it follows that, in fact, we obtain two more topologies on G, which are surely different (they have different weights); we denote these topologies by τ_0 and τ_1, respectively. Obviously, these two topologies are locally compact. The following theorem is one of the main results of this paper.

Theorem 0.1. *For the Abelian group $G = (\mathbb{Z}_p)^{\mathbb{N}}$, the following assertions hold.*

1. *The set of all Hausdorff group topologies on G contains a subset of cardinality $2^{2^{2^{\aleph_0}}}$ consisting of pairwise non-homeomorphic totally bounded topologies.*

2. *Under the continuum hypothesis,* τ_c*,* τ_d*,* τ_0*, and* τ_1 *are the only locally compact Hausdorff group topology on* G*. Only one of them,* τ_c*, is compact.*

As mentioned above, the general result of Podewski [17] affirms that there are $2^{2^{2^{\aleph_0}}}$ pairwise non-homeomorphic group topologies on G. The first assertion of the theorem shows that the set of all group topologies on G contains a subset of the same cardinality consisting of *totally bounded* pairwise non-homeomorphic topologies on G.

Although totally bounded topologies are very close to compact, under the continuum hypothesis, only four of these numerous topologies are locally compact, and only one of them is compact. This fact contrasts with Fuchs' result in [12] cited above.

Theorem 0.1 follows from more general statements proved below. Thus, its first assertion is, in fact, a general property of Abelian groups and follows directly from Theorem 1.3 proved in Section 1.

Compact and locally compact topologies are studied in Section 2 in a more general situation. However, in Section 2, we essentially use the algebraic structure of G, namely, the fact that each of its element has order p. The second assertion of the theorem follows directly from Corollary 2.4 and, according to this corollary, is equivalent to the continuum hypothesis. Note that the negation of the continuum hypothesis drastically changes the situation with locally compact topologies; see Proposition 2.3.

Section 1.1 considers the set of all totally bounded topologies on the group G. This subset, which has the largest possible cardinality in the set of all topologies on G, has a number of interesting structures. For example, we can single out lattices of topologies. This, in turn, allows us to construct chains of decreasing totally bounded topologies on G. Moreover, there exist chains of maximum length, which have maximal or minimal elements. These structures are studied in Section 1.1.

One of our main tools for studying Abelian topological groups is, as usual, the character theory founded by Pontryagin in [18]. This theory has been developed most fully for locally compact Abelian groups, for which Pontryagin's duality theorem is valid (see [19]). Among other classes of topological groups whose connection with their character groups remains strong enough are the already mentioned classes of totally bounded and locally bounded groups. The relationship between these groups and their character groups was studied by Comfort and Ross [5].

For other classes of Abelian topological groups, connection with character groups is not so pronounced; nevertheless, character groups remain an important tool for studying them.

In conclusion, we mention that our interest in this topic was aroused by a series of questions asked to us by R. I. Grigorchuk. We thank him for numerous fruitful discussions. We are also grateful to O. V. Sipacheva for useful conversations.

1. The Structure of the Set of All Topologies

Let $\mathbf{m}$ be a finite or infinite cardinal. In what follows, we use the notation $\prod_{\mathbf{m}} \mathbb{Z}_p = \prod\limits_{\alpha\in A} (\mathbb{Z}_p)_\alpha$, where the cardinality of the index set is $|A| = \mathbf{m}$; accordingly, $\sum_{\mathbf{m}} \mathbb{Z}_p = \bigoplus\limits_{\alpha\in A} (\mathbb{Z}_p)_\alpha$.

Given another cardinal $\mathbf{k}$, we set

$$F_{\mathbf{k},\mathbf{m}}(p) = \prod_{\mathbf{k}} \mathbb{Z}_p \times \sum_{\mathbf{m}} \mathbb{Z}_p. \tag{2}$$

In the case where p fixed or is clear from the context, we denote groups of the form (2) by $F_{\mathbf{k},\mathbf{m}}$.

