Asymptotic distributions associated to piecewise quasi-polynomials

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1. Introduction

Let V be a finite dimensional real vector space equipped with a lattice Λ. Let $P \subset V$ be a rational polyhedron. The Euler-Maclaurin formula ([4], [2]) gives an asymptotic estimate, when k goes to ∞, for the Riemann sum $\sum_{\lambda \in kP\cap\Lambda} \varphi(\lambda/k)$ of the values of a test function φ at the sample points $\frac{1}{k}\Lambda \cap P$ of P, with leading term $k^{\dim P} \int_P \varphi$. Here we consider the slightly more general case of a weighted sum. Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. We consider, for $k \geq 1$, the distribution

$$\langle \Theta(P; q)(k), \varphi\rangle = \sum_{\lambda \in kP\cap\Lambda} q(\lambda, k)\varphi(\lambda/k)$$

and we show (Proposition 1.2) that the function $k \mapsto \langle \Theta(P; q)(k), \varphi\rangle$ admits an asymptotic expansion when k tends to ∞ in powers of $1/k$ with coefficients periodic functions of k.

We extend this result to an algebra $\mathcal{S}(\Lambda)$ of piecewise quasi-polynomial functions on $\Lambda \oplus \mathbb{Z} \subset V \oplus \mathbb{R}$. A function $m(\lambda, k)$ ($\lambda \in \Lambda, k \in \mathbb{Z}$) in $\mathcal{S}(\Lambda)$ is supported in an union of polyhedral cones in $V \oplus \mathbb{R}$. The main feature of a function $m(\lambda, k)$ in $\mathcal{S}(\Lambda)$ is that $m(\lambda, k)$ is entirely determined by its large behavior in k. We associate to $m(\lambda, k)$ a formal series $A(m)$ of distributions on V encoding the asymptotic behavior of $m(\lambda, k)$ when k tends to ∞.

The motivating example is the case where M is a projective manifold, and $\mathcal{L}$ the corresponding ample bundle. If T is a torus acting on M, then write, for $t \in T$,

$$\sum_{i=0}^{\dim M} (-1)^i \mathrm{Tr}(t, H^i(M, \mathcal{O}(\mathcal{L}^k))) = \sum_{\lambda} m(\lambda, k)t^{\lambda}$$

where λ runs over the lattice Λ of characters of T. The corresponding asymptotic expansion of the distribution $\sum_{\lambda} m(\lambda, k)\delta_{\lambda/k}$ is an important object associated to M involving the Duistermaat-Heckmann measure and the Todd class of M, see [9] for its determination. The determination of similar asymptotics in the more general case of twisted Dirac operators is the object of a forthcoming article [7].

Thus let $m \in \mathcal{S}(\Lambda)$, and consider the sequence

$$\Theta(m)(k) = \sum_{\lambda \in \Lambda} m(\lambda, k)\delta_{\lambda/k}$$

of distributions on V and its asymptotic expansion $A(m)$ when k tends to ∞. Let T be the torus with lattice of characters Λ. If $g \in T$ is an element of finite order, then $m^g(\lambda, k) := g^\lambda m(\lambda, k)$ is again in $\mathcal{S}(\Lambda)$. Our main result (Theorem 1.8) is that the piecewise quasi-polynomial function m is entirely determined by the collections of asymptotic expansions $A(m^g)$, when g varies over the set of elements of T of finite order.

We also prove (Proposition 2.1) a functorial property of $A(m)$ under pushforward.

We use these results to give new proofs of functoriality of the formal quantization of a symplectic manifold [5] or, more generally, of a spinc manifold [6].

For these applications, we also consider the case where V is a Cartan subalgebra of a compact Lie group, and anti-invariant distributions on V of a similar nature.

1.1. **Piecewise polynomial functions.**

Let V be a real vector space equipped with a lattice Λ. Usually, an element of V is denoted by ξ, and an element of Λ by λ. In this article, a cone C will always be a closed convex polyhedral cone, and $0 \in C$.

Let Λ^* be the dual lattice, and let $g \in T := V^*/\Lambda^*$. If $G \in V^*$ is a representative of g and $\lambda \in \Lambda$, then we denote $g^\lambda = e^{2i\pi\langle G,\lambda\rangle}$.

A periodic function m on Λ is a function such that there exists a positive integer D (we do not fix D) such that $m(\lambda_0 + D\lambda) = m(\lambda_0)$ for $\lambda, \lambda_0 \in \Lambda$. The space of such functions is linearly generated by the functions $\lambda \mapsto g^\lambda$ for $g \in T$ of finite order. By definition, the algebra of quasi-polynomial functions on Λ is generated by polynomials and periodic functions on Λ. If V_0 is a rational subspace of V, the restriction of m to $\Lambda_0 := \Lambda \cap V_0$ is a quasi-polynomial function on Λ_0. The space of quasi-polynomial functions is graded: a quasi-polynomial function homogeneous of degree d is a linear combination of functions $t^\lambda h(\lambda)$ where $t \in T$ is of finite order, and h an homogeneous polynomial on V of degree d. Let $q(\lambda)$ be a quasi-polynomial function on Λ. There is a sublattice Γ of Λ of finite index d_Γ such that for any given $\gamma \in \Lambda$, we have $q(\lambda) = p_\gamma(\lambda)$ for any $\lambda \in \gamma + \Gamma$ where $p_\gamma(\xi)$

is a (uniquely determined) polynomial function on V. Then define $q_{pol}(\xi) = \frac{1}{d_\Gamma}\sum_{\gamma\in\Lambda/\Gamma} p_\gamma(\xi)$, a polynomial function on V. This polynomial function is independent of the choice of the sublattice Γ. Then $q(\lambda) - q_{pol}(\lambda)$ is a linear combination of functions of the form $t^\lambda h(\lambda)$ with $h(\lambda)$ polynomial and $t \neq 1$.

Using the Lebesgue measure associated to Λ, we identify generalized functions on V and distributions on V. If θ is a generalized function on V, we may write $\int_V \theta(\xi)\varphi(\xi)d\xi$ for its value on the test function φ. If R is a rational affine subspace of V, R inherits a canonical translation invariant measure. If P is a rational polyhedron in V, it generates a rational affine subspace of V, and $\int_P \varphi$ is well defined for φ a smooth function with compact support.

