A VARIATION ON A CONJECTURE OF FABER AND FULTON

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Abstract. In this paper we study the geometry of GIT configurations of n ordered points on $\mathbb{P}^1$ both from the the birational and the biregular viewpoint. In particular, we prove the analogue of the F-conjecture for GIT configurations of points on $\mathbb{P}^1$, that is we show that every extremal ray of the Mori cone of effective curves on the quotient $(\mathbb{P}^1)^n /\!\!/ PGL(2)$, taken with the symmetric polarization, is generated by a one dimensional boundary stratum of the moduli space. On the way to this result we develop some technical machinery that we use to compute the canonical divisor and the Hilbert polynomial of $(\mathbb{P}^1)^n /\!\!/ PGL(2)$ in its natural embedding.

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Introduction

In one of the most celebrated papers [DM69] in the history of algebraic geometry Deligne and Mumford proved that there exists an irreducible scheme $\mathcal{M}_{g,n}$ coarsely representing the moduli functor of n-pointed genus g smooth curves. Furthermore, they provided a compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ adding the so-called Deligne-Mumford stable curves as boundary points. Afterwards other compactifications of $\mathcal{M}_{g,n}$ have been introduced, see for instance [Has03].

In this paper we are interested in the compactification of $\mathcal{M}_{0,n}$ given by the GIT quotient $\Sigma_m := (\mathbb{P}^1)^{m+3} /\!\!/ PGL(2)$ of configurations of $n = m+3$ ordered points on $\mathbb{P}^1$ with respect to the symmetric polarization on $(\mathbb{P}^1)^n$. The aim of our notation is to stress the dimension of the space. In particular, for $m = 3$ we obtain the celebrated Segre cubic 3-fold $\Sigma_3 \subset \mathbb{P}^4$ [Do15].

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The moduli spaces $\overline{\mathcal{M}}_{g,n}$ are among the most studied objects in algebraic geometry. Despite this, many natural questions about their biregular and birational geometry remain unanswered.

The F-conjecture, where as far as we know F stays for Fulton and Faber, for $\overline{\mathcal{M}}_{g,n}$ is one of the long-standing conjectures in the field. Its statement is the following:

Conjecture. [GKM02, Conjecture 0.2] *A divisor on $\overline{\mathcal{M}}_{g,n}$ is ample if and only if it has positive intersection with all 1-dimensional strata. In other words, any effective curve in $\overline{\mathcal{M}}_{g,n}$ is numerically equivalent to an effective combination of 1- strata.*

In other words, as we stated in the abstract, any extremal ray of the Mori cone of effective curves $\mathrm{NE}(\overline{\mathcal{M}}_{g,n})$ is generated by a one dimensional boundary stratum. In [GKM02] A. Gibney, S. Keel and I. Morrison managed to reduce the conjecture for any genus g to the genus zero case, that is $\overline{\mathcal{M}}_{0,n}$. Anyway, so far the conjecture is known up to $n = 7$ [KMc96, Thm. 1.2(3)] (see [FGL16] for a more recent general account), and the very intricate combinatorics of the moduli space $\overline{\mathcal{M}}_{0,n}$ for higher n seem an obstacle pretty difficult to avoid if one wants to try his chance.

In this paper we go in a somewhat orthogonal direction. Instead of trying to show the conjecture for $n > 7$, we modify it slighlty by allowing a little coarser moduli space into the picture. In fact, we consider the GIT quotient Σ_m. This quotient offers an alternate compactification of $\mathcal{M}_{0,n}$, which is a little coarser than $\overline{\mathcal{M}}_{0,n}$ on the boundary, and in fact it is the target a birational morphism $\overline{\mathcal{M}}_{0,n} \to \Sigma_m$, that we will recall in the body of the paper, see also [Bo11].

Nevertheless, also Σ_m has a stratification of its boundary locus, similar to that of $\overline{\mathcal{M}}_{0,n}$, and one can ask exactly the same question of Conjecture . We tackle this problem taking advantage of the fact that Σ_m is a Mori Dream Space, while we know that $\overline{\mathcal{M}}_{0,n}$ is not a Mori Dream Space for $n \geq 10$ [CT15, Corollary 1.4], [GK16, Theorem 1.1], [HKL16, Addendum 1.4]. Mori Dream Spaces, introduced by Y. Hu and S. Keel [HK00], form a class of algebraic varieties that behave very well from the point of view of the minimal model program. In particular, their cones of curves and divisors are polyhedral and finitely generated.

