Superexponential growth or decay in the heat equation with a logarithmic nonlinearity

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Abstract

We consider the heat equation with a logarithmic nonlinearity, on the real line. For a suitable sign in front of the nonlinearity, we establish the existence and uniqueness of solutions of the Cauchy problem, for a well-adapted class of initial data. Explicit computations in the case of Gaussian data lead to various scenarii which are richer than the mere comparison with the ODE mechanism, involving (like in the ODE case) double exponential growth or decay for large time. Finally, we prove that such phenomena remain, in the case of compactly supported initial data.

1 Introduction

In this work we consider the nonnegative solutions $u(t,x)$ of the heat equation with a logarithmic nonlinearity, namely

$$\partial_t u = \partial_{xx} u + \lambda\, u \ln\left(u^2\right), \quad t>0,\ x\in\mathbb{R}, \tag{1.1}$$

where $\lambda > 0$ is a given parameter. Our primary goal is to investigate the large time behavior of the solutions, since this study reveals mechanisms which seem interesting to us. Equation (1.1) shares some similarities with *bistable* equations modelling an *Allee effect* in population dynamics, but does not seem to correspond clearly to any model proposed in e.g. biology or chemistry. On the other hand, (1.1) has challenging aspects from the mathematical point of view. Our results may be extended to the multidimensional case, leading to a more technical setting. We have chosen to stick to the one-dimensional case to simplify the presentation, thus highlighting the main mechanisms.

Associated with (1.1) is the following energy

$$\mathcal{E}[u](t) := \frac{1}{2}\int_{\mathbb{R}} (\partial_x u)^2(t,x)dx + \int_{\mathbb{R}} \frac{\lambda}{2} u^2\left(1-\ln\left(u^2\right)\right)(t,x)dx. \tag{1.2}$$

Formally, solutions to (1.1) satisfy

$$\frac{d\mathcal{E}[u]}{dt} = -\int_{\mathbb{R}} (\partial_t u)^2(t,x)dx \leq 0.$$

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Many features make (1.1) interesting from a mathematical point of view. First, the nonlinearity is not Lipschitzean, which causes difficulties already at the level of the local Cauchy problem. Also, the second term in the energy (1.2) has no definite sign, which makes a priori estimates a delicate issue. Next, (1.1) supports the existence of Gaussian solutions. Last, the Cauchy problem may exhibit superexponential growth or decay.

1.1 The Cauchy problem

Such a logarithmic nonlinearity has been introduced in Physics in the context of wave mechanics and optics [5, 6]. From a mathematical point of view, the Cauchy problem for logarithmic Schrödinger equations and logarithmic wave equations have been studied in [9, 8]: in the case of the logarithmic Schrödinger equation, it is shown that a unique, global weak solution can be constructed in a subset of H^1 (in any space dimension), whichever the sign of λ. For the three dimensional wave equation and a suitable sign for λ, a similar result is available. Due to the lack of regularity of the nonlinearity, solutions are constructed by compactness methods, and uniqueness is a rather unexpected property: in the case of Schrödinger equation, it is a consequence of an elegant estimate in complex analysis noticed in [9], while for the three dimensional wave equation, it follows from fine properties of the wave equation and a general result concerning the trace (see [9] or [14]).

In the context of the heat equation like (1.1), the presence of a logarithmic nonlinearity has been considered in [10], in the case of a bounded domain Ω, with Dirichlet boundary conditions. They construct global solutions in $H^1_0(\Omega)$, and exhibit some classes of solution growing or decaying (at least) exponentially in time, thanks to variational arguments (potential well method). On the other hand, it seems very delicate, if possible, to construct a solution to (1.1) by compactness methods on the whole line $\mathbb{R}$. Also, uniqueness is missing in the Cauchy theory developed in [10]. We will see that this issue can be overcome by changing functional spaces in which the Cauchy problem is studied.

Definition 1.1 (Notion of solution)**.** *Let u_0 be continuous and bounded, with continuous and bounded derivative, and bounded and piecewise continuous second derivative. A (global) solution to* (1.1) *starting from u_0 is a function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ which is continuous and bounded on $[0,T] \times \mathbb{R}$, for which u_t, u_x and u_{xx} exist and are continuous on $(0,T] \times \mathbb{R}$, such that $u(t,x)$ solves* (1.1) *on $(0,T] \times \mathbb{R}$ (for any $T > 0$), and $u_{|t=0} = u_0$. In addition, we require that $u(t,x)$ is uniformly bounded as $|x| \to \infty$ for $t \in [0,T]$.*

Proposition 1.2 (Global well-posedness for (1.1))**.** *Let $u_0 \geq 0$ be as in Definition 1.1. Then* (1.1) *has a unique solution u starting from u_0, in the sense of Definition 1.1.*

1.2 Superexponential growth vs. decay

In [16], the presence of the logarithmic nonlinearity is motivated as some limiting case for a nonlinearity of the form $\lambda u^{1+\varepsilon}$ in the limit $\varepsilon \to 0$ (in the growth regime $u \to \infty$), still in a bounded domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions. The authors show in particular that time periodic solutions are highly unstable, in the sense that a small perturbation of the initial data can lead to double exponential growth or double exponential decay in time, see [16, Theorem 1.1]. Some of our results are qualitatively similar (superexponential growth or decay), with the Gaussian steady state (2.1) acting as a separation comparable to the time periodic solutions in [16]. Nevertheless, let us mention two main differences. First, our results are valid on the whole line $\mathbb{R}$, and not (only) on a bounded domain. On the other hand, we provide in Section 5 initial data leading to superexponential growth or decay but that cannot be handled by [16, Theorem 1.1]. Roughly speaking, as can be seen from the proof, initial data of [16, Theorem 1.1] are multiples of the separating time periodic solution — which is comparable to the present Remark 4.4— whereas initial data in Section 5 are allowed to "cross" the separating Gaussian steady state (2.1) (see also Corollary 3.10).