Endowing the first factor with the compact product topology and the second with the discrete topology, we turn $F_{\mathbf{k},\mathbf{m}}$ into an Abelian topological group. In what follows, we assume that at least one of the cardinals $\mathbf{k}$ and $\mathbf{m}$ is infinite and the finite cardinals can take only the value 0. The easy-to-prove properties of this Abelian topological group are collected in the following proposition.

Proposition 1.1. *The topological group $F_{\mathbf{k},\mathbf{m}}$ is locally compact; it has cardinality* $\max\{2^{\mathbf{k}}, \mathbf{m}\}$ *and weight* $\max\{\mathbf{k}, \mathbf{m}\}$*. The group of characters of $F_{\mathbf{k},\mathbf{m}}$ is isomorphic to the group $F_{\mathbf{m},\mathbf{k}}$. For different pairs of cardinals* $(\mathbf{k}, \mathbf{m})$*, the topological spaces $F_{\mathbf{k},\mathbf{m}}$ are nonhomeomorphic.*

Below we give general facts about topologies on Abelian groups generated by subgroups of the character group.

We denote the entire group of characters of an Abelian group A, i.e., the group of all possible homomorphisms of A to the circle $S^1 = \{e^{it}\}$, by $A^\star$. If A is endowed with a group topology, then we denote the group of topological characters, i.e., continuous homomorphisms, of the Abelian topological group A to the circle by A^*. Note that A^* is always a subgroup of $A^\star$; if the topology of A is discrete, then these two groups coincide.

As usual, a subset $X \subset A^\star$ is said to separate vectors if, for any different $v, w \in A$, there exists an $h \in X$ such that $h(v) \neq h(w)$. Any set $X \subset A^\star$ generates a diagonal map

$$\Delta_X : A \longrightarrow (S^1)^X. \tag{3}$$

A set X separates vectors if and only if the diagonal map (3) is a monomorphism. Note that X separates vectors if and only if so does the subgroup $\langle X \rangle$ generated by this set.

As is known [19], for any nonzero element $v \in A$, there exists a character $\phi_v \in A^\star$ such that $\phi_v(v) \neq 1$. Uniting such characters over all elements $v \in A$, we obtain a set $X \subset A^\star$ separating the elements of A. Let $H = \langle X \rangle$ be the subgroup generated by this set; as mentioned above, H separates points as well. If $|A| = \mathbf{m}$, where $\mathbf{m}$ is an infinite cardinal, then, obviously, we have $|H| \leq \mathbf{m}$. Note also that the diagonal map Δ_H implements an isomorphism of the group A onto its image in $\prod_{h \in H} (S^1)_h$.

Given a subgroup $H \subset A^\star$, we denote the minimal topology on A with respect to which all maps in H are continuous by τ_H. This topology coincides with the minimal topology with respect to which the diagonal map (3) is continuous (the image is assumed to be endowed with the product topology). Obviously, τ_H is Hausdorff if and only if H separates points. Recall that a topology τ on a group G is said to be totally bounded if, for any neighborhood U of zero, there exists a finite set $\{0, g_1, \ldots, g_k\}$ of elements of G such that the family $\{U, g_1 + U, \ldots, g_k + U\}$ covers the entire group.

Topologies determined by various subgroups $H \subset A^\star$ were studied by Comfort and Ross [5]; we collect the results of [5] needed later on in the following theorem.

Theorem 1.2. [5] 1. *For any subgroup $H \subset A^\star$, the topology τ_H is totally bounded, and for each totally bounded topology τ on A, there exists a subgroup $H \subset A^\star$ such that $\tau = \tau_H$.*

2. *The identity map $id : (A, \tau_{H_2}) \longrightarrow (A, \tau_{H_1})$ is continuous if and only if $H_1 \subset H_2$.*

Now, we prove a general statement on the number of totally bounded topologies on an Abelian group.

