The Log Minimal Model Program for horospherical varieties via moment polytopes

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Abstract

In [Pas15], we described the Minimal Model Program in the family of $\mathbb{Q}$-Gorenstein projective horospherical varieties, by studying a family of polytopes defined from the moment polytope of an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor of the variety we begin with. Here, we summarize the results of [Pas15] and we explain how to generalize them in order to describe the Log Minimal Model Program for pairs (X, Δ) where X is a projective horospherical G-variety and Δ is a B-stable $\mathbb{Q}$-divisor (where G is a connected reductive algebraic group and B a Borel subgroup of G).

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1 Introduction

In this paper, we work over the complex numbers.

Let G be a connected reductive algebraic group. Let X be a normal G-variety. We say that X is horospherical if there exists $x \in X$ such that $G \cdot x$ is open in X and x is fixed by a maximal unipotent radical of G.

Note that if X is horospherical then there exists a Borel subgroup B of G such that $B \cdot x$ is open in X, ie X is a spherical variety. Moreover, if X is projective and horospherical, then X is the closure of the G-orbit of a sum of highest weight vectors in the projectivization of a multiplicity free G-module (see [Pas15, Proposition 2.11]). This point of view motivates the classification of projective horospherical varieties in terms of polytopes (called moment polytopes). In this paper, we propose to describe different variations of the Minimal Model Program (MMP) for projective horospherical varieties (including projective toric varieties) via moment polytopes.

We first recall what are the MMP and its log version: the Log MMP.

Definition 1. Let X be a normal projective variety. Let Δ be a $\mathbb{Q}$-divisor (not necessarily $\mathbb{Q}$-Cartier). Let K_X be a canonical divisor of X. We say that (X, Δ) is a log pair if $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

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Note that the divisor Δ is sometimes supposed to be effective in the Log MMP theory. Here, we do not necessarily make this assumption.

We denote by $NE(X)$ the cone of numerical classes of effective curves on X, by $\overline{NE}(X)$ its closure and by $\overline{NE}(X)_{K_X+\Delta<0}$ (respectively $\overline{NE}(X)_{K_X+\Delta>0}$) the open half-space of $\overline{NE}(X)$ defined by curves that are negative (respectively positive) along the divisor $K_X + \Delta$.

Then we summarized, in Figure 1, the principle of the Log MMP (without $\mathbb{Q}$-factorial assumption).

The different versions of MMP and Log MMP depend on the choice of the family $\mathcal{H}$:

- $\mathbb{Q}$-factorial MMP (or generally just called MMP), when $\mathcal{H}$ is the set of log pairs $(X, 0)$ such that X is $\mathbb{Q}$-factorial.
- non-$\mathbb{Q}$-factorial MMP (or $\mathbb{Q}$-Gorenstein MMP), when $\mathcal{H}$ is the set of log pairs $(X, 0)$ such that X is $\mathbb{Q}$-Gorenstein.
- $\mathbb{Q}$-factorial Log MMP (or generally just called Log MMP), when $\mathcal{H}$ is the set of log pairs (X, Δ) such that X is $\mathbb{Q}$-factorial.
- non-$\mathbb{Q}$-factorial Log MMP, when $\mathcal{H}$ is the set of all log pairs (X, Δ).

In [Pas15], the first two families was considered for horospherical varieties by reducing to the description of one-parameter families of polytopes. Moreover, in [Pas14], the non-$\mathbb{Q}$-factorial MMP and Log MMP were also detailed in general before to discuss on the case of spherical varieties.

In this paper, we consider the last two families when X is horospherical.

Note that, in these two families, we can distinguish several subfamilies according to types of singularities: terminal, canonical, Kawamata log terminal (klt), purely log terminal, divisorial log terminal, weakly log-terminal and log canonical (lc). For more details about these different types of singularities see for example [Fuj07]. Here, we will only deals with klt and log canonical singularities, whose definitions (see Definition 7) do not depend on the log resolution (by assuming that the coefficient of Δ are at most one in the definition of lc singularities).

The results that we obtain in this paper can be summarized in the following theorem.

Definition 2. A horospherical pair is a log pair (X, Δ) such that X is a horospherical G-variety and Δ is B-stable, where G is a connected reductive algebraic group and B is a Borel subgroup of G.

In this paper, we state and discuss the following result.

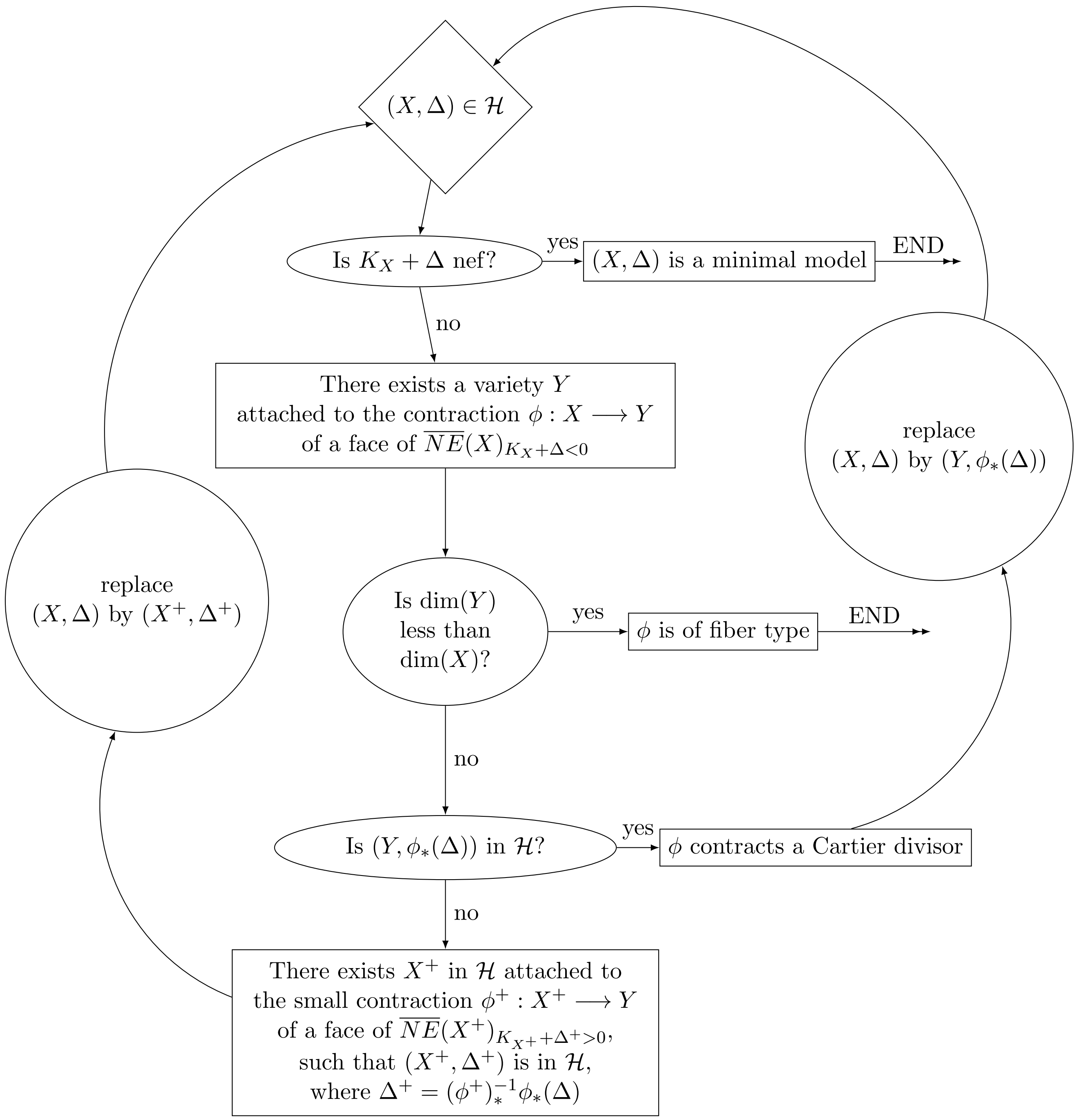
Theorem 1. *Let (X, Δ) be projective horospherical pair. Suppose that $(K_X + \Delta)$ is non-zero.*