Let us recall that, in his seminal work [12], Fujita considered solutions $u(t,x)$ to the nonlinear heat equation

$$\partial_t u = \Delta u + u^{1+p}, \quad t > 0,\ x \in \mathbb{R}^N, \tag{1.3}$$

supplemented with a nonnegative and nontrivial initial data. For $p > 0$ solutions of the underlying ordinary differential equation (ODE) problem — namely $\frac{dn}{dt} = n^{1+p}$, $n(0) = n_0 > 0$ — blow up in finite time. The dynamics of the partial differential equation (1.3) is more complex and rich. Precisely, there is a critical exponent $p_F := \frac{2}{N}$, referred to as the *Fujita exponent*, such that: If $0 < p \leq p_F$ then any solution blows up in finite time, like those of the ODE. On the other hand, if $p > p_F$ there is a balance between diffusion and reaction. Solutions with large initial data blow up in finite time whereas solutions with small initial data are global in time and go extinct as $t \to \infty$. Those facts are proved in [12], except the critical case $p = p_F$ which is studied in [15] when $N = 1, 2$, in [17] when $N \geq 3$, and in [21] via a direct and simpler approach.

Concerning equation (1.1), the underlying ODE problem

$$\frac{dn}{dt} = 2\lambda n \ln n, \quad n(0) = n_0 > 0, \tag{1.4}$$

is globally solved as

$$n(t) = e^{(\ln n_0)e^{2\lambda t}}. \tag{1.5}$$

As $t \to \infty$, $n(t) \to 0$ if $0 < n_0 < 1$ (extinction) whereas $n(t) \to \infty$ if $n_0 > 1$ (blow up in infinite time). Hence the dynamics of the ODE (1.4) already shares some similarities with the Fujita supercritical regime $p > p_F$ for the PDE (1.3). The dynamics of the PDE (1.1) is much richer than the mechanism of (1.4), and our main goal is to understand its long time behavior for initial data u_0 "crossing" the equilibrium 1.

Notice also that the composition of exponential functions in (1.5) is a strong indication that possible extinction or growth phenomena are strong, and can thus hardly be captured numerically. In practice, the superexponential growth may appear like a blow-up phenomenon, while the superexponential decay may be understood like a finite time extinction.

When possible, a second goal is to estimate these rates of convergence.

To give a flavor of the results established in the sequel, recall that the authors in [10] consider (1.1) (possibly in multidimension) on a bounded domain, with Dirichlet boundary conditions. By variational arguments, they exhibit classes of initial data whose evolution under (1.1) leads to (at least) exponential decay in L^2, and another class of initial data whose evolution leads to unboundedness of the L^2 norm in large time. As a consequence of our analysis on the whole line $\mathbb{R}$, we actually provide more precise information on those phenomena for the equation on a bounded domain, say (α, β).

Proposition 1.3 (Growth/decay rates in a bounded domain)**.** *Let $\alpha < \beta$ and $\Omega = (\alpha, \beta)$. Consider the mixed problem*

$$\begin{cases} \partial_t u = \partial_{xx} u + u \ln(u^2), \quad t > 0, \ x \in \Omega, \\ u_{|\partial\Omega} = 0, \quad t > 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1.6}$$

There are nonnegative initial data $u_0 \in C_c^1(\Omega)$ such that (1.6) *has a unique solution, whose L^2 and L^∞ norms decay at least like a double exponential in time,*

$$\exists C, \eta > 0, \quad \|u(t)\|_{L^2(\Omega)} \leq |\Omega|^{1/2} \|u(t)\|_{L^\infty(\Omega)} \leq C e^{-\eta e^{2t}}.$$

There are nonnegative initial data $u_0 \in C_c^1(\Omega)$ such that (1.6) *has a unique solution, whose L^2 and L^∞ norms grow at least like a double exponential in time,*

$$\exists C, \eta > 0, \quad |\Omega|^{1/2} \|u(t)\|_{L^\infty(\Omega)} \geq \|u(t)\|_{L^2(\Omega)} \geq C e^{\eta e^{2t}}.$$

1.3 Changing the sign of the nonlinearity

Let us observe that, for $\lambda > 0$, the problem

$$\partial_t u = \partial_{xx} u - 2\lambda \, u \ln u, \quad t > 0, \ x \in \mathbb{R}, \tag{1.7}$$

is of different nature. Indeed for the underlying ODE,

$$n'(t) = -2\lambda n(t) \ln n(t),$$

the equilibrium 0 is (very) unstable, while 1 is stable. Hence, by the comparison principle, solutions are *a priori* bounded between 0 and $\max(1, \|u_0\|_{L^\infty})$. Moreover, by comparison with Fisher-KPP equations, much can be said on the long time behavior of the Cauchy problem. For instance, consider a nontrivial compactly supported initial data $0 \leq u_0 \leq 1$. For any $r > 0$, we can construct