Theorem 1.3. *Let A be an Abelian group of cardinality $|A| = \mathbf{m}$, where $\mathbf{m}$ is an infinite cardinal. Then, on A, there are precisely $2^{2^{\mathbf{m}}}$ totally bounded Hausdorff group topologies. Among them there are $2^{2^{\mathbf{m}}}$ pairwise nonhomeomorphic topologies.*

Proof. Let $A^\star$ be the group of all characters of A; by Kakutani's theorem, we have $|A^\star| = 2^{\mathbf{m}}$. Fix a subgroup $H \lhd A^\star$ separating the points of characters. As mentioned above, this subgroup can always be chosen so that $|H| \leq \mathbf{m}$. Let $B = A^\star/H$; then $|B| = 2^{\mathbf{m}}$.

Given a subgroup $C \lhd B$, we set $\bar{C} = p^{-1}(C)$, where $p : A \longrightarrow B$ is the natural projection. Since $H \lhd \bar{C}$, it follows that the topology $\tau_{\bar{C}}$ on A is Hausdorff. Thus, to each subgroup of B we have assigned a totally bounded Hausdorff group topology on A. These topologies are different for different subgroups. Indeed, according to Theorem 1.2, the identity map is not a homeomorphism with respect to topologies corresponding to different subgroups. In turn, Lemma 1.4 implies that the cardinality of the set of such topologies is $2^{2^{\mathbf{m}}}$.

Finally, let us count nonequivalent topologies among those obtained above. By nonequivalent topologies we mean topologies that cannot be mapped to each other by a homeomorphism, not necessarily consistent with the group structure.

Since $|A| = \mathbf{m}$, it follows that the set of all self-maps of A has cardinality $\mathbf{m}^{\mathbf{m}} = 2^{\mathbf{m}}$ (as a set). It follows that any topology on A is equivalent to at most $2^{\mathbf{m}}$ other topologies on this set. Thus, we obtain a partition of $2^{2^{\mathbf{m}}}$ different totally bounded group topologies into classes of equivalent topologies, each of which has cardinality at most $2^{\mathbf{m}}$. Therefore, the number of classes is precisely $2^{2^{\mathbf{m}}}$, and topologies from different classes are pairwise nonequivalent. This completes the proof of the theorem. □

Lemma 1.4. *Let $\mathbf{m}$ be an infinite cardinal, and let B be an Abelian group of cardinality $2^{\mathbf{m}}$. Then the cardinality of the set of pairwise different subgroups in B equals $2^{2^{\mathbf{m}}}$.*

Proof. The upper bound for the cardinality of the set of subgroups in B is obvious and coincides with the cardinality of the set of all subsets in B. Let us prove the lower bound.

Let $T = \mathrm{Tors}(B)$ be the torsion subgroup of B; then $B' = B/T$ is torsion-free. The cardinality of at least one of the groups T and B' equals $2^{\mathbf{m}}$, and the presence of the required number of pairwise different subgroups in T or in B' implies the presence of the same number of subgroups in B. Thus, it suffices to prove the lemma for periodic groups and torsion-free groups separately. Consider two cases.

1. Suppose that the group B is periodic. Since B decomposes into the direct sum of its primary components [10], it follows that at least one primary component has cardinality $2^{\mathbf{m}}$, and it suffices to prove the required assertion for the case of a primary p-group, where p is a prime.

Suppose that B is a p-group. Consider the subgroup $B(p)$ consisting of all elements of order p. Obviously, $|B(p)| = 2^{\mathbf{m}}$, so that it suffices to prove the required assertion for the group $B(p)$. In this case, $B(p)$ is a $\mathbb{Z}_p$-vector space, and it has a Hamel basis of cardinality $2^{\mathbf{m}}$. Different subsets of this basis generate different subgroups of $B(p)$ and, therefore, of B. This completes the proof in the case of periodic groups.

2. Finally, suppose that B is torsion-free. Since $|B| = 2^{\mathbf{m}}$, it follows that the (Prüfer) rank of B equals $2^{\mathbf{m}}$. Thus, there is a free $\mathbb{Z}$-module of rank $2^{\mathbf{m}}$ in B. As above, different subsets in a family of generators of such a module generate different subgroups in the module and, therefore, in B. □

1.1. **The Structure of the Family of Totally Bounded Topologies.** In what follows, we essentially use the structure of the groups $F_{\mathbf{k},\mathbf{m}}(p)$ and, in particular, the fact that these group can be treated as $\mathbb{Z}_p$-vector spaces. The characters of such groups are simply the elements of the dual space. In what follows, we use this dual understanding of characters without mention.