Then, for any choice of an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor of X, we can construct a family of polytopes, depending of one non-negative rational parameter, such that: each polytope is either empty or the moment polytope of a projective horospherical variety; there are finitely many horospherical varieties that correspond to a polytope of the family; and all these horospherical varieties are the ones that appear in the log MMP.

In particular the Log MMP described by Figure 1 works and ends as soon as $-(K_X + \Delta)$ is non-zero, if $\mathcal{H}$ is the family of klt horospherical pairs, if $\mathcal{H}$ is the family of lc horospherical pairs, or if $\mathcal{H}$ is the family of horospherical pairs (X, Δ) such that $(K_X + \Delta)$ is effective. We can also restrict these families to log pairs with Δ effective.

Moreover, we can explicitly describe each step of the Log MMP until it ends.

Figure 1: Log MMP in a family $\mathcal{H}$ of log pairs



The strategy is the same as in [Pas15] and the proof of Theorem 1 is very similar to the proof of [Pas15, Theorem 1.1]. Hence, in Section 2, we follow the notation of [Pas15] and we recall briefly the definitions and properties that we need here. Then, in Section 3, we explain why the proofs of [Pas15] can be easily adapted to the log case. And we conclude and give examples in Section 4.

2 Projective horospherical varieties and polytopes

2.1 Notation

We will not recall here the long Luna-Vust theory of horospherical embeddings (classification in terms of colored fans). For more details on horospherical varieties, we refer the reader to [Pas08], and for basic results on Luna-Vust theory of spherical embeddings, we refer to [Kno91].

In this paper, we will only describe and use another a classification of projective horospherical varieties in terms of moment polytope, which was first introduced in [Pas15, Section 2.3].

We fix a connected reductive algebraic group G, a Borel subgroup B of G and a maximal torus T in B.

Then G/H always denotes a horospherical homogeneous space such that H contains the unipotent radical B. And if X is a horospherical variety, we denote by G/H the horospherical homogeneous space as above that is isomorphic to the open G-orbit of X, and we say that X is a G/H-embedding. (Note that Borel subgroups are all conjugated in G, so that the assumption on H can be done without loss a generality.)

Here, in order to simplify, we say here that a G/H-embedding is a normal algebraic G-variety with an open G-orbit isomorphic to G/H. (See [Pas15, section 2.1] for the precise definition and the notion of isomorphism of G/H-embeddings.)

The normalizer of H in G is a parabolic subgroup of G, we denote it by P.

Let S be the set of simple roots of (G, B, T). Then, we denote by R the subset of S of simple roots of P. Let $X(T)$ (respectively $X(T)^+$) be the lattice of characters of T (respectively the set of dominant characters). Similarly, we define $X(P)$ and $X(P)^+ = X(P) \cap X(T)^+$. Note that the lattice $X(P)$ and the cone $X(P)^+$ are generated by the fundamental weights ϖ_α with $\alpha \in S\backslash R$ and the weights of the center of G.

We denote by M the sublattice of $X(P)$ consisting of characters of P vanishing on H. (The rank of M is called the rank of G/H.) Let $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be the dual of M.

For any free lattice $\mathbb{L}$, we denote by $\mathbb{L}_{\mathbb{Q}}$ the $\mathbb{Q}$-vector space $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$.

For any $\alpha \in S\backslash R$, we define

$$W_{\alpha,P} := \{m \in X(P)_{\mathbb{Q}} \mid \langle m, \alpha^\vee \rangle = 0\}.$$

Note that these hyperplanes correspond to the walls of the dominant chamber $X(P)^+$.

2.2 Divisors of horospherical varieties and moment polytopes

In this section, we explain how to construct a polytope from an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor of a projective horospherical variety.

First, let us justify why we only consider B-stable divisors.

Proposition 2. *([Bri89, Section 2.2]) Any divisor of any G/H-embedding is linearly equivalent to a B-stable divisor.*

Now, we describe the B-stable prime divisors of a G/H-embedding X. We denote by $X_1, \dots, X_r$ the G-stable prime divisors of X. The other B-stable prime divisors (ie those that are not G-stable) are the closures in X of B-stable prime divisors of G/H (which are called colors of G/H). They are indexed by the simple roots α in $S\backslash R$ and we denote them by D_α (see [Pas08, Section 2] for an explicit description of these divisors).

Hence, any B-stable divisor (respectively $\mathbb{Q}$-divisor) of X can be written as follows: $\sum_{i=1}^r d_i X_i + \sum_{\alpha \in S\backslash R} d_\alpha D_\alpha$ with the d_i's and the d_α's in $\mathbb{Z}$ (respectively $\mathbb{Q}$).

All these B-stable prime divisors have an image in the lattice N as follows. Note that the lattice M is in bijection with the set of B-semi-invariant rational functions of G/H up to a scalar.Any B-stable divisor induces a B-stable valuation and then, by restriction to B-semi-invariant rational functions, it induces a point in N. We denote by x_i the corresponding point of N associated to X_i, it is a non-zero primitive element of the lattice N. For any simple root $\alpha \in S\backslash R$, the point associated to D_α is the restriction (may be zero) of the coroot $\alpha^\vee$ to M, which we denote by $\alpha_M^\vee$ (see [Pas08, Section 2]).

In [Bri89], there are characterizations of Cartier, $\mathbb{Q}$-Cartier and ample B-stable divisors of spherical varieties and also a description of the global sections of such divisors in terms of polytopes, that permits to give the following definition and result.

In the rest of the section, we fix a projective G/H-embedding X and a $\mathbb{Q}$-divisor $D = \sum_{i=1}^r d_i X_i + \sum_{\alpha \in S\backslash R} d_\alpha D_\alpha$ of X. And we suppose that D is $\mathbb{Q}$-Cartier and ample.

Definition 3. The pseudo-moment polytope of (X, D) is

$$\tilde{Q}_D := \{m \in M_\mathbb{Q} \mid \langle m, x_i \rangle \geq -d_i, \forall i \in \{1, \dots, r\} \text{ and } \langle m, \alpha_M^\vee \rangle \geq -d_\alpha, \forall \alpha \in \mathcal{F}_X\}.$$

Let $v^0 := \sum_{\alpha \in S\backslash R} d_\alpha \varpi_\alpha$. The moment polytope of (X, D) is $Q_D := v^0 + \tilde{Q}_D$.

Note that v^0 is not necessarily in $M_\mathbb{Q}$, but only in $X(P)_\mathbb{Q}$.