$$g_r : [0,1] \to \mathbb{R} \text{ concave with } g_r > 0 \text{ on } (0,1), \ g_r(0) = g_r(1), \ r = g_r'(0) > 0 > g_r'(1),$$

which is referred as to a Fisher-KPP nonlinearity, and such that $g_r(u) \leq -2\lambda u \ln u$. By the comparison principle, we deduce that $u_r(t,x) \leq u(t,x) \leq 1$, where u_r is the solution of

$$\partial_t u_r = \partial_{xx} u_r + g_r(u_r),$$

starting from u_0. But it is known [4] that the spreading speed of this Fisher-KPP equation, with compactly supported data, is $c_r^* := 2\sqrt{g_r'(0)} = 2\sqrt{r}$, meaning that

$$\text{if } c > c_r^* \text{ then } u_r(t,x) \to 0 \text{ uniformly in } \{|x| \geq ct\} \text{ as } t \to \infty,$$

$$\text{if } c < c_r^* \text{ then } u_r(t,x) \to 1 \text{ uniformly in } \{|x| \leq ct\} \text{ as } t \to \infty.$$

Since this is true for any $r > 0$ we get that

$$\text{for any } c > 0, \ u(t,x) \to 1 \text{ uniformly in } \{|x| \leq ct\} \text{ as } t \to \infty,$$

that is convergence to 1 with a superlinear speed.

The organization of the paper is as follows. In Section 2, we enquire on steady states, proving existence of a unique (Gaussian) nontrivial one. The well-posedness of the Cauchy problem is established in Section 3. The long time behavior (superexponential growth, decay or convergence to the steady state) is studied in Section 4 (Gaussian initial data and consequences), and Section 5 (more general data and consequences).

2 Steady states

It is readily checked that the only constant steady states of (1.1) are $u \equiv 0$ and $u \equiv 1$.

Proposition 2.1 (Steady state)**.** *There is a unique (up to translation) nonnegative nontrivial steady state φ solving* (1.1) *and satisfying $\varphi(\pm\infty) = 0$. It is the Gaussian given by*

$$\varphi(x) = e^{\frac{1}{2}} e^{-\frac{\lambda}{2}x^2}. \tag{2.1}$$

Proof. Let $u = u(x) \geq 0$ be a nontrivial solution to (1.1), that is

$$u''(x) + 2\lambda u(x) \ln u(x) = 0, \quad \forall x \in \mathbb{R}, \tag{2.2}$$

with $u(\pm\infty) = 0$. If $u(x_0) = 0$ for some $x_0 \in \mathbb{R}$ then $u \equiv 0$ from the strong maximum principle. Hence $u > 0$. Next, we multiply the equation by u', integrate and infer that there is $C \in \mathbb{R}$ such that

$$\left(u'\right)^2(x) + \lambda u^2(x)(2\ln u(x) - 1) = C, \quad \forall x \in \mathbb{R}.$$

From the above identity and since $u(\pm\infty) = 0$, we deduce that $u'(\pm\infty)$ must exist in $\mathbb{R}$ and, thus, be equal to 0 (otherwise we cannot have $u(\pm\infty) = 0$). Hence $C = 0$ and

$$\left(u'\right)^2(x) = \lambda u^2(x)(1 - 2\ln u(x)), \quad \forall x \in \mathbb{R}. \tag{2.3}$$

If $x \mapsto 1 - 2\ln u(x)$ never vanishes then this identity implies that u' has a constant sign, which contradicts $u(\pm\infty) = 0$. Hence, there exists $x_0 \in \mathbb{R}$ such that $2\ln u(x_0) = 1$, and thus $u'(x_0) = 0$ (from (2.3)), $u''(x_0) < 0$ (from (2.2)). In the sequel, we work on $[x_0, +\infty)$, the arguments being similar on $(-\infty, x_0]$.

Assume that there is $x_1 > x_0$ such that $u'(x_1) = 0$. From (2.3), $2\ln u(x_1) = 1$, and there must be a point $x^* \in (x_0, x_1)$ where u reaches a minimum strictly smaller than $e^{\frac{1}{2}}$, which contradicts (2.3). Hence $u' < 0$ on $(x_0, +\infty)$. It therefore follows from (2.3) that $-u'(x) = \sqrt{\lambda} u(x)\sqrt{1 - 2\ln u(x)}$ for $x \geq x_0$. Separating variables we get

$$-\sqrt{\lambda}(x - x_0) = \int_{u(x_0)}^{u(x)} \frac{du}{u\sqrt{1 - 2\ln u}} = -\sqrt{1 - 2\ln u(x)},$$

since $u(x_0) = e^{\frac{1}{2}}$. We end up with $u(x) = e^{\frac{1}{2}} e^{-\frac{\lambda}{2}(x - x_0)^2}$, which completes the proof. $\square$

3 Cauchy problem

As emphasized in the introduction, the Cauchy problem associated to (1.1) is not trivial, for two reasons:

- Local well-posedness: the nonlinearity is not Lipschitzean.
- Global well-posedness: the potential energy in (1.2) has no definite sign.

The first aspect implies that constructing a solution certainly requires compactness arguments, and uniqueness is not granted. The second aspect shows that to have a solution defined for all $t \geq 0$, it may be helpful that the first step yields this property "for free". This is the strategy adopted in [10], where, on a bounded domain Ω, with Dirichlet boundary conditions, the authors construct a solution in $H_0^1(\Omega)$ by Galerkin approximation. However, uniqueness is not established in this case.