Consider the case where the group A is a $\mathbb{Z}_p$-vector space of dimension equal to an infinite cardinal $\mathbf{m}$. In this case, Theorem 1.3 is substantially simpler and has a transparent combinatorial interpretation. The subgroup $H \lhd A^\star$ of characters separating points is a subspace of the same dimension $\mathbf{m}$. Next, we choose a Hamel basis $\mathcal{B}$ in H and extend it to a Hamel basis $\mathcal{B}^\star$ in the entire space $A^\star$. Let $M = \mathcal{B}^\star \setminus \mathcal{B}$; obviously, we have $|M| = 2^{\mathbf{m}}$.

Each set $N \subset M$ now determines a subgroup in A being the linear span of the system of vectors $\{N \cup \mathcal{B}\}$. The set of pairwise different subgroups thus defined has cardinality $2^{2^{\mathbf{m}}}$ and can be used in Theorem 1.3 for the special case where A is a $\mathbb{Z}_p$-vector space. The Hausdorff group topologies on A thus obtained have an additional structure: on this set of topologies, there arise lattices.

The construction described above shows that, for the set $M = \mathcal{B}^\star \setminus \mathcal{B}$ of cardinality $2^{\mathbf{m}}$, the lattice $(2^M; \subset, \cup, \cap)$ of subsets is embedded in the lattice of totally bounded linearly invariant topologies on the space A. Therefore, the structures on the lattice $(2^M; \subset, \cup, \cap)$ can be transferred to topologies. Thus, any linearly ordered set M generates a chain of length $2^{\mathbf{m}}$ of one-to-one continuous maps. In the family of topologies under consideration, there is the minimal topology $\tau_{\min} = \tau_{H_{\mathcal{B}}}$ and the maximal topology $\tau_{\max} = \tau_{A^\star}$.

Thus, on a vector space A of dimension $\mathbf{m}$, there are chains of length $2^{\mathbf{m}}$ of one-to-one continuous maps, and there are no longer chains. Such chains carry various combinatorial structures, which correspond to different types of ordering of the vectors in the Hamel basis.

Moreover, the existence of $2^{2^{\mathbf{m}}}$ independent subsets in M implies the existence of $2^{2^{\mathbf{m}}}$ totally bounded linearly invariant Hausdorff topologies on A such that the identity map is not continuous with respect to any pair of these topologies.

Proposition 1.5. *Let A be a vector space of infinite dimension $\mathbf{m}$. Then any separating subspace $H \subset V^\star$ contains a separating subspace of dimension at most $\mathbf{m}$.*

Proof. Indeed, given any pair of different vectors in the space A, it suffices to take a linear functional on H separating them. The dimension of the linear span of such functionals does not exceed $\mathbf{m}^2 = \mathbf{m}$. □

Proposition 1.6. *Let τ be a totally bounded linearly invariant Hausdorff topology on a vector space A of infinite dimension $\mathbf{m}$. Then there is a set M of cardinality $2^{\mathbf{m}}$ such that, on the set of totally bounded linearly invariant Hausdorff topologies on A, there is a lattice isomorphic to $(2^M; \subset, \cup, \cap)$. Moreover, the initial topology τ is either minimal* *or maximal in this lattice. In particular, any totally bounded linearly invariant Hausdorff topology on A can be included in a linear chain of length $2^{\mathbf{m}}$ of different topologies.*

Proof. Consider the space H of linear functionals generating the given topology. If the dimension of H is less than $2^{\mathbf{m}}$, then we take the Hamel basis $\mathcal{B}$ in H, extend it to a Hamel basis $\mathcal{B}^\star$ in $A^\star$, and set $M = \mathcal{B}^\star \setminus \tilde{\mathcal{B}}$. In this case, $(2^M; \subset, \cup, \cap)$ is the required lattice, and the given topology is minimal in this lattice.