Proposition 3. *[Pas15, Corollary 2.8]*

1. *The pseudo-moment polytope $\tilde{Q}_D$ of (X, D) is of maximal dimension in $M_\mathbb{Q}$.*

2. *The moment polytope Q_D is contained in the dominant chamber $X(P)^+$ and it is not contained in any wall $W_{\alpha,P}$ for $\alpha \in S\backslash R$.*

3. *There is a bijection between faces of Q_D (or $\tilde{Q}_D$) and G-orbits of X (preserving the respective orders). In particular, the G-stable primes divisors X_i are in bijection with the facets of Q_D that are not contained in any wall $W_{\alpha,P}$ for $\alpha \in S\backslash R$ (the bijection maps X_i to the facet of $\tilde{Q}_D$ defined by $\langle m, x_i \rangle = -d_i$).*

4. *The divisor D can be computed from the pair $(Q, \tilde{Q})$ as follows: the coefficients d_α with $\alpha \in S\backslash R$ are given by the translation vector in $X(P)^+$ that maps $\tilde{Q}$ to Q; and for any $i \in \{1, \dots, r\}$, the coefficient d_i is given by $-\langle v_i, x_i \rangle$ for any element $v_i \in M_\mathbb{Q}$ in the facet of $\tilde{Q}$ associated to X_i.*

2.3 Classification of projective horospherical varieties in terms of polytopes

In this section, we write the classification of projective G/H-embeddings in terms of G/H-polytopes (defined below in Definition 4) and we also give a similar classification of polarized projective horospherical varieties.

Definition 4. Let Q be a polytope in $X(P)^+_{\mathbb{Q}}$ (not necessarily a lattice polytope). We say that Q is a G/H-polytope, if its direction is $M_{\mathbb{Q}}$ and if it is contained in no wall $W_{\alpha,P}$ with $\alpha\in S\backslash R$.

Let Q and Q' be two G/H-polytopes in $X(P)^+_{\mathbb{Q}}$. Consider any polytopes $\tilde{Q}$ and $\tilde{Q}'$ in $M_{\mathbb{Q}}$ obtained by translations from Q and Q' respectively. We say that Q and Q' are equivalent G/H-polytopes if the following conditions are satisfied.

1. There exist an integer j and $2j$ affine half-spaces $\mathcal{H}_1^+,\dots,\mathcal{H}_j^+$ and $\mathcal{H'}_1^+,\dots,\mathcal{H'}_j^+$ of $M_{\mathbb{Q}}$ (respectively delimited by the affine hyperplanes $\mathcal{H}_1,\dots,\mathcal{H}_j$ and $\mathcal{H'}_1,\dots,\mathcal{H'}_j$) such that $\tilde{Q}$ is the intersection of the $\mathcal{H}_i^+$, $\tilde{Q}'$ is the intersection of the $\mathcal{H'}_i^+$, and for all $i\in\{1,\dots,j\}$, $\mathcal{H}_i^+$ is the image of $\mathcal{H'}_i^+$ by a translation.

2. With notation of the previous item, for all subset J of $\{1,\dots,j\}$, the intersections $\cap_{i\in J}\mathcal{H}_i\cap Q$ and $\cap_{i\in J}\mathcal{H'}_i\cap Q'$ have the same dimension.

3. Q and Q' intersect exactly the same walls $W_{\alpha,P}$ of $X(P)^+$ (with $\alpha\in S\backslash R$).

Proposition 4. *[Pas15, Proposition 2.10] The map from (isomorphism classes of) projective G/H-embeddings to the set of equivalence classes of G/H-polytopes that maps X to the class of the moment polytope of (X,D), where D is any ample $\mathbb{Q}$-Cartier B-stable $\mathbb{Q}$-divisor, is a well-defined bijection.*

Since isomorphism classes of horospherical homogeneous spaces are in bijection with pairs (P,M) where P is a parabolic subgroup of G containing B and M is a sublattice of $X(P)$ (see [Pas15, Proposition 2.4]), we can give the following alternative classification.

Definition 5. A moment quadruple is a quadruple $(P,M,Q,\tilde{Q})$ where P is a parabolic subgroup of G containing B, M is a sublattice of $X(P)$, Q is a polytope in $X(P)^+_{\mathbb{Q}}$ and $\tilde{Q}$ is a polytope in $M_{\mathbb{Q}}$ that satisfy the three following conditions.

1. There exists (a unique) $\varpi\in\mathfrak{X}(P)_{\mathbb{Q}}$ such that $Q=\varpi+\tilde{Q}$.

2. The polytope $\tilde{Q}$ is of maximal dimension in $M_{\mathbb{Q}}$ (ie its interior in $M_{\mathbb{Q}}$ is not empty).

3. The polytope Q is not contained in any wall $W_{\alpha,P}$ for $\alpha\in S\backslash R$.

Definition 6. A polarized horospherical variety (respectively G/H-embedding) is a pair (X,D) such that X is a projective horospherical variety (respectively G/H-embedding) and D is an ample $\mathbb{Q}$-Cartier B-stable $\mathbb{Q}$-divisor. We say that (X,D) is isomorphic to (X',D') if there is an isomorphism from X to X' of embeddings of the same homogeneous space such that D identifies with to D' under this isomorphism.

Corollary 5. *The map from the set of isomorphic classes of polarized projective horospherical varieties to the set of classes of moment triples, that maps (X,D) to $(P,M,Q_D,\tilde{Q}_D)$ is a bijection.*

2.4 G-equivariant morphisms between projective horospherical varieties and polytopes

The existence of dominant G-equivariant morphisms between projective horospherical varieties can be characterized in terms of colored fans [Kno91] but also in terms of moment polytopes.

Consider two horospherical homogeneous spaces G/H and G/H'. Let (X,D) be a polarized G/H-embedding, let (X',D') be a polarized G/H'-embedding. By Corollary 5, (X,D) corresponds to a moment quadruple $(P,M,Q,\tilde{Q})$ and (X',D') corresponds to a moment quadruple $(P',M',Q',\tilde{Q}')$. (We also denote by R' the set of simple roots of P' and by N' the dual lattice of M'.)

A first necessary condition for the existence of a dominant G-equivariant morphism from X to X', is the existence of a projection π from G/H to G/H'. In particular $H'\supset H$, $P'\supset P$ and $R'\supset R$. The projection π induces an injective morphism π^* from M' to M. We suppose that this necessary condition is satisfied in the rest of the section and we identify M' with $\pi^*(M')$.

We first need to define a map ψ from the set of facets of $\tilde{Q}$ to the set of faces of $\tilde{Q}'$ (including $\tilde{Q}'$ itself).

First, note a general fact on polytopes: if $\mathcal{P}$ is a polytope in $\mathbb{Q}^r$, then for any affine half-space $\mathcal{H}^+$ delimited by an affine hyperplane $\mathcal{H}$ in $\mathbb{Q}^r$, there exists a unique face F of $\mathcal{P}$ and a point $x\in\mathbb{Q}^r$ such that F is defined by $x+\mathcal{H}$ (i.e. $F=\mathcal{P}\cap(x+\mathcal{H})$ and $\mathcal{P}\subset v+\mathcal{H}^+$). Then, for any facet F of $\tilde{Q}$, let $\mathcal{H}^+$ be the affine half-space in $M_{\mathbb{Q}}$, delimited by an affine hyperplane $\mathcal{H}$, such that $F=\mathcal{H}\cap\tilde{Q}$ and $\tilde{Q}\subset\mathcal{H}^+$. If $\mathcal{H}^+\cap M'_{\mathbb{Q}}\neq M'_{\mathbb{Q}}$, it is an affine half-space in $M'_{\mathbb{Q}}$ and, applying the above fact to $\mathcal{P}=\tilde{Q}'$, it gives a unique face F' of $\tilde{Q}'$. We set $\psi(F)=F'$. And if $\mathcal{H}^+\cap M'_{\mathbb{Q}}=M'_{\mathbb{Q}}$, we set $\psi(F)=\tilde{Q}'$.