In this section, we prove Proposition 1.2, by showing that it fits perfectly into the framework of the PhD thesis of J. C. Meyer [19]. Instead of working in spaces where the energy (1.2) is well defined, we adopt the approach of [19], see also [20]. Consider more generally the Cauchy problem

$$u_t = u_{xx} + f(u), \quad 0 < t \leq T,\ x \in \mathbb{R}, \quad u_{|t=0} = u_0, \tag{3.1}$$

so we can emphasize which are the suitable assumptions of the nonlinearity f described in [19]. Notice that, in [19, 20], the standard examples, motivated by models from Chemistry, are of the form $f(u) = \pm(u^p)^+$, $0 < p < 1$, and $f(u) = (u^p)^+((1-u)^q)^+$, $0 < p, q < 1$.

The generalization of Definition 1.1, as introduced in [19], is the following.

Definition 3.1 (Notion of solution)**.** *Let u_0 be continuous and bounded, with continuous and bounded derivative, and bounded and piecewise continuous second derivative. A solution to* (3.1) *is a function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ which is continuous and bounded on $[0,T] \times \mathbb{R}$, for which u_t, u_x and u_{xx} exist and are continuous on $(0,T] \times \mathbb{R}$, such that $u(t,x)$ satisfies* (1.1)*. In addition, we require that $u(t,x)$ is uniformly bounded as $|x| \to \infty$ for $t \in [0,T]$.*

Notation 3.2. *Following [19, 20], we denote by* $\mathrm{BPC}^2(\mathbb{R})$ *the set of such initial data.*

Two notions are crucial, and correspond exactly to the type of logarithmic nonlinearity considered in the present paper.

Definition 3.3 (Hölder continuity)**.** *Let $\alpha \in (0,1)$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be* α-Hölder continuous *if for any closed bounded interval $E \subset \mathbb{R}$, there exists a constant $k_E > 0$ such that for all $x, y \in E$,*

$$|f(x) - f(y)| \leq k_E |x-y|^\alpha.$$

A notion weaker than the standard notion of Lipschitz continuity turns out to be rather interesting, as we will see below.

Definition 3.4 (Upper Lipschitz continuity)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be* upper Lipschitz continuous *if f is continuous, and for any closed bounded interval $E \subset \mathbb{R}$, there exists a constant $k_E > 0$ such that for all $x, y \in E$, with $y \geq x$,*

$$f(y) - f(x) \leq k_E(y-x).$$

Essentially, this property suffices to have a comparison principle, hence a uniqueness result for (3.1).

Example 3.5. *In the case of* (1.1)*, $f(u) = \lambda u \ln(u^2)$. First, f is α-Hölder continuous for any $\alpha \in (0,1)$. Indeed, for $y > x > 0$, we have*

$$|f(y) - f(x)| = 2\lambda \left| (y-x)\ln y + x \ln\left(1 + \frac{y-x}{x}\right) \right| \leq 2\lambda |y-x|(|\ln y| + 1),$$

so that $\frac{|f(y)-f(x)|}{|y-x|^\alpha} \leq 2\lambda|y-x|^{1-\alpha}(|\ln y|+1) \leq 2\lambda|y|^{1-\alpha}(|\ln y|+1)$, *which remains bounded as* $y \to 0$. *On the other hand, even though* f *is not Lipschitz continuous, we check that for* $\lambda > 0$ *(the case of interest in the present paper),* f *is* upper *Lipschitz continuous. Indeed, for* $x, y \in E$ *bounded, with* $y > x > 0$, *Taylor formula yields*

$$\begin{aligned} f(y) - f(x) &= (y-x) \int_0^1 f'\left(x + \theta(y-x)\right) d\theta \\ &= 2\lambda(y-x) \int_0^1 (1+\ln)\left(x+\theta(y-x)\right) d\theta \\ &\leq 2\lambda(y-x) 2\lambda \left(1 + \sup_{z\in E} \ln z\right). \end{aligned}$$

The last factor remains bounded as $x \to 0$. *It would not be if the infimum was considered:* f *is not Lipschitz continuous.*

Definition 3.6 (Sub- and super-solutions)**.** *Let* $\underline{u}, \overline{u} : [0,T] \times \mathbb{R}$ *be continuous on* $[0,T]\times\mathbb{R}$ *and such that* $\underline{u}_t, \underline{u}_x, \underline{u}_{xx}, \overline{u}_t, \overline{u}_x, \overline{u}_{xx}$ *exist and are continuous on* $(0,T]\times\mathbb{R}$. *If*

$$\begin{aligned} &\underline{u}_t - \underline{u}_{xx} - f\left(\underline{u}\right) \leq 0 \leq \overline{u}_t - \overline{u}_{xx} - f\left(\overline{u}\right), \quad 0 < t \leq T,\ x \in \mathbb{R}, \\ &\underline{u}(0,x) \leq u_0(x) \leq \overline{u}(0,x), \quad \forall x \in \mathbb{R}, \end{aligned}$$

and $\underline{u}, \overline{u}$ *are uniformly bounded as* $|x| \to \infty$ *for* $t \in [0,T]$, *then* $\underline{u}$ *is called a* regular sub-solution, *and* $\overline{u}$ *is called a* regular super-solution *to* (3.1)*.*

Theorem 3.7 (Comparison; Theorem 7.1 from [19, 20])**.** *Let* f *be upper Lipschitz continuous. If* $\underline{u}$ *and* $\overline{u}$ *and regular sub and super-solutions on* $[0,T]\times\mathbb{R}$, *respectively, then*

$$\underline{u}(t,x) \leq \overline{u}(t,x), \quad \forall (t,x) \in [0,T]\times\mathbb{R}.$$

Theorem 3.8 (Uniqueness; Theorem 7.2 from [19, 20])**.** *Let* f *be upper Lipschitz continuous. Then, for any* $T > 0$, (3.1) *has at most one solution in* $[0,T]\times\mathbb{R}$.