We say that a distribution $\theta(k)$ depending of an integer k is periodic in k if there exists a positive integer D such that for any test function φ on V, and $k_0, k \in \mathbb{Z}$, $\langle\theta(k_0 + Dk), \varphi\rangle = \langle\theta(k_0), \varphi\rangle$. Then there exists (unique) distributions θ_ζ indexed by D-th roots of unity such that $\langle\theta(k), \varphi\rangle = \sum_{\zeta,\zeta^D=1} \zeta^k\langle\theta_\zeta, \varphi\rangle$.

Let $(\Theta(k))_{k\geq 1}$ be a sequence of distributions. We say that $\Theta(k)$ admits an asymptotic expansion (with periodic coefficients) if there exists $n_0 \in \mathbb{Z}$ and a sequence of distributions $\theta_n(k), n \geq 0$, depending periodically of k, such that for any test function φ and any non negative integer N, we have

$$\langle\Theta(k), \varphi\rangle = k^{n_0}\sum_{n=0}^{N}\frac{1}{k^n}\langle\theta_n(k), \varphi\rangle + o(k^{n_0-N}).$$

We write

$$\Theta(k) \equiv k^{n_0}\sum_{n=0}^{\infty}\frac{1}{k^n}\theta_n(k).$$

The distributions $\theta_n(k)$ are uniquely determined.

Given a sequence $\theta_n(k)$ of periodic distributions, and $n_0 \in \mathbb{Z}$, we write formally $M(\xi, k)$ for the series of distributions on V defined by

$$\langle M(\xi, k), \varphi\rangle = k^{n_0}\sum_{n=0}^{\infty}\frac{1}{k^n}\int_V \theta_n(k)(\xi)\varphi(\xi)d\xi.$$

We can multiply $M(\xi, k)$ by quasi-polynomial functions $q(k)$ of k and smooth functions $h(\xi)$ of ξ and obtain the formal series $q(k)h(\xi)M(\xi, k)$ of the same form with n_0 changed to $n_0 + \text{degree}(q)$.

Let $E = V \oplus \mathbb{R}$, and we consider the lattice $\tilde{\Lambda} = \Lambda \oplus \mathbb{Z}$ in E. An element of $\tilde{\Lambda}$ is written as (λ, k) with $\lambda \in \Lambda$ and $k \in \mathbb{Z}$. We consider quasi-polynomial functions $q(\lambda, k)$ on $\tilde{\Lambda}$. As before, this space is graded. We call the degree of a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$ the total degree. A quasi-polynomial function $q(\lambda, k)$ is of total degree d if it is a linear combination of functions $(\lambda, k) \mapsto j(k)t^\lambda k^a h(\lambda)$ where $j(k)$ is a periodic function of k, $t \in T$ of finite order, a a non negative integer, and h an homogeneous polynomial on V of degree b, with b such that $a + b = d$.

Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. We construct $q_{pol}(\xi, k)$ on $V \times \mathbb{Z}$, and depending polynomially on ξ as before. We choose a sublattice of finite index d_Γ in Λ and functions $p_\gamma(\xi, k)$ depending polynomially on $\xi \in V$ and quasi-polynomial in k such that $q(\lambda, k) = p_\gamma(\lambda, k)$ if $\lambda \in \gamma + \Gamma$. Then $q_{pol}(\xi, k) = \frac{1}{d_\Gamma} \sum_{\gamma \in \Lambda/\Gamma} p_\gamma(\xi, k)$. We say that $q_{pol}(\xi, k)$ is the polynomial part (relative to Λ) of q. If q is homogeneous of total degree d, then the function $(k, \xi) \mapsto q_{pol}(k\xi, k)$ is a linear combination of functions of the form $j(k)k^d s(\xi)$ where $j(k)$ is a periodic function of k and $s(\xi)$ a polynomial function of ξ.

Proposition 1.1. *Let P be a rational polyhedron in V with non empty interior. Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$ homogeneous of total degree d. Let $q_{pol}(\xi, k)$ be its polynomial part. Let $k \geq 1$. The distribution*

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP} q(\lambda, k)\varphi(\lambda/k)$$

admits an asymptotic expansion when $k \to \infty$ of the form

$$k^{\dim V} k^d \sum_{n=0}^{\infty} \frac{1}{k^n} \langle \theta_n(k), \varphi \rangle.$$

Furthermore, the term $k^d \langle \theta_0(k), \varphi \rangle$ is given by

$$k^d \langle \theta_0(k), \varphi \rangle = \int_P q_{pol}(k\xi, k)\varphi(\xi) d\xi$$

where q_{pol} is the polynomial part (with respect to Λ) of q.

Proof. Let $q(\lambda, k) = j(k)k^a g^\lambda h(\lambda)$ be a quasi-polynomial function of total degree d. Let

$$\langle \Theta_0^g(P)(k), \varphi \rangle = \sum_{\lambda \in kP \cap \Lambda} g^\lambda \varphi(\lambda/k). \tag{1.1}$$

If $\Theta_0^g(P)(k)$ admits the asymptotic expansion $M(\xi, k)$, then $\Theta(P; q)(k)$ admits the asymptotic expansion $j(k)k^a h(k\xi)M(\xi, k)$. So it is sufficient to consider the case where $q(\lambda, k) = g^\lambda$ and the distribution $\Theta_0^g(P)(k)$.

We now proceed as in [2] for the case $g = 1$ and sketch the proof. By decomposing the characteristic function $[P]$ of the polyhedron P

in a signed sum of characteristic functions of tangent cones, via the Brianchon Gram formula, then decomposing furthermore each tangent cone in a signed sum of cones C_a of the form $\Sigma_a \times R_a$ with Σ_a is a translate of a unimodular cone and R_a a rational space, we are reduced to study this distribution for the product of the dimension 1 following situations.

$V = \mathbb{R}, \Lambda = \mathbb{Z}$ and one of the following two cases:

- $P = \mathbb{R}$
- $P = s + \mathbb{R}_{\geq 0}$ with s a rational number.

For example, if $P = [a, b]$ is an interval in $\mathbb{R}$ with rational end points a, b, we write $[P] = [a, \infty] + [-\infty, b] - [\mathbb{R}]$.

For $P = \mathbb{R}$, and ζ a root of unity, it is easy to see that

$$\langle \Theta^{\zeta}(k), \varphi \rangle = \sum_{\mu \in \mathbb{Z}} \zeta^{\mu} \varphi(\mu/k)$$

is equivalent to $k \int_{\mathbb{R}} \varphi(\xi) d\xi$ if $\zeta = 1$ or is equivalent to 0 if $\zeta \neq 1$.