See [Pas15, Example 2.15] to illustrate this definition.

We can now characterize the existence of dominant G-equivariant morphisms.

Proposition 6. *[Pas15, Proposition 2.16 and Corollary 2.17] Under the above necessary condition, there exists a dominant G-equivariant morphism from X to X', if and only if*

1. *for any set $\mathcal{G}$ of facets of $\tilde{Q}$, $\cap_{F\in\mathcal{G}}F\neq\emptyset$ implies $\cap_{F\in\mathcal{G}}\psi(F)\neq\emptyset$, and*
2. *for any $\alpha\in S\backslash R$ such that $Q\cap W_{\alpha,P}\neq\emptyset$, we have $Q'\cap W_{\alpha,P}\neq\emptyset$.*

Suppose there exists a dominant G-equivariant morphism ϕ from X to X'. Let $\mathcal{O}$ be the G-orbit in X associated to a face $\cap_{F\in\mathcal{G}}F$ (where $\mathcal{G}$ is a set of facets). Then $\phi(\mathcal{O})$ is the G-orbit in X' associated to the face $\cap_{F\in\mathcal{G}}\psi(F)$ of Q'.

Remark that, in Proposition 6 (1), we could replace the pseudo-moment polytope by the moment polytope (by extension of the definition of ψ).

2.5 Curves in horospherical varieties

Here we describe effective curves of a projective horospherical variety X and their intersections with ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

We denote by $N_1(X)$ the group of numerical classes of 1-cycles of the variety X. Recall that $NE(X)$ is the convex cone in $N_1(X)$ generated by effective 1-cycles.

Proposition 7. *[Pas15, Section 2.5] Let X be a projective horospherical variety. Let Q be any moment polytope of X (with any choice of an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D of X).*

There exist B-stable rational curves C_μ and $C_{\alpha,v}$ in X (which do not depend on the choice of Q), indexed by edges μ of Q and by pairs (α, v) with $\alpha \in S\backslash R$ and v a vertex of Q that is not in the wall $W_{\alpha,P}$, such that:

1. *the classes $[C_\mu]$ and $[C_{\alpha,v}]$ of these curves generate $NE(X)$, which is then closed and polyhedral;*
2. *for any edge μ of the moment polytope Q_D, the intersection number $D.C_\mu$ is the integral length of μ, ie the length of μ divided by the length of the primitive element in the direction of μ;*
3. *for any pair (α, v) as above, we have $D.C_{\alpha,v} = \langle v, \alpha^\vee\rangle$.*

3 Log MMP via a one-parameter family of polytopes

We begin with any horospherical pair (X, Δ) (see Definition 2).

3.1 The one-parameter family of polytopes

We construct the one-parameter family of polytopes that permits to run the Log MMP from (X, Δ) as follows. We use the notation of Section 2.

We write $\Delta = \sum_{i=0}^{r} \delta_i X_i + \sum_{\alpha\in S\backslash R} \delta_\alpha D_\alpha$. Moreover, an anticanonical divisor of X is $-K_X = \sum_{i=1}^{r} X_i + \sum_{\alpha\in S\backslash R} a_\alpha D_\alpha$ where $a_\alpha = \langle 2\rho^P, \alpha^\vee\rangle$ such that ρ^P is the sum of positive roots of G that are not roots of P [Bri89].

For any rational number $\epsilon > 0$ small enough, the divisor $D + \epsilon(K_X + \Delta)$ is still ample (and $\mathbb{Q}$-Cartier by hypothesis), so that $(X, D + \epsilon(K_X + \Delta))$ defines a moment polytope Q^ϵ and a pseudo-moment polytope $\tilde{Q}^\epsilon$. Then we extend naturally the definition for any rational number $\epsilon > 0$. More precisely, for any $\epsilon > 0$, we define $\tilde{Q}^\epsilon := \{x \in M_\mathbb{Q} \mid Ax \geq \tilde{B} + \epsilon\tilde{C}\}$ and $Q^\epsilon := v^\epsilon + \tilde{Q}^\epsilon$ where the matrices A, $\tilde{B}$, $\tilde{C}$ and the vector v^ϵ are defined below.

Recall that $x_1, \dots, x_r$ denote the primitive elements of N associated to the G-stable prime divisors X_i of X. We choose an order in $S\backslash R$ and we then denote by $\alpha_1, \dots, \alpha_s$ its elements. We fix a basis $\mathcal{B}$ of M and we denote by $\mathcal{B}^\vee$ the dual basis in N.

Now define $A \in \mathcal{M}_{r+s,n}(\mathbb{Q})$ whose first r lines are the coordinates of the vectors x_i in the basis $\mathcal{B}^\vee$ with $i \in \{1, \dots, r\}$ and whose last s lines are the coordinates of the vectors $\alpha_{jM}^\vee$ in $\mathcal{B}^\vee$ with $j \in \{1, \dots, s\}$.

Let $\tilde{B}$ be the column matrix such that the pseudo-moment polytope of D is defined by $\{x \in M_\mathbb{Q} \mid Ax \geq \tilde{B}\}$. In fact, if $D = \sum_{i=1}^{r} b_i X_i + \sum_{\alpha\in S\backslash R} b_\alpha D_\alpha$, then $\tilde{B}$ is the column matrix associated to the vector $(-b_1, \dots, -b_r, -b_{\alpha_1}, \dots, -b_{\alpha_s})$.

Similarly, the column matrix $\tilde{C}$ corresponds to the vector $(1-\delta_1, \dots, 1-\delta_r, a_{\alpha_1}-\delta_{\alpha_1}, \dots, a_{\alpha_s}-\delta_{\alpha_s})$.

Finally, define $v^\epsilon := \sum_{\alpha\in S\backslash R}(b_\alpha + \epsilon(\delta_\alpha - a_\alpha))\varpi_\alpha$ (which is not necessarily in $M_\mathbb{Q}$).

Note that, if $M_\mathbb{Q} = X(P)_\mathbb{Q}$, we can also write the moment polytopes as follows $Q^\epsilon := \{x \in M_\mathbb{Q} \mid Ax \geq B + \epsilon C\}$ where B and C are respectively the column matrices associated to the vectors

$(-b_1+\langle x_1,\sum_{\alpha\in S\backslash R}b_\alpha\varpi_\alpha\rangle,\dots,-b_r+\langle x_r,\sum_{\alpha\in S\backslash R}b_\alpha\varpi_\alpha\rangle,0,\dots,0)$ and $(1-\delta_1+\langle x_1,\sum_{\alpha\in S\backslash R}(\delta_\alpha-a_\alpha)\varpi_\alpha\rangle,\dots,1-\delta_r+\langle x_r,\sum_{\alpha\in S\backslash R}(\delta_\alpha-a_\alpha)\varpi_\alpha\rangle,0,\dots,0)$. Moreover, even if $M_\mathbb{Q}\neq X(P)_\mathbb{Q}$, it is easy to see that the s last inequalities defining $\tilde{Q}^\epsilon$ are equivalent to the fact that Q^ϵ is in $X^+(P)$.