The following statement is a slight modification from the original, where we add a uniqueness assumption to simplify the presentation.

Theorem 3.9 (Existence; Theorem 8.1 and Corollary 8.6 from [19, 20])**.** *Suppose that* f *is* α*-Hölder continuous for some* $\alpha \in (0,1)$, *and let* $u_0 \in \mathrm{BPC}^2(\mathbb{R})$. *Suppose that uniqueness holds for* (3.1)*. Then* (3.1) *has a (unique) solution* $u : [0,T^*[\times\mathbb{R}$. *In addition, either* $T^* = \infty$, *or* $\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}$ *is unbounded as* $t \to T^*$.

Proof of Proposition 1.2. As emphasized above, the nonlinearity in (1.1) is both α-Hölder continuous (for any $\alpha \in (0,1)$) and upper Lipschitz continuous. Therefore, Theorem 3.8 implies uniqueness, and Theorem 3.9 yields a (unique) maximal solution $u \in C([0,T^*)\times\mathbb{R})$.

We conclude by showing that the solution is global ($T^* = \infty$) thanks to a suitable a priori estimate. The solution of the ODE (1.4) starting from $\|u_0\|_{L^\infty}$, namely

$$\overline{u}(t) = e^{\ln\|u_0\|_{L^\infty} e^{2\lambda t}}.$$

is a super-solution, while the zero function is obviously a sub-solution. Theorem 3.7 implies that

$$0 \leq u(t,x) \leq \overline{u}(t), \quad \forall t \in [0,T^*).$$

We conclude that $T^* = \infty$, and the result follows. □

Corollary 3.10 (Initial data comparable to 1)**.** *Let $u_0 \in \mathrm{BPC}^2(\mathbb{R})$, $u_0 \geq 0$, and $\varepsilon \in (0,1)$.*

- *If $u_0(x) \geq 1+\varepsilon$ for all $x \in \mathbb{R}$, then u grows at least like a double exponential in time:*

$$u(t,x) \geq e^{\ln(1+\varepsilon)e^{2\lambda t}}, \quad \forall t \geq 0,\ \forall x \in \mathbb{R}.$$

- *If $u_0(x) \leq 1-\varepsilon$ for all $x \in \mathbb{R}$, then u decays at least like a double exponential in time:*

$$u(t,x) \leq e^{\ln(1-\varepsilon)e^{2\lambda t}}, \quad \forall t \geq 0,\ \forall x \in \mathbb{R}.$$

Proof. This corollary is a straightforward consequence of Proposition 1.2, the comparison principle (Theorem 3.7), and the ODE case (1.4)–(1.5). □

4 Large time behavior: Gaussian data

Families of Gaussian solutions for nonlinear (and nonlocal) equations can be found in [7], [3, 2], in the context of evolutionary genetics. In the case of a logarithmic nonlinearity, for the Schrödinger equation, it was observed in [6] that the flow preserves the Gaussian structures, and so the resolution of the partial differential equation boils down to the resolution of ordinary differential equations; see [8] for more details. It is not surprising that the same holds in the case of (1.1), and we have indeed:

Proposition 4.1 (Gaussian solutions)**.** *Let $b_0 > 0$ and $a_0 > 0$ be given. The solution of* (1.1) *starting from the Gaussian*

$$u_0(x) = b_0 e^{-\frac{a_0}{2}x^2}, \tag{4.1}$$

is the Gaussian given by

$$u(t,x) = b(t)e^{-\frac{a(t)}{2}x^2} := e^{\psi(t)e^{2\lambda t}} e^{-\frac{a(t)}{2}x^2}, \tag{4.2}$$

where

$$\psi(t) = \ln b_0 - \frac{a_0}{2}\frac{\ln\lambda - \ln(a_0 + (\lambda - a_0)e^{-2\lambda t})}{\lambda - a_0}, \tag{4.3}$$

with the natural continuation $\psi(t) = \ln b_0 - \frac{1}{2}(1 - e^{-2\lambda t})$ if $a_0 = \lambda$, and

$$a(t) = \lambda \frac{a_0 e^{2\lambda t}}{\lambda - a_0 + a_0 e^{2\lambda t}}. \tag{4.4}$$

Proof. We plug the ansatz (4.2) into equation (1.1), we identify the x^0 and the x^2 coefficients to obtain two ordinary differential equations. The first one is the logistic equation

$$a'(t) = 2a(t)(\lambda - a(t)),$$

whose solution, starting from $a(0) = a_0$, is given by (4.4). The second one is

$$b'(t) = 2\lambda b(t)\ln b(t) - a(t)b(t).$$

Denoting $\phi(t) := \ln b(t)$ the above is recast

$$\phi'(t) = 2\lambda\phi(t) - a(t),$$

whose solution, starting from $\phi(0) = \ln b_0$, is

$$\phi(t) = \left(\ln b_0 - \int_0^t e^{-2\lambda s} a(s)ds\right) e^{2\lambda t}. \tag{4.5}$$