We now study the case where $P = s + \mathbb{R}_{\geq 0}$. Let

$$\langle \Theta^{\zeta}(k), \varphi \rangle = \sum_{\mu \in \mathbb{Z}, \mu - ks \geq 0} \zeta^{\mu} \varphi(\mu/k)$$

and let us compute its asymptotic expansion.

For $r \in \mathbb{R}$, the fractional part $\{r\}$ is defined by $\{r\} \in [0, 1[, r - \{r\} \in \mathbb{Z}$. If μ is an integer greater or equal to ks, then $\mu = ks + \{-ks\} + u$ with u a non negative integer.

We consider first the case where $\zeta = 1$. This case has been treated for example in [3] (Theorem 9.2.2), and there is an Euler-Maclaurin formula with remainder which leads to the following asymptotic expansion.

The function $z \mapsto \frac{e^{xz}}{e^z - 1}$ has a simple pole at $z = 0$. Its Laurent series at $z = 0$ is

$$\frac{e^{xz}}{e^z - 1} = \sum_{n=-1}^{\infty} B_{n+1}(x) \frac{z^n}{(n+1)!}$$

where $B_n(x)$ $(n \geq 0)$ are the Bernoulli polynomials.

If s is rational, and $n \geq 0$, the function $k \mapsto B_n(\{-ks\})$ is a periodic function of k with period the denominator of s, and

$$\sum_{\mu \in \mathbb{Z}, \mu \geq ks} \varphi(\frac{\mu}{k}) \equiv k(\int_s^{\infty} \varphi(\xi) d\xi - \sum_{n=1}^{\infty} \frac{1}{k^n} \frac{B_n(\{-ks\})}{n!} \varphi^{(n-1)}(s)).$$

This formula is easily proven by Fourier transform. Indeed, for $f(\xi) = e^{i\xi z}$, the series $\sum_{\mu \geq ks} f(\mu/k)$ is $\sum_{u \geq 0} e^{isz} e^{i\{-ks\}z/k} e^{iuz/k}$. It is convergent if z is in the upper half plane, and the sum is

$$F(z)(k) = -e^{isz}\frac{e^{i\{-ks\}z/k}}{e^{iz/k}-1}.$$

So the Fourier transform of the tempered distribution $\Theta^{\zeta=1}(k)$ is the boundary value of the holomorphic function $z \mapsto F(z)(k)$ above. We can compute the asymptotic behavior of $F(z)(k)$ easily when k tends to ∞, since $\{-ks\} \leq 1$, and z/k becomes small.

Rewriting $[P]$ as the signed sum of the characteristic functions of the cones C_a, we see that the distribution $\Theta_0^g(P)(k)$ for $g = 1$ is equivalent to

$$k^{\dim V}(\sum_{n=0}^{\infty}\frac{1}{k^n}\theta_n(k))$$

with θ_0 independent of k, and given by $\langle\theta_0, \varphi\rangle = \int_P \varphi(\xi)d\xi$.

Now consider the case where $\zeta \neq 1$. Then

$$\sum_{\mu\in\mathbb{Z},\mu\geq ks}\zeta^\mu\varphi(\mu/k) = \sum_{u\geq 0}\zeta^{ks+\{-ks\}}\zeta^u\varphi(s+\{-ks\}/k+u/k).$$

The function $k \mapsto \zeta^{ks+\{-ks\}}$ is a periodic function of k with period ed if $\zeta^e = 1$ and ds is an integer. If $\zeta \neq 1$, the function $z \mapsto \frac{e^{xz}}{\zeta e^z-1}$ is holomorphic at $z = 0$. Define the polynomials $B_{n,\zeta}(x)$ via the Taylor series expansion:

$$\frac{e^{xz}}{\zeta e^z - 1} = \sum_{n=0}^{\infty} B_{n+1,\zeta}(x)\frac{z^n}{(n+1)!}.$$

It is easily seen by Fourier transform that $\sum_{\mu\in\mathbb{Z},\mu\geq ks}\zeta^\mu\varphi(\mu/k)$ is equivalent to

$$-k\zeta^{ks+\{-ks\}}\sum_{n=1}^{\infty}\frac{1}{k^n}\frac{B_{n,\zeta}(\{-ks\})}{n!}\varphi^{(n-1)}(s).$$

In particular, $\Theta^\zeta(k)$ admits an asymptotic expansion in non negative powers of $1/k$ and each coefficient of this asymptotic expansion is a periodic distribution supported at s.

Rewriting $[P]$ in terms of the signed cones C_a, we see that indeed if $g \in T$ is not 1, one of the corresponding ζ in the reduction to a product of one dimensional cones is not 1, and so

$$\Theta_0^g(P)(k) \equiv k^{\dim V-1}(\sum_{n=0}^{\infty}\frac{1}{k^n}\theta_n(k)).$$

So we obtain our proposition.

□

Consider now P a rational polyhedron, with possibly empty interior. Let C_P be the cone of base P in $E = V \oplus \mathbb{R}$,

$$C_P := \{(t\xi, t), t \geq 0, \xi \in P\}.$$

Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. We consider again

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP \cap \Lambda} q(\lambda, k)\varphi(\lambda/k).$$

Consider the vector space E_P generated by the cone C_P in E. It is clear that $\Theta(P; q)$ depends only of the restriction r of q to $E_P \cap (\Lambda \oplus \mathbb{Z})$. This is a quasi-polynomial function on E_P with respect to the lattice $E_P \cap (\Lambda \oplus \mathbb{Z})$. We assume that the quasi-polynomial function r is homogeneous of degree d_0. This degree might be smaller that the total degree of q. Consider the affine space R_P generated by P in V. Let $E_P^{\mathbb{Z}} = E_P \cap (V \oplus \mathbb{Z})$. If $\xi \in R_P, k \in \mathbb{Z}$, then $(k\xi, k) \in E_P^{\mathbb{Z}}$. We will see shortly (Definition 1.3) that we can define a function $(\xi, k) \mapsto r_{pol}(\xi, k)$ for $(\xi, k) \in E_P^{\mathbb{Z}}$, and that the function $(\xi, k) \mapsto r_{pol}(k\xi, k)$ on $R_P \times \mathbb{Z}$ is a linear combination of functions of the form $k^{d_0} j(k)s(\xi)$ where $j(k)$ is a periodic function of k and $s(\xi)$ a polynomial function of ξ, for ξ varying on the affine space R_P.

We now can state the general formula.