Remark 1. In [Pas15], the definition of the family of polytopes seams to be more complicated. Indeed we extended step by step the family of pseudo-moment polytopes to any rational number $\epsilon>0$, by erasing a line i of A, $\tilde{B}$ and $\tilde{C}$ with $i\in\{1,\cdots,r\}$ as soon as this line does not correspond to a facet of $\tilde{Q}^\epsilon$. It gives the same family of polytopes because of the convexity of the set of ϵ's such that the line i corresponds to a facet of $\tilde{Q}^\epsilon$. But we had to give this complicated construction to define the good equivalence relation in this family of polytopes so that the equivalence classes of G/H-polytopes (Definition 4) in the family $(Q^\epsilon)_{\epsilon\in[0,\epsilon_{max}[}$ and the equivalence classes ([Pas15, Definition 3.14]) in the family $(\tilde{Q}^\epsilon)_{\epsilon\in[0,\epsilon_{max}[}$ are the same ([Pas15, Proposition 4.1]), where ϵ_{max} is the minimum (may be $+\infty$) non-negative rational number ϵ such that Q^ϵ is not a G/H-polytope. If ϵ_{max} is finite, it is a positive rational number by [Pas15, Corollary 3.16].

Remark 2. Note also that, if $\tilde{C}\geq 0$ and non-zero, ϵ_{max} is finite (and rational). Indeed, since $\tilde{C}\geq 0$, we get easily that for any $\epsilon>0$, $\tilde{Q}^\epsilon\subset\tilde{Q}^0$. But $\tilde{C}\neq 0$ and $\tilde{Q}^0$ is bounded, then there exists an index i in $\{1,\dots,m+r\}$ such that $\tilde{C}_i>0$, so that the set $\tilde{Q}^0\cap\{X\in M_\mathbb{Q}\mid A_iX\geq\tilde{B}_i+\epsilon\tilde{C}_i\}$ is empty for ϵ big enough (even if $A_i=0$). Hence, $\tilde{Q}^\epsilon$ is empty for ϵ big enough, in particular Q^ϵ cannot be a G/H-polytope for any $\epsilon>0$.

Moreover, if ϵ_{max} is finite, $Q^{\epsilon_{max}}$ is neither empty nor a G/H-polytope and for any $\epsilon>\epsilon_{max}$, $Q^{\epsilon_{max}}$ is empty [Pas15, Remark 3.18].

Remark 3. The construction above of the families (Q^ϵ) and $(\tilde{Q}^\epsilon)$ can be made for any B-stable $\mathbb{Q}$-divisor D' instead of $K_X+\Delta$. And then, D' is $\mathbb{Q}$-Cartier if and only if for ϵ small enough the polytopes Q^ϵ are G/H-polytopes equivalent to Q^0. Indeed, if D' is $\mathbb{Q}$-Cartier, then for ϵ small enough, $D+\epsilon D'$ is $\mathbb{Q}$-Cartier and ample, and Q^ϵ is the moment polytope of $(X,D+\epsilon D')$, in particular it is equivalent to the moment polytope Q^0 of (X,D). Inversely, let ϵ small enough such that Q^ϵ is a G/H-polytope equivalent to Q^0. Then the pair $(Q^\epsilon,\tilde{Q}^\epsilon)$ corresponds to a unique polarized variety (X,D'') where D'' is an ample $\mathbb{Q}$-Cartier B-stable $\mathbb{Q}$-divisor. Using 4. of Proposition 3, we can compute that $D''=D+\epsilon D'$, in particular D' is $\mathbb{Q}$-Cartier.

3.2 Construction of pairs, contractions and flips

We apply [Pas15, Corollary 3.16] to the family $(\tilde{Q}^\epsilon)_{\epsilon\in\mathbb{Q}_{\geq 0}}$. Then there exist non-negative integers $k,j_0,\dots,j_k$, rational numbers $\alpha_{i,j}$ for $i\in\{0,\dots,k\}$ and $j\in\{0,\dots,j_i\}$ and $\alpha_{k,j_k+1}=\epsilon_{max}\in\mathbb{Q}_{>0}\cup\{+\infty\}$ ordered as follows with the convention that $\alpha_{i,j_i+1}=\alpha_{i+1,0}$ for any $i\in\{0,\dots,k-1\}$,

1. $\alpha_{0,0}=0$;
2. for any $i\in\{0,\dots,k\}$, and for any $j<j'$ in $\{0,\dots,j_i+1\}$ we have $\alpha_{i,j}<\alpha_{i,j'}$;

such that the equivalent classes of G/H-embeddings of pseudo-moment polytopes of the family $(Q^\epsilon)_{\epsilon\in[0,\epsilon_{max}[}$ are:

1. $X_{i,j}$ for any $i\in\{0,\dots,k\}$ and $j\in\{0,\dots,j_i\}$, respectively associated to moment polytopes Q^ϵ with $\epsilon\in]\alpha_{i,j},\alpha_{i,j+1}[$;
2. $Y_{i,j}$ for any $i\in\{0,\dots,k\}$ and $j\in\{1,\dots,j_i\}$, respectively associated to the moment polytope $Q^{\alpha_{i,j}}$;

All these varieties have the same B-stable and not G-stable prime divisors as X because they have the same open G-orbit G/H. Moreover, any of their G-stable prime divisors corresponds to a G-stable prime divisor of X (but the inverse is not necessarily true). Indeed a G-stable prime divisor of one of these G/H-embeddings corresponds to a facet F, of the corresponding moment polytope, that is not in any wall of the dominant chamber. In particular, if $\tilde{F}$ denotes the facet of $\tilde{Q}$ obtained by translation of F, $\tilde{F}$ is defined by a hyperplane associated to one of the r first lines of A, $\tilde{B}$ and $\tilde{C}$. We still denote by X_k, with $k \in \{1,\dots,r\}$, the G-stable prime divisor of $X_{i,j}$ or $Y_{i,j}$ when it is still a divisor of $X_{i,j}$ or $Y_{i,j}$.

Then, we define

$$\Delta_{X_{i,j}} := \sum_{k,\,X_k \text{ is a divisor of } X_{i,j}} \delta_k X_k + \sum_{\alpha\in S\backslash R} \delta_\alpha D_\alpha$$

and $\Delta_{Y_{i,j}}$ similarly.

Note that for $\Delta = K_X$ we would have $\Delta_{X_{i,j}} = K_{X_{i,j}}$ and $\Delta_{Y_{i,j}} = K_{Y_{i,j}}$.

Remark 4. For any i and j, for any $\epsilon \in]\alpha_{i,j},\alpha_{i,j+1}[$ the pair $(Q^\epsilon,\tilde{Q}^\epsilon)$ corresponds to the polarized variety $(X_{i,j}, D+\epsilon(K_{X_{i,j}}+\Delta_{X_{i,j}}))$ by the bijection of Corollary 5, and $(Q^{\alpha_{i,j}},\tilde{Q}^{\alpha_{i,j}})$ corresponds to the polarized variety $(X_{i,j}, D+\epsilon(K_{Y_{i,j}}+\Delta_{Y_{i,j}}))$.