Next, using (4.4) we get

$$\begin{aligned}\int_0^t e^{-2\lambda s} a(s)ds &= \int_0^t \frac{\lambda a_0 e^{-2\lambda s}}{a_0 + (\lambda - a_0)e^{-2\lambda s}} ds \\ &= \begin{cases} \dfrac{a_0}{-2(\lambda - a_0)} \left(\ln\left(a_0 + (\lambda - a_0)e^{-2\lambda t}\right) - \ln\lambda\right) & \text{if } a_0 \neq \lambda \\ \dfrac{1}{2}(1 - e^{-2\lambda t}) \quad \text{if } a_0 = \lambda, \end{cases}\end{aligned}$$

which we plug into (4.5) to get (4.3). □

Clearly, the sign of $\psi_\infty := \lim_{t\to\infty} \psi(t)$ decides between (superexponential) decay and growth of the Cauchy problem starting from a Gaussian data, the critical case $\psi_\infty = 0$ leading to convergence to the steady state.

Corollary 4.2 (Gaussian data: three scenarii)**.** *Let $b_0 > 0$ and $a_0 > 0$ be given. Define*

$$\psi_\infty := \ln b_0 - \frac{a_0}{2} \frac{\ln\lambda - \ln a_0}{\lambda - a_0}, \tag{4.6}$$

with the natural continuation $\psi_\infty = \ln b_0 - \frac{1}{2}$ if $a_0 = \lambda$. Denote by $u(t,x)$ the Gaussian solution of Proposition 4.1.

(i) If $\psi_\infty < 0$, then there is superexponential decay in the sense that

$$\|u(t,\cdot)\|_{L^\infty} = \max_{x\in\mathbb{R}} u(t,x) \underset{t\to\infty}{\sim} e^{\frac{1}{2}} e^{\psi_\infty e^{2\lambda t}}, \quad \|u(t,\cdot)\|_{L^1} \underset{t\to\infty}{\sim} \sqrt{\frac{2\pi}{\lambda}} e^{\frac{1}{2}} e^{\psi_\infty e^{2\lambda t}}.$$

(ii) If $\psi_\infty = 0$, then there is convergence to the steady state of Proposition 2.1 in the sense that

$$\|u(t,\cdot) - \varphi\|_{L^\infty} + \|u(t,\cdot) - \varphi\|_{L^1} \underset{t\to\infty}{\longrightarrow} 0.$$

(iii) If $\psi_\infty > 0$, then there is superexponential growth in the sense that, for all $R > 0$,

$$\min_{|x|\le R} u(t,x) \underset{t\to\infty}{\sim} e^{\frac{1}{2}} e^{\psi_\infty e^{2\lambda t}} e^{-\frac{\lambda}{2}R^2}, \quad \|u(t,\cdot)\|_{L^1} \underset{t\to\infty}{\sim} \sqrt{\frac{2\pi}{\lambda}} e^{\frac{1}{2}} e^{\psi_\infty e^{2\lambda t}}.$$

Proof. One just has to use the asymptotic expansion $\psi(t) = \psi_\infty + \frac{1}{2}e^{-2\lambda t} + \mathcal{O}(e^{-4\lambda t})$ as $t \to \infty$, and perform straightforward estimates. □

In view of the comparison principle (Theorem 3.7) and of Corollary 4.2, we infer:

Corollary 4.3 (Initial data comparable to a Gaussian)**.** *Let $u_0 \in \mathrm{BPC}^2(\mathbb{R})$, $u_0 \geq 0$, and $\varepsilon \in (0,1)$. Let $a_0, b_0 > 0$, and denote again*

$$\psi_\infty = \ln b_0 - \frac{a_0}{2} \frac{\ln\lambda - \ln a_0}{\lambda - a_0},$$

with the natural continuation $\psi_\infty = \ln b_0 - \frac{1}{2}$ if $a_0 = \lambda$.

- *If $\psi_\infty < 0$ and $u_0(x) \leq b_0 e^{-a_0 x^2/2}$, then u decays at least like a double exponential in time,*

$$\|u(t,\cdot)\|_{L^\infty} \leq 2e^{\frac{1}{2}} e^{\psi_\infty e^{2\lambda t}}, \quad \|u(t,\cdot)\|_{L^1} \leq \sqrt{\frac{4\pi}{\lambda}} e^{\frac{1}{2}} e^{\psi_\infty e^{2\lambda t}}, \qquad \text{as } t \to \infty.$$

- *If $\psi_\infty > 0$ and $u_0(x) \geq b_0 e^{-a_0 x^2/2}$, then u grows locally at least like a double exponential in time: for all $R > 0$,*

$$\min_{|x|\leq R} u(t,x) \geq \frac{1}{2} e^{\frac{1}{2}} e^{\psi_\infty e^{2\lambda t}} e^{-\frac{\lambda}{2}R^2}, \quad \|u(t,\cdot)\|_{L^1} \geq \sqrt{\frac{\pi}{\lambda}} e^{\frac{1}{2}} e^{\psi_\infty e^{2\lambda t}}, \qquad \text{as } t \to \infty.$$