Proposition 1.2. *Let P be a rational polyhedron in V. Let $q(\lambda, k)$ be a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. Let r be its restriction to $E_P \cap (\Lambda \oplus \mathbb{Z})$ and r_{pol} the "polynomial part" of r on $E_P \cap (V \oplus \mathbb{Z})$. Assume that the quasi-polynomial function r is homogeneous of degree d_0. Let $k \geq 1$. The distribution*

$$\langle \Theta(P; q)(k), \varphi \rangle = \sum_{\lambda \in kP} q(\lambda, k)\varphi(\lambda/k)$$

admits an asymptotic expansion when $k \to \infty$ of the form

$$k^{\dim P} k^{d_0} \sum_{n=0}^{\infty} \frac{1}{k^n} \langle \theta_n(k), \varphi \rangle.$$

Furthermore, the term $k^{d_0}\langle \theta_0(k), \varphi \rangle$ is given by

$$k^{d_0}\langle \theta_0(k), \varphi \rangle = \int_P r_{pol}(k\xi, k)\varphi(\xi)d\xi.$$

Proof. We will reduce the proof of this proposition to the case treated before of a polyhedron with interior. Let $\text{lin}(P)$ be the linear space parallel to R_P, and $\Lambda_0 := \Lambda \cap \text{lin}(P)$. If R_P contains a point $\beta \in \Lambda$, then E_P is isomorphic to $\text{lin}(P) \oplus \mathbb{R}$ with lattice $\Lambda_0 \oplus \mathbb{Z}$. Otherwise, we will have to dilate R_P. More precisely, let $I_P = \{k \in \mathbb{Z}, kR_P \cap \Lambda \neq \emptyset\}$.

This is an ideal in $\mathbb{Z}$. Indeed if $k_1 \in I_P, k_2 \in I_P$, $\alpha_1, \alpha_2 \in R_P$ are such that $k_1\alpha_1 \in \Lambda$, $k_2\alpha_2 \in \Lambda$, then $\alpha_{1,2} = \frac{1}{n_1k_1+n_2k_2}(n_1k_1\alpha_1 + n_2k_2\alpha_2)$ is in R_P, and $(n_1k_1 + n_2k_2)(\alpha_{1,2}) \in \Lambda$. Thus there exists a smallest $k_0 > 0$ generating the ideal I_P. We see that our distribution $\Theta(P; p)(k)$ is identically equal to 0 if k is not in I_P. Let $\delta_{I_P}(k)$ be the function of k with

$$\delta_{I_P}(k) = \begin{cases} 0 & \text{if } k \notin I_P, \\ 1 & \text{if } k = uk_0 \in I_P. \end{cases}$$

This is a periodic function of k of period k_0. We choose $\alpha \in R_P$ such that $k_0\alpha \in \Lambda$. We identify E_P to $\mathrm{lin}(P) \oplus \mathbb{R}$ by the map $T_\alpha(\xi_0, t) = (\xi_0 + tk_0\alpha, tk_0)$. In this identification, the lattice $(\Lambda \oplus \mathbb{Z}) \cap E_P$ becomes the lattice $\Lambda_0 \oplus \mathbb{Z}$. Consider $P_0 = k_0(P - \alpha)$, a polyhedron with interior in $\mathrm{lin}(P)$. Let $q^\alpha(\gamma, u) = r(\gamma + uk_0\alpha, uk_0)$. This is a quasi-polynomial function on $\Lambda_0 \oplus \mathbb{Z}$. Its total degree is d_0. We have defined its polynomial part $q^\alpha_{pol}(\xi, u)$ for $\xi \in \mathrm{lin}(P)$, $u \in \mathbb{Z}$.

Definition 1.3. *Let $(\xi, k) \in E_P^{\mathbb{Z}}$. Define:*

$$r_{pol}(\xi, k) = \begin{cases} 0 & \text{if } k \notin I_P, \\ q^\alpha_{pol}(\xi - uk_0\alpha, u) & \text{if } k = uk_0 \in I_P. \end{cases}$$

The function $r_{pol}(\xi, k)$ does not depend of the choice of α. Indeed, if $\alpha, \beta \in R_P$ are such that $k_0\alpha, k_0\beta \in \Lambda$, then $q^\beta(\gamma, u) = q^\alpha(\gamma + uk_0(\beta - \alpha), u)$. Then we see that $q^\beta_{pol}(\xi, u) = q^\alpha_{pol}(\xi + uk_0(\beta - \alpha), u)$. Furthermore, the function $(k, \xi) \mapsto r_{pol}(k\xi, k)$ is of the desired form, a linear combination of functions $\delta_{I_P}(k)j(k)k^{d_0}s(\xi)$ with $s(\xi)$ polynomial functions on R_P.

If φ is a test function on V, we define the test function φ_0 on $\mathrm{lin}(P)$ by $\varphi_0(\xi_0) = \varphi(\frac{\xi_0}{k_0} + \alpha)$. We see that

$$\langle \Theta(P; q)(uk_0), \varphi \rangle = \langle \Theta(P_0; q^\alpha)(u), \varphi_0 \rangle. \tag{1.2}$$

Thus we can apply Proposition 1.1. We obtain

$$\langle \Theta(P; q)(uk_0), \varphi \rangle \equiv u^{\dim P} u^{d_0} \sum_{n=0}^{\infty} \frac{1}{u^n} \langle \omega_n(u), \varphi_0 \rangle.$$

We have

$$u^{d_0} \langle \omega_0(u), \varphi_0 \rangle = \int_{P_0} q^\alpha_{pol}(u\xi_0, u)\varphi_0(\xi_0)d\xi_0 = \int_{P_0} q^\alpha_{pol}(u\xi_0, u)\varphi(\frac{\xi_0}{k_0} + \alpha)$$

When ξ_0 runs in $P_0 = k_0(P - \alpha)$, $\xi = \frac{\xi_0}{k_0} + \alpha$ runs over P. Changing variables, we obtain

$$u^{d_0}\langle\omega_0(u), \varphi_0\rangle = k^{d_0}\int_P r_{pol}(k\xi, k)\varphi(\xi)d\xi.$$

Thus we obtain our proposition. □

Let P be a rational polyhedron in V and q a quasi-polynomial function on $\Lambda \oplus \mathbb{Z}$. We do not assume that P has interior in V. We denote by $[C_P]$ the characteristic function of C_P. Then the function $q(\lambda, k)[C_P](\lambda, k)$ is zero if (λ, k) is not in C_P or equal to $q(\lambda, k)$ if (λ, k) is in C_P. We denote it by $q[C_P]$. The space of functions on $\Lambda \oplus \mathbb{Z}$ we will study is the following space.