If ϵ_{max} is finite, the polytope $Q^{\alpha_{max}}$ also defines a projective horospherical G-variety Z. Indeed, we can apply Corollary 5 to a quadruple $(P^1, M^1, Q^{\alpha_{max}}, \tilde{Q}^{\alpha_{max}})$ to get a polarized horospherical variety. We define P^1 and M^1 such that $(P^1, M^1, Q^{\alpha_{max}}, \tilde{Q}^{\alpha_{max}})$ is a moment quadruple as follows: $M^1_{\mathbb{Q}}$ is the minimal vector subspace containing $\tilde{Q}^{\alpha_{max}}$ and then $M^1 := M^1_{\mathbb{Q}} \cap M$; P^1 is the parabolic subgroup containing B with simple roots R^1 that is the union of R with the set of $\alpha \in S\backslash R$ such that $Q^{\alpha_{max}}$ is contained in the wall $W_{\alpha,P}$. Then the moment quadruple $(P^1, M^1, Q^{\alpha_{max}}, \tilde{Q}^{\alpha_{max}})$ corresponds to a polarized horospherical variety (Z, D_Z). In particular, Z is a G/H^1-embedding where H^1 is the subgroup of P^1 defined as the intersection of kernels of the characters of P^1 in M^1.

Remark that, by definition, $M^1 \subset M$ and $R \subset R^1$ so that we have a projection $\pi : G/H \longrightarrow G/H^1$. Note also that the above choice of P^1 and H^1 is the unique such that $(P^1, M^1, Q^{\alpha_{max}}, \tilde{Q}^{\alpha_{max}})$ is a moment quadruple and π has connected fibers.

Now, by Proposition 6, we get dominant G-equivariant morphisms:

1. $\phi_{i,j} : X_{i,j-1} \longrightarrow Y_{i,j}$ for any $i \in \{0,\dots,k\}$ and $j \in \{1,\dots,j_i\}$;
2. $\phi^+_{i,j} : X_{i,j} \longrightarrow Y_{i,j}$ for any $i \in \{0,\dots,k\}$ and $j \in \{1,\dots,j_i\}$;
3. $\phi_i : X_{i,j_i} \longrightarrow X_{i+1,0}$ for any $i \in \{0,\dots,k-1\}$;
4. and, if ϵ_{max} is finite, $\phi : X_{k,j_k} \longrightarrow Z$.

With the same proofs as in [Pas15, Sections 4.3 and 4.4] (in particular by replacing the condition on X to be $\mathbb{Q}$-Gorenstein by the condition on $K_X + \Delta$ to be $\mathbb{Q}$-Cartier), we get the following results.

Proposition 8. *1. For any $i \in \{0,\dots,k\}$ and $j \in \{1,\dots,j_i\}$, the curves C that are contracted by the morphism $\phi_{i,j}$ satisfy $(K_{X_{i,j-1}}+\Delta_{X_{i,j-1}})\cdot C < 0$; for any $i \in \{0,\dots,k-1\}$, the curves C that are contracted by the morphism ϕ_i satisfy $(K_{X_{i,j_i}}+\Delta_{X_{i,j_i}})\cdot C < 0$; and, if ϵ_{max} is finite, the curves C that are contracted by the morphism ϕ satisfy $(K_{X_{k,j_k}}+\Delta_{X_{k,k_j}})\cdot C < 0$.*

2. *For any $i \in \{0, \dots, k\}$ and $j \in \{1, \dots, j_i\}$, the curves C that are contracted by the morphism $\phi^+_{i,j}$ satisfy $(K_{X_{i,j}} + \Delta_{X_{i,j}}) \cdot C > 0$.*

3. *For any $i \in \{0, \dots, k-1\}$, the morphism ϕ_i contracts at least a G-stable divisor of X_{i,j_i}.*

Proposition 9. *The pairs $(X_{i,j}, \Delta_{X_{i,j}})$ with $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$ are horospherical pairs (ie $K_{X_{i,j}} + \Delta_{X_{i,j}}$ is $\mathbb{Q}$-Cartier). (And the pairs $(Y_{i,j}, \Delta_{Y_{i,j}})$ with $i \in \{0, \dots, k\}$ and $j \in \{0, \dots, j_i\}$ are not horospherical pairs (ie $K_{Y_{i,j}} + \Delta_{Y_{i,j}}$ is not $\mathbb{Q}$-Cartier).)*

Note that, for any $i \in \{0, \dots, k-1\}$, the $\mathbb{Q}$-Cartier divisor $\phi_i^*(K_{X_{i+1,0}} + \Delta_{X_{i+1,0}}) - (K_{X_{i,j_i}} + \Delta_{X_{i,j_i}})$ is supported in the exceptional locus of ϕ_i, then ϕ_i contracts a Cartier divisor.

3.3 $\mathbb{Q}$-factorial Log MMP

For D general, the log MMP works also for the family of $\mathbb{Q}$-factorial horospherical pairs. And for D general, from a $\mathbb{Q}$-factorial horospherical pair, all the contractions that appears above in the log MMP are contractions of extremal rays.

Proposition 10. *Let (X, Δ) be a horospherical pair such that X is $\mathbb{Q}$-factorial. Choose D such that the vector $\tilde{B}$ is in the open set*

$$\bigcup_{I \subset \{1,\dots,r+s\},\, |I|>n} \pi_I^{-1}(\mathbb{Q}^{|I|} \setminus \mathrm{Im}(A_I)),$$

where π_I is the canonical projection of $\mathbb{Q}^{r+s}$ to its vector subspace corresponding to the coordinates in I.

Then, for any $i \in \{0, \dots, k\}$ and any $j \in \{0, \dots, j_i\}$, the variety $X_{i,j}$ is $\mathbb{Q}$-factorial.

The proof is exactly the same as the proof of [Pas15, Proposition 4.6].

Remark 5. The open set where $\tilde{B}$ is chosen, is clearly not empty and dense in $\mathbb{Q}^{r+s}$, because for any I of cardinality greater than n, the image of A_I is of codimension at least one. And, since X is $\mathbb{Q}$-factorial, any vector $\tilde{B} \in \mathbb{Q}^{r+s}$ gives a $\mathbb{Q}$-Cartier divisor.

Proposition 11. *Let (X, Δ) be a horospherical pair such that X is $\mathbb{Q}$-factorial If D is general in the set of ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, all morphisms $\phi_{i,j}$, $\phi^+_{i,j}$, ϕ_i and ϕ (if ϵ_{max} is finite) defined in Section 3.2 are contractions of rays of the corresponding effective cones $NE(X_{i,j})$.*

The proof is the same as the proof [Pas15, Proposition 4.8] by replacing K_Y by $K_Y + \Delta_Y$ and all$K_{X_{i,j}}$ by $K_{X_{i,j}} + \Delta_{X_{i,j}}$.

Remark 6. Proposition 11 is not true without the hypothesis of $\mathbb{Q}$-factoriality (at least for flips): see [Pas15, Example 5.6] (such an example also exists for toric varieties of dimension 3).

3.4 General fibers of contractions of fiber type

We assume here that ϵ_{max} is finite.

then, we can also describe the general fibers of the morphism $\phi : X_{k,j_k} \longrightarrow Z$ defined in Section 3.2. We may assume that $k = 0$ and $j_0 = 0$, in particular $X_{k,j_k} = X$.