Suppose that the dimension of H equals $2^{\mathbf{m}}$. According to Proposition 1.5, H contains a separating subspace $Q \subset H$ of dimension $\mathbf{m}$. We take a Hamel basis $\mathcal{B}$ in Q and extend it to a Hamel basis $\tilde{\mathcal{B}}$ in H. As above, we set $M = \tilde{\mathcal{B}} \setminus \mathcal{B}$. The desired lattice is again $(2^M; \subset, \cup, \cap)$, and the topology determined by H is maximal in it. □

Proposition 1.7. *Let $\mathbf{k}$ be an infinite cardinal, and let $A = F_{k,0} = \prod_{\mathbf{k}} \mathbb{Z}_p$. Then the subspace $H_0 \subset A^\star$ generated by the coordinate projections of A is a minimal separating space.*

Proof. Obviously, the subspace H_0 itself is separating. Suppose that there exists a separating subspace $H \subset H_0$. Consider the one-to-one continuous map generated by the embeddings $(A, \tau_{H_0}) \longrightarrow (A, \tau_H)$. The topology of τ_{H_0} coincides with the Tychonoff product topology on A and, therefore, is compact. Since a continuous one-to-one map of a compact space to any Hausdorff space is a homeomorphism, the separating subspace H cannot be proper. □

The following result shows that there are no other minimal separating subspaces.

Theorem 1.8. *For a $\mathbb{Z}_p$-vector space A, a subspace $H \subset A^\star$ is a minimal separating space if and only if the natural map $\mathbf{e} : A \longrightarrow H^\star$ is an isomorphism.*

Proof. Obviously, the natural map $\mathbf{e}$ is a monomorphism if and only if $H \subset A^\star$ is a separating subspace.

Suppose that $\mathbf{e}$ is an isomorphism and $H_1 \subset H$ is a proper subspace. Since the map $p : H^\star \longrightarrow H_1^\star$ of the dual spaces has nontrivial kernel, it follows that so does the natural map $\mathbf{e}_1 = p \circ \mathbf{e}$ in $H_1^\star$. Therefore, H_1 does not separate points, i.e., H is minimal.

Now, let us suppose that H is a minimal separating space and show that $\mathbf{e}$ is an isomorphism. Take a nonzero element $\phi \in H^\star$ and let $\ker \phi = H_1 \subset H$. By minimality, H_1 is not separating; hence there exists a nonzero vector $v \in A$ such that $\mathbf{e}(v)(x) = x(v) = 0$ for all $x \in H_1$. Since H separates points, it follows that there exists a $y \in H$ for which $y(v) \neq 0$. Without loss of generality, we can assume that $y(v) = 1$.

The subspace H_1 has codimension 1, because this is the kernel of a linear functional. Therefore, H_1 and y generate H; in particular, $\phi(y) \neq 0$. Finally, for any vector $z \in H$, we have $z = x + ty$, where $x \in H_1$. This implies

$$\phi(z) = t\phi(y) = t\phi(y)y(v) = ty(\phi(y)v) = (x + ty)(\phi(y)v) = \mathbf{e}(\phi(y)v)(z).$$

Thus, $\phi = \mathbf{e}(\phi(y)v)$, so that $\mathbf{e}$ is an epimorphism and, by the remark at the beginning of the proof, an isomorphism. □

Corollary 1.9. *Let A be a $\mathbb{Z}_p$-vector space for which $H \subset A^\star$ is a minimal separating subspace of infinite dimension $\mathbf{k}$. Then there exists an isomorphism $i : \prod_{\mathbf{k}} \mathbb{Z}_p \longrightarrow A$ such that $i^\star(H) \subset (\prod_{\mathbf{k}} \mathbb{Z}_p)^\star$ coincides with the subspace generated by the basis projections of $\prod_{\mathbf{k}} \mathbb{Z}_p$.*