Definition 1.4. *We define the space $\mathcal{S}(\Lambda)$ to be the space of functions on $\Lambda \oplus \mathbb{Z}$ linearly generated by the functions $q[C_P]$ where P runs over rational polyhedrons in V and q over quasi-polynomial functions on $\Lambda \oplus \mathbb{Z}$.*

The representation of m as a sum of functions $q[C_P]$ is not unique. For example, consider $V = \mathbb{R}$, $P = \mathbb{R}, P_+ := \mathbb{R}_{\geq 0}, P_- := \mathbb{R}_{\leq 0}, P_0 := \{0\}$, then $[C_P] = [C_{P_+}] + [C_{P_-}] - [C_{P_0}]$.

Example 1.5. *An important example of functions $m \in \mathcal{S}(\Lambda)$ is the following. Assume that we have a closed cone C in $V \oplus \mathbb{R}$, and a covering $C = \cup_\alpha C_\alpha$ by closed cones. Let m be a function on $C \cap (\Lambda \oplus \mathbb{Z})$, and assume that the restriction of m to $C_\alpha \cap (\Lambda \oplus \mathbb{Z})$ is given by a quasipolynomial function q_α. Then, using exclusion-inclusion formulae, we see that $m \in \mathcal{S}(\Lambda)$.*

Definition 1.6. *If $m(\lambda, k)$ belongs to $\mathcal{S}(\Lambda)$, and $g \in T$ is an element of finite order, then define*

$$m^g(\lambda, k) = g^\lambda m(\lambda, k).$$

The function m^g belongs to $\mathcal{S}(\Lambda)$.

If $m \in \mathcal{S}(\Lambda)$, and $k \geq 1$, we denote by $\Theta(m)(k)$ the distribution on V defined by

$$\langle\Theta(m)(k), \varphi\rangle = \sum_{\lambda\in\Lambda} m(\lambda, k)\varphi(\lambda/k),$$

if φ is a test function on V. The following proposition follows immediately from Proposition 1.2.

Proposition 1.7. *If $m(\lambda, k) \in \mathcal{S}(\Lambda)$, the distribution $\Theta(m)(k)$ admits an asymptotic expansion $A(m)(\xi, k)$.*

The function $m(\lambda, k)$ can be non zero, while $A(m)(\xi, k)$ is zero. For example let $V = \mathbb{R}$, $P = \mathbb{R}$ and $m(\lambda, k) = (-1)^\lambda$. Then $\Theta(m)(k)$ is the distribution on $\mathbb{R}$ given by $\mathrm{T}(k) = \sum_{\lambda=-\infty}^{\infty}(-1)^\lambda \delta_{\lambda/k}, k \geq 1$ and this is equivalent to 0. However, here is an unicity theorem.

Theorem 1.8. *Assume that $m \in \mathcal{S}(\Lambda)$ is such that $A(m^g) = 0$ for all $g \in T$ of finite order, then $m = 0$.*

Proof. We start by the case of a function $m = q[C_P]$ associated to a single polyhedron P and a quasi-polynomial function q. Assume first that P is with non empty interior P^0. If q is not identically 0, we write $q(\lambda, k) = \sum_{g\in T} g^{-\lambda} p_g(\lambda, k)$ where $p_g(\lambda, k)$ are polynomials in λ. If d is the total degree of q, then all the polynomials $p_g(\lambda, k)$ are of degree less or equal than d. We choose $t \in T$ such that $p_t(\lambda, k)$ is of degree d. If we consider the quasi-polynomial $q^t(\lambda, k)$, then its polynomial part is $p_t(\lambda, k)$ and the homogeneous component $p_t^{top}(\lambda, k)$ of degree d is not zero. We write $p_t^{top}(\xi, k) = \sum_{\zeta,a} \zeta^k k^a p_{\zeta,a}(\xi)$ where $p_{\zeta,a}(\xi)$ is a polynomial in ξ homogeneous of degree $d - a$. Testing against a test function φ and computing the term in $k^{d+\dim V}$ of the asymptotic expansion by Proposition 1.1, we see that $\sum_{\zeta,a} \zeta^k k^d \int_P p_{\zeta,a}(\xi)\varphi(\xi)d\xi = 0$. This is true for any test function φ. So, for any ζ, we obtain $\sum_a p_{\zeta,a}(\xi) = 0$. Each of the $p_{\zeta,a}$ being homogeneous of degree $d - a$, we see that $p_{\zeta,a} = 0$ for any a, ζ. Thus $p_t^{top} = 0$, a contradiction. So we obtain that $q = 0$, and $m = q[C_P] = 0$. Remark that to obtain this conclusion, we may use only test functions φ with support contained in the interior P^0 of P.

Consider now a general polyhedron P and the vector space $\mathrm{lin}(P)$. Let us prove that $m(\lambda, k) = q(\lambda, k)[C_P](\lambda, k)$ is identically 0 if $A(m^g) = 0$ for any $g \in T$ of finite order. Using the notations of the proof of Proposition 1.2, we see that $m(\lambda, k) = 0$, if k is not of the form uk_0. Furthermore, if $q^\alpha(\gamma, u) = q(\gamma + uk_0\alpha, uk_0)$, it is sufficient to prove that $q^\alpha = 0$. Let $P_0 = k_0(P - \alpha)$, a polyhedron with interior in V_0. Consider $m_0 = q^\alpha[C_{P_0}]$. We consider T as the character group of Λ, so T surjects on T_0. Let $g \in T$ of finite order and such that $g^{k_0\alpha} = 1$, and let g_0 be the restriction of g to Λ_0. Using Equation 1.1, we then see that

$$\langle \Theta(m^g)(uk_0), \varphi\rangle = \langle \Theta(m_0^{g_0})(u), \varphi_0\rangle.$$

Any $g_0 \in T_0$ of finite order is the restriction to Λ_0 of an element $g \in T$ of finite order and such that $g^{k_0\alpha} = 1$. So we conclude that the asymptotic expansion, when u tends to ∞, of $\Theta(m_0^{g_0})(u)$ is equal to 0 for any $g_0 \in T_0$ of finite order. Remark again that we need only to know that $\langle \Theta(m^g)(k), \varphi\rangle \equiv 0$ for test functions φ such that the support

S of φ is contained in a very small neighborhood of compact subsets of P contained in the relative interior of P^0.