Theorem 12. *Let (X,Δ) be a horospherical pair, with X a G/H-embedding. Let D be an ample $\mathbb{Q}$-Cartier B-stable $\mathbb{Q}$-divisor on X. suppose that there exists a positive rational ϵ_1 such that for any $\epsilon\in[0,\epsilon_1[$ the G/H-polytope Q^ϵ is equivalent to $Q=Q^0$ and Q^{ϵ_1} is not a G/H-polytope.*

Let H^1, P^1, R^1 and M^1 defined as in Section 3.2. In particular Q^{ϵ_1} is a G/H^1-polytope. Denote by Z the associated G/H^1-embedding and by ϕ the G-equivariant morphism from X to Z.

Then the general fibers of ϕ are either the flag variety P^1/P or a projective horospherical variety F_ϕ. Moreover in the second case, F_ϕ is a L^1/H^2-embedding, where $L^1:=H^1/R_u(H^1)$ and $H^2:=H/R_u(H^1)$ ($R_u(H^1)$ denoting the unipotent radical of H^1 in G). And a moment polytope of F_ϕ is the projection of Q in $X(P)_{\mathbb{Q}}/M^1_{\mathbb{Q}}$.

Assume now that X is $\mathbb{Q}$-factorial. Then, for general D, the general fibers P^1/P or F_ϕ have Picard number one.

The proof and the description of F_ϕ are the same as in [Pas15, Section 4.6], where we only used the matrices A, $\tilde{B}$, $\tilde{C}$ and the vector v^0, and then never used that $\tilde{C}$ is defined from K_X, so that we can replace K_X by any other $\mathbb{Q}$-Cartier divisor (under the assumptions of the beginning of the section).

4 Conclusion and examples

Let (X,Δ) be a horospherical pair. Suppose that $-(K_X+\Delta)$ is non-zero and effective. Pick an ample $\mathbb{Q}$-Cartier B-stable $\mathbb{Q}$-divisor of X. Then the family (Q^ϵ) of polytopes defined in Section 3.1 describes the Log MMP from (X,Δ). Moreover, it preserves some singularities of X and (X,Δ).

First give the definition of klt and lc singularities that we use here. It is equivalent to [KM98, Definition 2.34].

Definition 7. Let (X,Δ) be a log pair such that $\lceil\Delta\rceil\leq 1$ (ie the coefficients of Δ are at most 1). A log resolution of (X,Δ) is a proper birational map $\phi:V\longrightarrow X$ where V is smooth and such that $\mathrm{Exc}(\phi)+\phi_*^{-1}(\Delta)$ is a divisor whose support has simple normal crossings.

The pair (X,Δ) has klt (respectively lc) singularities if there exists a log resolution $\phi:V\longrightarrow X$ such that every coefficient of the divisor $K_V-\phi^*(K_X+\Delta)$ of V is greater than -1 (respectively at least -1).

In the case of horospherical varieties, we can construct log resolutions by using Bott-Samelson resolutions and then we get the following characterization.

Theorem 13. *[Pas16] A horospherical pair (X,Δ) has klt singularities if and only if $\lfloor\Delta\rfloor\leq 0$ (ie the coefficient of Δ are less than 1). A horospherical pair (X,Δ) has lc singularities if and only if $\lceil\Delta\rceil\leq 1$.*

In [Pas16], we proved the first statement with the assumption that Δ is effective. But it is easy to see that this assumption is not necessary. And it is also not difficult to use the same proof with "$\geq$" instead of "$>$" to get the second statement.

Hence, we deduce easily that the family (Q^ϵ) gives the different types of MMP as follows.

1. if X is $\mathbb{Q}$-factorial, we get the $\mathbb{Q}$-factorial Log MMP;
2. if (X,Δ) has klt singularities, we get the Log MMP for klt pairs (ie any log pair $(X_{i,j},\Delta_{i,j})$ has klt singularities);

3. if (X, Δ) has log canonical singularities, we get the Log MMP for lc pairs (ie any log pair $(X_{i,j}, \Delta_{i,j})$ has lc singularities).

Note also that the results above are still true if we assume in the definition of pairs that Δ is effective.

Remark 7. The assumption on $-(K_X + \Delta)$ to be effective is too restrictive: it could happen, without this assumption, that the family (Q^ϵ) describes the Log MMP until its end (see the end of Example 2). Indeed, the optimal assumption to do is that ϵ_{max} (defined in Remark 1) is finite. We have "$-(K_X + \Delta)$ is non-zero and effective" implies "ϵ_{max} is finite" (see Remark 2), but the inverse is not true.

Moreover, if $\epsilon_{max} = +\infty$, the family (Q^ϵ) describes a beginning (may be "empty") of the Log MMP that never ends.

Remark also that $-(K_X + \Delta)$ is effective as soon as (X, Δ) has lc singularities.

Example 1. Let $G = \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^*$, let $P = B$ be the product of the subgroup of upper triangular matrices of $\mathrm{SL}_2(\mathbb{C})$ by $\mathbb{C}^*$ and let $M = X(B)$. We denote by ϖ_α the fundamental weight of G associated to the unique simple root of (G, B). And we denote by ϖ_0 the weight of B defined by the projection on $\mathbb{C}^*$. Then $M = \mathbb{Z}\varpi_\alpha \oplus \mathbb{Z}\varpi_0$ and $X(B)^+ = \mathbb{N}\varpi_\alpha \oplus \mathbb{Z}\varpi_0$.

The pair (P, M) defines (up to isomorphism) a horospherical homogeneous space G/H of dimension 3. We define

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} -b_1 \\ b_1 \\ -b_3 \\ 0 \end{pmatrix},$$

where b_1 and b_3 are rational numbers such that $b_3 > 0$.

We can check that the polytope $Q^0 := \{X \in M_\mathbb{Q} \mid AX \geq B\}$ is a triangle that intersects the line $\mathbb{Q}\varpi_0$ in exactly one vertex. Then the moment triple (P, M, Q^0, Q^0) corresponds to the polarized G/H-embedding (X, D) where the ample $\mathbb{Q}$-Cartier B-stable $\mathbb{Q}$-divisor D is of the form $b_1X_1 - b_1X_2 + b_3X_3 + 0D_\alpha$. Note that the three G-stable divisors of X correspond to $x_1 = (1, -1)$, $x_2 = (2, 1)$ and $x_3 = (-1, 0)$ respectively, and that $\alpha_M^\vee$ is $(1, 0)$, in the dual basis of the basis $(\varpi_\alpha, \varpi_0)$ of M.

Now define $Q^\epsilon := \{X \in M_\mathbb{Q} \mid AX \geq B + \epsilon C\}$, where $C \in M_{4,1}(\mathbb{Q})$. We can check that, Q^ϵ is a triangle that intersects the line $\mathbb{Q}\varpi_0$ in exactly one vertex for any ϵ small enough, if and only if $C_1 + C_2 = 0$. Here, an anticanonical divisor of X is $-K_X = X_1 + X_2 + X_3 + 2D_\alpha$. In particular, we can compute with Remark 3 that K_X is not $\mathbb{Q}$-Cartier.

More generally, if $\Delta = \delta_1X_1 + \delta_2X_2 + \delta_2X_3 + \delta_\alpha D_\alpha$ is a B-stable $\mathbb{Q}$-divisor of X, then $K_X + \Delta$ is $\mathbb{Q}$-Cartier if and only if $4 + \delta_1 + \delta_2 - 3\delta_\alpha = 0$. In particular, if Δ is effective, (X, Δ) does never have klt or even lc singularities.