Proof. Indeed, it suffices to choose a Hamel basis in H; this determines an isomorphism $H \simeq \sum_{\mathbf{k}} \mathbb{Z}_p$, whence $A \simeq H^\star \simeq \prod_{\mathbf{k}} \mathbb{Z}_p$. □

We define the *logarithm* of a cardinal $\mathbf{m}$ by

$$\mathrm{Ln}(\mathbf{m}) = \{\mathbf{n} : \mathbf{m} = 2^{\mathbf{n}}\}.$$

Obviously, $\mathbf{k} \in \mathrm{Ln}(2^{\mathbf{k}})$ for any cardinal $\mathbf{k}$. Moreover, the set $\mathrm{Ln}(\aleph_0)$ is empty by Cantor's theorem. Finally, the generalized continuum hypothesis ($2^{\aleph_\alpha} = \aleph_{\alpha+1}$) implies $\mathrm{Ln}(2^{\mathbf{k}}) = \{\mathbf{k}\}$.

Corollary 1.10. *If the dimension* $\mathbf{m}$ *of a vector space* A *satisfies the condition* $Ln(\mathbf{m}) = \emptyset$*, then* $A^\star$ *contains no minimal separating subspace.*

2. Locally Compact Topologies

In this section, we analyze the set of locally compact topologies arising on the group $\prod_{\mathbf{n}} \mathbb{Z}_p$, where $\mathbf{n}$ is an infinite cardinal.

Let G be an infinite compact Abelian group; then, according to Kakutani's theorem [7], we have $|G| = 2^{\mathbf{m}}$, where $m = |G^*|$. Since the character group G^* is discrete, it follows that if any element of G has order p, then the group G is isomorphic to one of the groups $G \simeq \prod_{\mathbf{n}} \mathbb{Z}_p$, where $\mathbf{n} \in \mathrm{Ln}(|G|)$, with the natural product topology. For different $\mathbf{n} \in \mathrm{Ln}(|G|)$, the topological groups $\prod_{\mathbf{n}} \mathbb{Z}_p$ are nonhomeomorphic, because, according to Proposition 1.1, the weight of $\prod_{\mathbf{n}} \mathbb{Z}_p$ equals $\mathbf{n}$. Finally, note that under the generalized continuum hypothesis there exists only one compact topology on each direct product $\prod_{\mathbf{n}} \mathbb{Z}_p$. Thus, the following assertion is valid.

Proposition 2.1. *If* $\mathbf{n}$ *is an infinite cardinal, then the cardinality of the set of compact Hausdorff group topologies on the group* $G \simeq \prod_{\mathbf{n}} \mathbb{Z}_p$ *equals* $|Ln(|G|)|$*. Moreover, each* $\mathbf{n}' \in Ln(|G|)$ *corresponds to precisely one compact topology, and its weight equals* $\mathbf{n}'$*.*

Theorem 2.2. *Let* τ *be a locally compact Hausdorff topology on the group* $G \simeq \sum_{\mathbf{m}} \mathbb{Z}_p$*. Then the group* (G, τ) *is topologically isomorphic to* $F_{\mathbf{k},\mathbf{n}}$*, where the cardinals* $\mathbf{k}$ *and* $\mathbf{n}$ *satisfy the condition* $\max\{2^{\mathbf{k}}, \mathbf{n}\} = \mathbf{m}$.

Proof. All elements of the group G and, therefore, all elements of its character group G^* have order p. It follows that the group (G, τ) is totally disconnected.

According to van Dantzig's theorem [6], there exists a compact open subgroup $H \subset G$. Let $p : G \longrightarrow N = G/H$ be the natural projection, and let $i : H \longrightarrow G$ be an embedding. Since H is open, it follows that N is discrete and, therefore, $N \simeq \sum_{\mathbf{n}} \mathbb{Z}_p$, where $\mathbf{n}$ is a cardinal. According to Proposition 2.1, we have $H \simeq \prod_{\mathbf{k}} \mathbb{Z}_p$ for an appropriate cardinal $\mathbf{k}$ (naturally, the direct product is endowed with the compact product topology).