Remark 4.4. *Observe that, if $\psi_\infty = \ln b_0 - \frac{a_0}{2}\frac{\ln\lambda - \ln a_0}{\lambda - a_0} = 0$, then initial data $(1-\varepsilon)b_0 e^{-a_0x^2/2}$ ($0 < \varepsilon < 1$), and $(1+\varepsilon)b_0 e^{-a_0 x^2/2}$ ($\varepsilon > 0$), fall into the regime $\psi_\infty < 0$ (decay), and $\psi_\infty > 0$ (growth), respectively. Typical examples are $(1-\varepsilon)\varphi(x)$, $(1+\varepsilon)\varphi(x)$, where $\varphi(x) = e^{\frac{1}{2}} e^{-\lambda x^2/2}$ is the steady state from Proposition 2.1.*

5 Large time behavior in the case of more general data

We have seen that the comparison with the constant initial datum equal to one leads to a strong dichotomy (Corollary 3.10). The same is true by comparison with an initial Gaussian, leading to a larger variety of initial data (Corollary 4.3 and Remark 4.4). Now, we enquire on initial data that can be compared neither to 1 nor to a Gaussian. In this direction, we can prove superexponential decay for small initial data. To do so, we need the following standard estimate, which stems from Young inequality applied to the formula

$$e^{t\partial_{xx}} v_0(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/(4t)} v_0(y)dy.$$

Lemma 5.1. *For any initial data $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the solution of the Cauchy problem $\partial_t v = \partial_{xx} v$, $v_{|t=0} = v_0$, satisfies*

$$\|v(t,\cdot)\|_{L^\infty} \leq V(t) := \min\left(\|v_0\|_{L^\infty}, \frac{\|v_0\|_{L^1}}{\sqrt{4\pi t}}\right), \qquad \text{for any } t \geq 0.$$

Theorem 5.2 (Superexponential decay for small data)**.** *For a nonnegative initial datum u_0 in $\mathrm{BPC}^2(\mathbb{R})$, define $m_\infty := \|u_0\|_{L^\infty}$ and $m_1 := \|u_0\|_{L^1}$. Assume*

$$\begin{aligned} \psi^*_\infty &:= \ln m_\infty - \int_\tau^{+\infty} \lambda e^{-2\lambda s} \ln s \, ds + e^{-2\lambda\tau} \ln\sqrt{\tau} < 0, \\ &\text{where } \tau := \left(\frac{m_1}{\sqrt{4\pi}\, m_\infty}\right)^2. \end{aligned} \tag{5.1}$$

Then the solution of (1.1), starting from u_0, is decaying superexponentially in the sense that

$$0 \leq u(t,x) \leq \min\left(m_\infty, \frac{m_1}{\sqrt{4\pi t}}\right) e^{\psi(t)e^{2\lambda t}}, \tag{5.2}$$

where $\psi(t) \to \psi^*_\infty < 0$ as $t \to \infty$.

Remark 5.3. *The above criterion provides new initial data leading to superexponential decay. For instance, assume that u_0 has tails heavier than Gaussian (so that domination by a Gaussian cannot be used), that*

$$1 < m_\infty < e^{\int_1^{+\infty} \lambda e^{-2\lambda s} \ln s \, ds}, \tag{5.3}$$

(so that domination by the ODE cannot be used), and that $\tau = 1$. Then (5.1) *holds true, hence* (5.2)*. A typical example could be*

$$u_0(x) = m_\infty e^{-\alpha\left(\sqrt{1+x^2}-1\right)},$$

with (5.3) *and $\alpha > 0$ adjusted so that $\tau = 1$.*

Proof. Following [13] or [1], we look for a supersolution to (1.1) in the form $g(t)v(t,x)$, where $g(t) > 0$ is to be determined (with $g(0) = 1$), and $v(t,x)$ is the solution of the heat equation $\partial_t v = \partial_{xx} v$ with u_0 as initial datum. A straightforward computation shows that to construct a supersolution, it is enough to have

$$\frac{g'(t)}{g(t)} - 2\lambda \ln g(t) \geq 2\lambda \ln v(t,x).$$

By Lemma 5.1, it is therefore enough to have $g(t) = e^{\phi(t)}$ where

$$\phi'(t) - 2\lambda\phi(t) = 2\lambda \ln V(t), \quad \phi(0) = 0,$$

that is

$$\phi(t) = e^{2\lambda t} \int_0^t 2\lambda e^{-2\lambda s} \ln V(s) ds.$$

Observe that $V(t) = m_\infty$ when $t \leq \tau$ while $V(t) = \frac{m_1}{\sqrt{4\pi t}}$ when $t \geq \tau$. Cutting the above integral and performing straightforward computations, we end up with $g(t) = e^{\psi(t)e^{2\lambda t}}$, where

$$\psi(t) := (\ln m_\infty)(1 - e^{-2\lambda\tau}) + \ln \frac{m_1}{\sqrt{4\pi}}(e^{-2\lambda\tau} - e^{-2\lambda t}) - \int_\tau^t \lambda e^{-2\lambda s} \ln s \, ds,$$

which tends to ψ_∞^* as $t \to \infty$. It therefore follows from the comparison principle (Theorem 3.7) that $u(t,x) \leq g(t)v(t,x)$, which yields (5.2). □

In the context of bounded solutions, typically for Lipschitz *ignition* or *bistable* nonlinearities, some threshold results between extinction and convergence to an equilibrium, say 1, are known to exist [22], [11], [18]. For equation (1.1), we can prove a threshold result between decay and growth. We first need to construct compactly supported sub-solutions.