We now consider three cases. Note that the matrix C that defines the moment polytopes Q^ϵ as in Section 3 is $C = \begin{pmatrix} -1 - \delta_1 + \delta_\alpha \\ -3 - \delta_2 + 2\delta_\alpha \\ 3 - \delta_3 - \delta_\alpha \\ 0 \end{pmatrix}$.

- For $\Delta = -X_1 + D_\alpha$, the pair (X, Δ) has lc singularities. For any $\epsilon \in [0, \frac{b_3}{2}[$, the G/H-polytope Q^ϵ is equivalent to Q^0, and $Q^{\frac{b_3}{2}}$ is the point $(0, b_1 - \frac{b_3}{2})$ then the Log MMP gives a contraction of fiber type from X to a point. Note that here $\tilde{C} \geq 0$.

- For $\Delta = X_1 + X_2 + 2D_\alpha$, the pair (X, Δ) does not have lc singularities. For any $\epsilon \in [0, b_3[$, the G/H-polytope Q^ϵ is equivalent to Q^0, and Q^{b_3} is the point $(0, b_1)$ then the Log MMP also gives a contraction of fiber type from X to a point. Note that here $\tilde{C} \geq 0$.

- For $\Delta = \frac{5}{3}X_3 + \frac{4}{3}D_\alpha$, the pair (X, Δ) does not have lc singularities. For any $\epsilon \in [0, +\infty[$, the G/H-polytope Q^ϵ is equivalent to Q^0 then the Log MMP does not ends. Note that here $\tilde{C} \ngeq 0$.

We now consider another example with the same horospherical homogenous space G/H but from another G/H-embedding.

Example 2. Let G, P, M, and A as in Example 1. Let $B = \begin{pmatrix} -b_1 \\ -b_2 \\ -b_3 \\ 0 \end{pmatrix}$, where b_1, b_2 and b_3 are rational numbers such that $-b_1 - b_2 > 0$ and $b_1 + b_2 + 3b_3 > 0$.

We can check that the polytope $Q^0 := \{X \in M_\mathbb{Q} \mid AX \geq B\}$ is a triangle that is contained in the interior of $X(B)^+$.

Then the moment triple (P, M, Q^0, Q^0) corresponds to the polarized G/H-embedding (X, D) where the ample $\mathbb{Q}$-Cartier B-stable $\mathbb{Q}$-divisor D is of the form $b_1X_1 + b_2X_2 + b_3X_3 + 0D_\alpha$. (The three G-stable divisors of X still correspond to $x_1 = (1, -1)$, $x_2 = (2, 1)$ and $x_3 = (-1, 0)$ respectively.)

Now define $Q^\epsilon := \{X \in M_\mathbb{Q} \mid AX \geq B + \epsilon C\}$, where $C \in M_{4,1}(\mathbb{Q})$. We can check that, for any C and for any ϵ small enough, Q^ϵ is a triangle that is contained in the interior of $X(B)^+$. Hence, X is $\mathbb{Q}$-factorial by remark 3. In particular, for any B-stable $\mathbb{Q}$-divisor Δ of X, (X, Δ) is a horospherical pair and $(X, 0)$ has klt singularities.

One can consider the same three cases as in Example 1 and obtain similar descriptions of Log MMP (except that the contraction of fiber type goes to $G/B \simeq \mathbb{P}^1$ instead of a point).

We now consider another family of cases including the case of the pair $(X, 0)$. Let $\Delta = \delta_3X_3$, with $\delta_3 \in \mathbb{Q}$, then we distinguish four cases (here $\tilde{C} \geq 0$ if and only if $\delta_3 \leq 1$). Let $\epsilon_1 := \frac{b_1+b_2+3b_3}{5-3\delta_3}$, $\epsilon_2 := \frac{-b_1-b_2}{4}$ and $\epsilon_3 = \frac{b_3}{3-\delta_3}$.

- If $\delta_3 < \frac{5}{3}$ and $\epsilon_1 < \epsilon_2$, then for any $\epsilon \in [0, \epsilon_1[$, the G/H-polytope Q^ϵ is equivalent to Q^0, and Q^{ϵ_1} is a point in the interior of the dominant chamber. Hence the Log MMP gives a contraction of fiber type to $G/B \simeq \mathbb{P}^1$.

- If $\delta_3 < \frac{5}{3}$ and $\epsilon_1 = \epsilon_2$, then for any $\epsilon \in [0, \epsilon_1[$, the G/H-polytope Q^ϵ is equivalent to Q^0, and Q^{ϵ_1} is a point in the line $\mathbb{Q}\varpi_0$. Hence the Log MMP gives a contraction of fiber type to a point.

- If $\delta_3 < \frac{5}{3}$ and $\epsilon_1 > \epsilon_2$ or if $\frac{5}{3} \leq \delta_3 < 3$, then for any $\epsilon \in [0, \epsilon_2[$ (with ϵ_2 defined above), the G/H-polytope Q^ϵ is equivalent to Q^0, Q^{ϵ_2} is a triangle that intersects the line $\mathbb{Q}\varpi_0$ in exactly one vertex, and for any $\epsilon \in]\epsilon_2, \epsilon_3[$, the G/H-polytope Q^ϵ is a quadrilateral that intersects the line $\mathbb{Q}\varpi_0$ along an edge, and Q^{ϵ_3} is a segment in the line $\mathbb{Q}\varpi_0$. Hence the Log MMP begins here with a flip and ends with a contraction of fiber type to the $\mathbb{C}^*$-embedding $\mathbb{P}^1$.

  In particular, the family (Q^ϵ) describes the Log MMP until its end, even if $\tilde{C}$ is not non-negative ie even if $-(K_X + \Delta)$ is not effective.

- If $\delta_3 \geq 3$, for any $\epsilon \in [0, \epsilon_2[$, the G/H-polytope Q^ϵ is equivalent to Q^0, Q^{ϵ_2} is a triangle that intersects the line $\mathbb{Q}\varpi_0$ in exactly one vertex, and for any $\epsilon \in]\epsilon_2, +\infty[$, the G/H-polytope Q^ϵ is a quadrilateral that intersects the line $\mathbb{Q}\varpi_0$ along an edge. Hence the Log MMP begins here with a flip and but does not end.

Remark 8. We can imagine a Log MMP avoiding flips. For example, consider the pair $(X, 0)$ of Example 2, and the case where $\epsilon_1 > \epsilon_2$. Instead of considering the flip (ie G/H-polytopes Q^ϵ with $\epsilon \in]\epsilon_2, \frac{b_3}{3}[$), we can apply the program from the beginning to the G/H-embedding corresponding to the G/H-polytope Q^{ϵ_2}, which is the G/H-embedding of Example 1.

Remark that $(X, 0)$ is a pair with klt singularities with an effective $\mathbb{Q}$-divisor, but the G/H-embedding of Example 1 admits no pair (X, Δ) with klt singularities such that Δ is effective. But for any horospherical projective variety X, there exists a horospherical pair (X, Δ) with klt singularities. Indeed, pick any ample $\mathbb{Q}$-Cartier divisor D' of X. Up to linear equivalence, we can suppose that D' is B-stable and strictly effective (ie effective and the support of D' is the union of all prime B-stable divisors of X). Then there exists a positive integer k such that $\lfloor -K_X - mD' \rfloor \leq 0$, and $\Delta := -K_X - mD'$ suits. Note that the constructed pair is quite special because here $-(K_X + \Delta) = mD'$ is ample.

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