For any integer ℓ, denote by $\mathcal{S}_\ell(\Lambda)$ the subspace of functions $m \in \mathcal{S}(\Lambda)$ generated by the functions $q[C_P]$ with $\dim P \leq \ell$.

When $\ell = 0$, our polyhedrons are a finite number of rational points $f \in V$, the function $m(\lambda, k)$ is supported on the union of lines $(ud_f f, ud_f)$ if d_f is the smallest integer such that $d_f f$ is in Λ. Choose a test function φ with support near f. Then $u \mapsto \langle \Theta(m)(d_f u), \varphi \rangle$ is identical to its asymptotic expansion $m(ud_f f, ud_f)\varphi(f)$. Clearly we obtain that $m = 0$.

If $m \in \mathcal{S}_\ell(\Lambda)$ by inclusion-exclusion, we can write

$$m = \sum_{P;\dim(P)=\ell} q_P[C_P] + \sum_{H,\dim H<\ell} q_H[C_H]$$

and we can assume that the intersections of a polyhedron P occurring in the first sum, with any polyhedron P' occurring in the decomposition of m and different from P is of dimension strictly less than ℓ. Consider P in the first sum, so $\dim(P) = \ell$. We can thus choose test functions φ with support in small neighborhoods of K, with K a compact subset contained in the relative interior of P. Then

$$\langle \Theta(m^g)(k), \varphi \rangle = \langle \Theta(q_P^g[C_P])(k), \varphi \rangle.$$

The preceding argument shows that $q_P[C_P] = 0$. So $m \in \mathcal{S}_{\ell-1}(\Lambda)$. By induction $m = 0$. □

2. Composition of piecewise quasi-polynomial functions

Let V_0, V_1 be vector spaces with lattice Λ_0, Λ_1.

Let $C_{0,1}$ be a closed polyhedral rational cone in $V_0 \oplus V_1$ (containing the origin). Thus for any $\mu \in \Lambda_1$, the set of $\lambda \in V_0$ such that $(\lambda, \mu) \in C_{0,1}$ is a rational polyhedron $P(\mu)$ in V_0. Let P be a polyhedron in V. We assume that for any $\mu \in \Lambda_1$, $P \cap P(\mu)$ is compact. Thus, for $m = q_P[C_P] \in \mathcal{S}(\Lambda)$, and $c(\lambda, \mu)$ a quasi-polynomial function on $\Lambda_0 \oplus \Lambda_1$, we can compute

$$m_c(\mu, k) = \sum_{(\lambda,\mu)\in C_{0,1}} m(\lambda, k) c(\lambda, \mu).$$

Proposition 2.1. *The function m_c belongs to $\mathcal{S}(\Lambda_1)$.*

Before establishing this result, let us give an example, which occur for example in the problem of computing the multiplicity of a representation $\chi^\lambda \otimes \chi^\lambda$ of $SU(2)$ restricted to the maximal torus.

Example 2.2. *Let $V_0 = V_1 = \mathbb{R}$, and $\Lambda_0 = \Lambda_1 = \mathbb{Z}$. Let $P := [0, 2]$, and let*

$$q(\lambda, k) = \begin{cases} \frac{1}{2}(1 - (-1)^\lambda) & \text{if } 0 \leq \lambda \leq 2k \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$C_{0,1} = \{(x, y) \in \mathbb{R}^2; x \geq 0, -x \leq y \leq x\}$$

and

$$c(\lambda, \mu) = \frac{1}{2}(1 - (-1)^{\lambda-\mu}).$$

Let $\mu \geq 0$. Then

$$m_c(\mu, k) = \frac{1}{4} \sum_{0 \leq \lambda \leq 2k, \lambda \geq \mu} (1-(-1)^\lambda)(1-(-1)^{\lambda-\mu}) = (1 + (-1)^\mu)(k/2 - \mu/4).$$

So if $P_1 = [0, 2], P_2 := [-2, 0], P_3 := \{0\}$, we obtain

$$m_c = q_1[C_{P_1}] + q_2[C_{P_2}] + q_3[C_{P_3}]$$

with

$$\begin{cases} q_1(\mu, k) = (1 + (-1)^\mu)(k/2 - \mu/4), \\ q_2(\mu, k) = (1 + (-1)^\mu)(k/2 + \mu/4), \\ q_3(\mu, k) = -k. \end{cases}$$

We now start the proof of Proposition 2.1.

Proof. Write $c(\lambda, \mu)$ as a sum of products of quasi-polynomial functions $q_j(\lambda)$, $f_j(\mu)$, and $q_P(\lambda, k)$ a sum of products of quasi-polynomial functions $m_\ell(k)$, $h_\ell(\lambda)$. Then we see that it is thus sufficient to prove that, for $q(\lambda)$ a quasi-polynomial function of λ, the function

$$S(q)(\mu, k) = \sum_{\lambda \in kP \cap P(\mu)} q(\lambda) \tag{2.1}$$

belongs to $\mathcal{S}(\Lambda_1)$. For this, let us recall some results on families of polytopes $\mathfrak{p}(\mathbf{b}) \subset E$ defined by linear inequations. See for example [1], or [8].

Let E be a vector space, and $\omega_i, i = 1, \ldots, N$ be a sequence of linear forms on E. Let $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ be an element of $\mathbb{R}^N$. Consider the polyhedron $\mathfrak{p}(\mathbf{b})$ defined by the inequations

$$\mathfrak{p}(\mathbf{b}) = \{v \in E; \langle\omega_i, v\rangle \leq b_i, i = 1, \ldots, N\}.$$

We assume E equipped with a lattice L, and inequations ω_i defined by elements of L^*. Then if the parameters b_i are in $\mathbb{Z}^N$, the polytopes $\mathfrak{p}(\mathbf{b})$ are rational convex polytopes.

Assume that there exists $\mathbf{b}$ such that $\mathfrak{p}(\mathbf{b})$ is compact (non empty). Then $\mathfrak{p}(\mathbf{b})$ is compact (or empty) for any $\mathbf{b} \in \mathbb{R}^N$. Furthermore, there exists a closed cone $\mathcal{C}$ in $\mathbb{R}^N$ such that $\mathfrak{p}(\mathbf{b})$ is non empty if and only if $\mathbf{b} \in \mathcal{C}$. There is a decomposition $\mathcal{C} = \cup_\alpha \mathcal{C}_\alpha$ of $\mathcal{C}$ in closed polyhedral cones with non empty interiors, where the polytopes $\mathfrak{p}(\mathbf{b})$, for $\mathbf{b} \in \mathcal{C}_\alpha$, does not change of shape. More precisely:

• When $\mathbf{b}$ varies in the interior of $\mathcal{C}_\alpha$, the polytope $\mathfrak{p}(\mathbf{b})$ remains with the same number of vertices $\{s_1(\mathbf{b}), s_2(\mathbf{b}), \ldots, s_L(\mathbf{b})\}$.