Next, choosing a Hamel basis in the $\mathbb{Z}_p$-vector space N, we take a splitting $q : N \longrightarrow G$ of the projection p. The homomorphism q is automatically continuous, and its image is discrete in G.

Finally, consider the homomorphism $i \times q : H \times N \longrightarrow G$ defined by $i \times q(h, n) = i(h) + q(n)$. Obviously, $i \times q$ is an algebraic isomorphism. Since H is open, it follows that this is a local homeomorphism between $H \times N$ and G, which, in turn, implies that $i \times q$ is an isomorphism of topological groups.

Thus, we conclude that (G, τ) is isomorphic to $F_{\mathbf{k},\mathbf{n}}$ as a topological group. The equality $\max\{2^{\mathbf{k}}, \mathbf{n}\} = \mathbf{m}$ is now obvious. □

The following proposition shows what happens to locally compact topologies on the group $\sum_{\mathbf{n}} \mathbb{Z}_p$, where $\mathbf{n} = \aleph_1$ or $2^{\aleph_0}$, under the negation of the continuum hypothesis.

Proposition 2.3. *If $\aleph_1 < 2^{\aleph_0}$, then there is only one (discrete) locally compact topology on the group $\sum_{\aleph_1} \mathbb{Z}_p$. Under the same assumption, there exist at least five pairwise nonhomeomorphic locally compact topologies on the group $\sum_{2^{\aleph_0}} \mathbb{Z}_p$.*

Proof. According to Theorem 2.2, the group $\sum_{\aleph_1} \mathbb{Z}_p$ with a locally compact topology is isomorphic to a group $F_{\mathbf{k},\mathbf{n}}$. Taking into account the inequality $\aleph_1 < 2^{\aleph_0}$, we see that the only possible choice of the cardinals is $\mathbf{k} = 0$ and $\mathbf{n} = \aleph_1$, i.e., the topology is discrete.

The groups $F_{\mathbf{k},\mathbf{n}}$ are locally compact and, under the assumption $\aleph_1 < 2^{\aleph_0}$, algebraically isomorphic to $\sum_{2^{\aleph_0}} \mathbb{Z}_p$ for the following pairs $(\mathbf{k}, \mathbf{n})$ of cardinals:

$$(2^{\aleph_0}, 0),\ (2^{\aleph_0}, \aleph_0),\ (2^{\aleph_0}, \aleph_1),\ (2^{\aleph_0}, \aleph_2),\ (0, 2^{\aleph_0}).$$

Here the first topology is compact and the last one is discrete. All of these topologies are pairwise nonhomeomorphic by virtue of Proposition 1.1. □

Theorem 2.2 and Proposition 2.3 have the following immediate corollary.

Corollary 2.4. *The following conditions are equivalent:*
(1) *the continuum hypothesis holds: $\aleph_1 = 2^{\aleph_0}$;*
(2) *the group $\sum_{\aleph_1} \mathbb{Z}_p$ admits a compact group topology;*
(3) *the group $\sum_{\aleph_1} \mathbb{Z}_p$ admits a nondiscrete locally compact group topology;*
(4) *the group $\prod_{\aleph_0} \mathbb{Z}_p$ admits at most four locally compact group topologies;*
(5) *the group $\prod_{\aleph_0} \mathbb{Z}_p$ admits precisely four locally compact group topologies.*

In conclusion, we mention that the four locally compact topological groups in condition (5) are precisely as follows:

$F_{\aleph_0,0}$ (this is a compact topology of weight $\aleph_0$);

$F_{\aleph_0,\aleph_0}$ (a noncompact nondiscrete topology of weight $\aleph_0$);

$F_{\aleph_0,2^{\aleph_0}}$ (a noncompact nondiscrete topology of weight $2^{\aleph_0}$);

$F_{0,2^{\aleph_0}}$ (the discrete topology of weight $2^{\aleph_0}$).

This list completes the proof of the second assertion of Theorem 0.1 stated in the introduction.

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