Lemma 5.4 (High plateaux as sub-solutions)**.** *Let $L > 0$ and $0 < \varepsilon < L$ be given. Let $\Theta \in C^\infty([-L,L]) \cap \mathrm{BPC}^2(\mathbb{R})$ be such that*

$$\begin{aligned}
&\Theta(x) = 0 \textit{ for } |x| \geq L,\\
&\Theta > 0 \textit{ on } (-L,L),\\
&\Theta(\pm L) = \Theta'(\pm L) = \Theta''(\pm L) = 0,\\
&\gamma := \Theta'''(-L) = -\Theta'''(L) > 0,\\
&\Theta \equiv 1 \textit{ on } [-L+\varepsilon, L-\varepsilon].
\end{aligned}$$

Then there is $K_0 > 1$ such that, for any $K \geq K_0$, the function $\Theta_K := K\Theta$ satisfies

$$\Theta_K'' + 2\lambda\Theta_K \ln \Theta_K \geq 0, \quad \text{on } \mathbb{R}. \tag{5.4}$$

Hence, Θ_K is a sub-solution to (1.1).

Proof. By our assumption $\Theta(x) \sim \frac{1}{6}\gamma(x+L)^3$, $\Theta''(x) \sim \gamma(x+L)$ for $0 < x + L \ll 1$, where we thus have

$$\Theta_K''(x) + 2\lambda\Theta_K(x)\ln\Theta_K(x) \geq K(\Theta''(x) + 2\lambda\Theta(x)\ln\Theta(x)) \sim K\gamma(x+L).$$

As result, there is $\delta > 0$ such that (5.4) holds on $(-L, -L+\delta)$ and, by symmetry, on $(L-\delta, L)$.

Next, denoting $\theta^* := \min_{-L+\delta \leq x \leq L-\delta} \Theta(x) > 0$, we have, for $x \in [-L+\delta, L-\delta]$,

$$\begin{aligned}\Theta_K''(x) + 2\lambda\Theta_K(x)\ln\Theta_K(x) &= K(\Theta''(x) + 2\lambda\Theta(x)\ln\Theta(x) + 2\lambda\Theta(x)\ln K)\\ &\geq K(-\|\Theta'' + 2\lambda\Theta\ln\Theta\|_{L^\infty} + 2\lambda\theta^*\ln K)\end{aligned}$$

which is nonnegative if $K > 1$ is large enough. □

Theorem 5.5 (Threshold phenomena for compactly supported data)**.** *Let $0 < \varepsilon < L < L'$ be given. Select $K \geq K_0 > 1$, where K_0 is given by Lemma 5.4. Let $u_0 \in \mathrm{BPC}^2(\mathbb{R})$ be such that $u_0 > 0$ on $(-L', L')$ and $u_0 \equiv 0$ on $(-\infty, -L'] \cup [L', +\infty)$. For $M > 0$, we denote by $u_M(t,x)$ the solution of* (1.1) *starting from Mu_0.*

(i) There is $M_{\mathrm{decay}} > 0$ such that, for any $0 < M < M_{\mathrm{decay}}$, the solution $u_M(t,x)$ is decaying superexponentially in time.

(ii) There is $M_{\mathrm{growth}} > 0$ such that, for any $M > M_{\mathrm{growth}}$, the solution $u_M(t,x)$ grows locally superexponentially in time, in the sense that

$$u_M(t,x) \geq K e^{\left(\ln\frac{K+1}{K}\right)e^{2\lambda t}}, \quad \forall x \in [-L+\varepsilon, L-\varepsilon]. \tag{5.5}$$

Proof. The first point is a consequence of Corollary 3.10, that is comparison with the ODE, provided we choose $M_{\mathrm{decay}} = 1/\|u_0\|_{L^\infty}$. We now prove (ii). Since $\min_{-L\leq x\leq L} u_0(x) > 0$, there is $M_{\mathrm{growth}} > 0$ such that, for all $M \geq M_{\mathrm{growth}}$, $Mu_0 \geq (K+1)\Theta$. Next we take $n(t)$ as the solution of the underlying ODE starting from $n_0 := \frac{K+1}{K}$, see (1.4) and (1.5), that is

$$n(t) = e^{\left(\ln\frac{K+1}{K}\right)e^{2\lambda t}}.$$

Now we define

$$w(t,x) := \Theta_K(x)n(t) = K\Theta(x)n(t).$$

We have $w(0,\cdot) = (K+1)\Theta \leq Mu_0$ and

$$\begin{aligned}&\partial_t w(t,x) - \partial_{xx}w(t,x) - 2\lambda w(t,x)\ln w(t,x)\\ &= \Theta_K(x)n'(t) - \Theta_K''(x)n(t) - 2\lambda\Theta_K(x)n(t)\ln n(t) - 2\lambda\Theta_K(x)n(t)\ln\Theta_K(x)\\ &= n(t)(-\Theta_K''(x) - 2\lambda\Theta_K(x)\ln\Theta_k(x))\\ &\leq 0,\end{aligned}$$

by Lemma 5.4. It therefore follows from the comparison principle that $u_M \geq w$. In particular, since $\Theta \equiv 1$ on $[-L+\varepsilon, L+\varepsilon]$, we get (5.5). □

Proposition 1.3, concerned with a bounded domain, is a straightforward consequence of the above result. One just has to use a translation in space $c \neq 0$ if necessary so that $c + [-L', L'] \subset (\alpha, \beta)$, and note that the quantities involved in Theorem 5.5 control the L^2 norm on Ω.

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