• for each $1 \leq i \leq L$, there exists a cone C_i in E, such that the tangent cone to the polytope $\mathfrak{p}(\mathbf{b})$ at the vertex $s_i(\mathbf{b})$ is the affine cone $s_i(\mathbf{b}) + C_i$.

• the map $\mathbf{b} \to s_i(\mathbf{b})$ depends of the parameter $\mathbf{b}$, via linear maps $\mathbb{R}^N \to E$ with rational coefficients.

Furthermore -as proven for example in [1]- the Brianchon-Gram decomposition of $\mathfrak{p}(\mathbf{b})$ is "continuous" in $\mathbf{b}$ when b varies on C_α, in a sense discussed in [1].

Before continuing, let us give a very simple example, let b_1, b_2, b_3 be 3 real parameters and consider $\mathfrak{p}(b_1, b_2, b_3) = \{x \in \mathbb{R}, x \leq b_1, -x \leq b_2, -x \leq b_3\}$. So we are studying the intersection of the interval $[-b_2, b_1]$ with the half line $[-b_3, \infty]$. Then for $\mathfrak{p}(\mathbf{b})$ to be non empty, we need that $\mathbf{b} \in \mathcal{C}$, with

$$\mathcal{C} = \{\mathbf{b}; b_1 + b_2 \geq 0, b_1 + b_3 \geq 0\}.$$

Consider $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, with

$$\mathcal{C}_1 = \{\mathbf{b} \in \mathcal{C}; b_2 - b_3 \geq 0\},$$

$$\mathcal{C}_2 = \{\mathbf{b} \in \mathcal{C}; b_3 - b_2 \geq 0\}.$$

On $\mathcal{C}_1$ the vertices of $\mathfrak{p}(\mathbf{b})$ are $[-b_3, b_1]$, while on $\mathcal{C}_2$ the vertices of $\mathfrak{p}(\mathbf{b})$ are $[-b_2, b_1]$.

The Brianchon-Gram decomposition of $\mathfrak{p}(\mathbf{b})$ for $\mathbf{b}$ in the interior of $\mathcal{C}_1$ is $[-b_3, \infty] + [-\infty, b_1] - \mathbb{R}$. If $\mathbf{b} \in \mathcal{C}_1$ tends to the point $(b_1, b_2, -b_1)$ in the boundary of $\mathcal{C}$, we see the Brianchon-Gram decomposition tends to that $[b_1, \infty] + [-\infty, b_1] - \mathbb{R}$, which is indeed the polytope $\{b_1\}$.

Let $q(\gamma)$ be a quasi-polynomial function of $\gamma \in L$. Then, when $\mathbf{b}$ varies in $\mathcal{C}_\alpha \cap \mathbb{Z}^N$, the function

$$S(q)(\mathbf{b}) = \sum_{\gamma \in \mathfrak{p}(\mathbf{b}) \cap L} q(\gamma)$$

is given by a quasi-polynomial function of b. This is proven in [8], Theorem 3.8. In this theorem, we sum an exponential polynomial function $q(\gamma)$ on the lattice points of $\mathfrak{p}(\mathbf{b})$ and obtain an exponential polynomial function of the parameter b. However, the explicit formula shows that if we sum up a quasi-polynomial function of γ, then we obtain a quasi-polynomial function of $\mathbf{b} \in \mathbb{Z}^N$. Another proof follows from [1] (Theorem 54) and the continuity of Brianchon-Gram decomposition. In [1], only the summation of polynomial functions is studied, via a Brianchon-Gram decomposition, but the same proof gives the result for quasi-polynomial functions (it depends only of the fact that the vertices vary via rational linear functions of $\mathbf{b}$). The relations between partition polytopes $P_\Phi(\xi)$ (setting used in [8], [1]) and families of polytopes $\mathfrak{p}(\mathbf{b})$ is standard, and is explained for example in the introduction of [1].

Consider now our situation with $E = V$ equipped with the lattice Λ. The polytope $kP \subset V$ is given by a sequence of inequalities $\omega_i(\xi) \leq ka_i$, $i = 1, \ldots, I$, where we can assume $\omega_i \in \Lambda^*$ and $a_i \in \mathbb{Z}$ by eventually multiplying by a large integer the inequality. The polytope $P(\mu)$ is given by a sequence of inequalities $\omega_j(\xi) \leq \nu_j(\mu)$, $j = 1, \ldots, J$ where ν_j depends linearly on μ. Similarly we can assume $\nu_j(\mu) \in \mathbb{Z}$. Let

$$(\mu, k) \mapsto \mathbf{b}(\mu, k) = [ka_1, \ldots, ka_I, \nu_1(\mu), \ldots, \nu_J(\mu)]$$

a linear map from $\Lambda_1 \oplus \mathbb{Z}$ to $\mathbb{Z}^N$. Our polytope $kP \cap P(\mu)$ is the polytope $\mathfrak{p}(\mathbf{b}(k, \mu))$ and

$$S(q)(\mu, k) = \sum_{\lambda \in \mathfrak{p}(\mathbf{b}(k,\mu)) \cap \Lambda} q(\lambda) = S(q)(\mathbf{b}(\mu, k)).$$

Consider one of the cones $\mathcal{C}_\alpha$. Then $\mathbf{b}(\mu, k) \in \mathcal{C}_\alpha$, if and only if (μ, k) belongs to a rational polyhedral cone C_α in $V_1 \oplus \mathbb{R}$. If Q is a quasi-polynomial function of $\mathbf{b}$, then $Q(\mathbf{b}(\mu, k))$ is a quasi-polynomial function of (μ, k). Thus on each of the cones C_α, $S(q)(\mu, k)$ is given by a quasi-polynomial function of (μ, k). From Example 1.5, we conclude that $S(q)$ belongs to $\mathcal{S}(\Lambda_1)$. ☐

3. Piecewise quasi-polynomial functions on the Weyl chamber

For applications, we have also to consider the following situation.

Let G be a compact Lie group. Let T be a maximal torus of G, $\mathfrak{t}$ its Lie algebra, W be the Weyl group. Let $\Lambda \subset \mathfrak{t}^*$ be the weight lattice of T. We choose a system $\Delta^+ \subset \mathfrak{t}^*$ of positive roots, and let $\rho \in \mathfrak{t}^*$ be the corresponding element. We consider the positive Weyl chamber $\mathfrak{t}^*_{\geq 0}$ with interior $\mathfrak{t}^*_{>0}$.

We consider now $\mathcal{S}_{\geq 0}(\Lambda)$ the space of functions generated by the functions $q[C_P]$ with polyhedrons P contained in $\mathfrak{t}^*_{\geq 0}$. This is a subspace of $\mathcal{S}(\Lambda)$. If $t \in T$ is an element of finite order, the function $m^t(\lambda, k) = t^\lambda m(\lambda, k)$ is again in $\mathcal{S}_{\geq 0}(\Lambda)$.

If $m \in \mathcal{S}_{\geq 0}(\Lambda)$, we define the following anti invariant distribution with value on a test function φ given by

$$\langle \Theta_a(m)(k), \varphi \rangle = \frac{1}{|W|} \sum_{\lambda} m(\lambda, k) \sum_{w \in W} \epsilon(w) \varphi(w(\lambda + \rho)/k)$$

Proposition 3.1. *If for every $t \in T$ of finite order, we have $\Theta_a(m^t) \equiv 0$, then $m = 0$.*

Proof. Consider φ a test function supported in the interior of the Weyl chamber. Thus, for $\lambda \geq 0$, $\varphi(w(\lambda + \rho)/k)$ is not zero only if $w = 1$. So

$$\langle \Theta_a(m)(k), \varphi \rangle = \frac{1}{|W|} \sum_{\lambda \geq 0} m(\lambda, k) \varphi((\lambda + \rho)/k)$$

while

$$\langle \Theta(k), \varphi \rangle = \sum_{\lambda \geq 0} m(\lambda, k) \varphi(\lambda/k).$$

Let $(\partial_\rho \varphi)(\xi) = \frac{d}{d\epsilon} `\varphi(\xi + \epsilon\rho)|_{\epsilon=0}$ and consider the series of differential operators with constant coefficients $e^{\partial_\rho/k} = 1 + \frac{1}{k}\partial_\rho + \cdots$. We then see that, if $\langle A(\xi, k), \varphi \rangle$ is the asymptotic expansion of $\langle \Theta(k), \varphi \rangle$, the asymptotic expansion of $\langle \Theta_a(k), \varphi \rangle$ is $\langle A(\xi, k), e^{\partial_\rho/k}\varphi \rangle$. Proceeding as in the proof of Theorem 1.8, we see that if $\langle \Theta_a(m^t)(k), \varphi \rangle \equiv 0$ for all $t \in T$ of finite order, then $m(\lambda, k)$ is identically 0 when λ is on the interior of the Weyl chamber.

Consider all faces (closed) σ of the closed Weyl chamber. Define $\mathcal{S}_{\ell, \geq 0} \subset \mathcal{S}(\Lambda)$ to be the space of $m = \sum_{\sigma, \dim(\sigma) \leq \ell} m_\sigma$, where $m_\sigma \in \mathcal{S}_{\geq 0}(\Lambda)$ is such that $m_\sigma(\lambda, k) = 0$ if λ is not in σ. Let us prove by induction on ℓ that if $m \in \mathcal{S}_{\ell, \geq 0}$ and $\Theta_a^t(m^t) \equiv 0$, for all $t \in T$ of finite order, then $m = 0$.

If $\ell = 0$, then $m(\lambda, k) = 0$ except if $\lambda = 0$, and our distribution is

$$m(0, k) \sum_{w} \epsilon(w) \varphi(w\rho/k).$$

Now, take for example $\varphi(\xi) = \prod_{\alpha > 0}(\xi, H_\alpha)\chi(\xi)$ where χ is invariant with small compact support and identically equal to 1 near 0. Then $\langle \Theta_a(m), \varphi \rangle$ for k large is equal to $c\frac{1}{k^N} m(0, k)$, where N is the number of positive roots, and c a non zero constant. So we conclude that $m(0, k) = 0$.

Now consider $m = \sum_{\dim \sigma = \ell} m_\sigma + \sum_{\dim f < \ell} m_f$. Choose m_σ in the first sum. Let σ^0 be the relative interior of σ. Let Δ_0 be the set of roots α, such that $\langle H_\alpha, \sigma \rangle = 0$. Then $\mathfrak{t}^* = \mathfrak{t}_1^* \oplus \mathfrak{t}_0^*$, where $\mathfrak{t}_0^* = \sum_{\alpha \in \Delta^0} \mathbb{R}\alpha$ and $\mathfrak{t}_1^* = \mathbb{R}\sigma$. We write $\xi = \xi_0 + \xi_1$ for $\xi \in \mathfrak{t}^*$, with $\xi_0 \in \mathfrak{t}_0^*, \xi_1 \in \mathfrak{t}_1^*$. Then $\rho = \rho_0 + \rho_1$ with $\rho_1 \in \mathfrak{t}_1^*$ and $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha$. Let W_0 be the subgroup of the Weyl group generated by the reflections s_α with $\alpha \in \Delta_0$. It leaves stable σ.

Consider φ a test function of the form $\varphi_0(\xi_0)\varphi_1(\xi_1)$ with $\varphi_0(\xi_0) = \chi_0(\xi_0)\prod_{\alpha\in\Delta_0^+}\langle\xi_0, H_\alpha\rangle$ with $\chi_0(\xi_0)$ a function on $\mathfrak{t}_0^*$ with small support near 0 and identically 1 near 0, while $\varphi_1(\xi_1)$ is supported on a compact subset contained in σ^0.

For k large,

$$\langle\Theta_a^t, \varphi\rangle = \frac{1}{|W|} m_\sigma(\lambda, k) \sum_{w\in W_0} \phi(w(\lambda+\rho)/k).$$

So

$$\langle\Theta_a^t, \varphi\rangle = c_0 \frac{1}{k^{N_0}} \sum_{\lambda\in\sigma} m_\sigma(\lambda, k)\varphi_1((\lambda+\rho_1)/k).$$

As in the preceding case, this implies that $m_\sigma(\lambda, k) = 0$ for $\lambda \in \sigma^0$. Doing it successively for all σ entering in the first sum, we conclude that $m \in \mathcal{S}_{\geq 0,\ell-1}(\Lambda)$. By induction, we conclude that $m = 0$. □

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