By Proposition 2.7 if $n = m+3$ is even then $\mathrm{Pic}(\Sigma_m) \cong \mathbb{Z}$ and its cones of curves and divisors are 1-dimensional. On the other hand, if $n = m+3$ is odd then $\mathrm{Pic}(\Sigma_m) \cong \mathbb{Z}^{m+3}$ and the birational geometry of Σ_m gets more interesting. In this case we manage to describe the cones of nef and effective divisors of Σ_m and their dual cones of effective and moving curves. The first main result of this paper says that the analogue of the F-conjecture for GIT configurations of points holds.

Theorem 1. *If $n = m+3$ is odd then the Mori cone $\mathrm{NE}(\Sigma_m)$ is generated by classes of 1-dimensional boundary strata.*

In Theorem 2.33 we will also describe precisely what are these 1-dimensional strata. The main ingredients of the proof of Theorem 1 is a construction of our GIT quotients as images $\Sigma_m \subset \mathbb{P}^N$ of rational maps induced by certain linear systems on the projective space $\mathbb{P}^m$, due to C. Kumar [Ku00, Ku03], a careful analysis of the Mori chamber decomposition of the movable cone of certain blow-ups of the projective space and some quite refined projective geometry of the GIT quotients. More precisely C. Kumar realized Σ_m as the closure of the image of the rational map induced by the linear system $\mathcal{L}_{2g-1}$ of degree g hyersurfaces of

$\mathbb{P}^{2g-1}$ having multiplicity $g-1$ at $2g+1$ general points if $m = 2g-1$ is odd, and as the closure of the image of the rational map induced by the linear system $\mathcal{L}_{2g}$ of degree $2g+1$ hyersurfaces of $\mathbb{P}^{2g}$ having multiplicity $2g-1$ at $2g+2$ general points if $m = 2g$ is even. In particular, $N = h^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1})$ if $m = 2g-1$ is odd, and $N = h^0(\mathbb{P}^{2g}, \mathcal{L}_{2g})$ if $m = 2g$ is even.

The tools developed turn out to be useful in resolving a couple of other problems related to GIT quotients of configuration of points on $\mathbb{P}^1$. Furthermore, thanks to recent results on the dimension of linear system on the projective space due to M. C. Brambilla, O. Dumitrescu and E. Postinghel [BDP15, BDP16], we obtain some explicit formulas for the Hilbert polynomial of $\Sigma_m \subset \mathbb{P}^N$. These results are resumed in Corollaries 3.4, 3.5. We would like to mention that an inductive formula for the degree of these GIT quotients had already been given in [HMSV09], while a closed formula for the Hilbert function of GIT quotients of evenly weighted points on the line had been given in [HH14]. The main results on the geometry of $\Sigma_m \subset \mathbb{P}^N$ in Sections 2.1, 2.9, 4, and Corollaries 3.4, 3.5 can be summarized as follows.

Theorem 2. *Let us consider the GIT quotient $\Sigma_m \subset \mathbb{P}^N$. If $m = 2g-1$ is odd we have*

$$\operatorname{Pic}(\Sigma_{2g-1}) \cong \mathbb{Z},\ K_{\Sigma_{2g-1}} \cong \mathcal{O}_{\Sigma_{2g-1}}(-2),$$

and the the Hilbert polynomial of $\Sigma_{2g-1} \subseteq \mathbb{P}^N$ is given by

$$h_{\Sigma_{2g-1}}(t) = \binom{gt+2g-1}{2g-1} + \sum_{r=0}^{g-2}(-1)^{r+1}\binom{2g+1}{r+1}\binom{t(g-r-1)+2g-1-r-1}{2g-1}.$$

In particular

$$\deg(\Sigma_{2g-1}) = g^{2g-1} + \sum_{r=0}^{g-2}(-1)^{r+1}\binom{2g+1}{r+1}(g-r-1)^{2g-1}.$$

If $m = 2g$ is even we have

$$\operatorname{Pic}(\Sigma_{2g}) \cong \mathbb{Z}^{2g+3},\ K_{\Sigma_{2g}} \cong \mathcal{O}_{\Sigma_{2g}}(-1),$$

and the the Hilbert polynomial of $\Sigma_{2g} \subseteq \mathbb{P}^N$ is given by

$$h_{\Sigma_{2g}}(t) = \binom{(2g+1)t+2g}{2g} + \sum_{r=0}^{\lfloor\frac{2g-1}{2}\rfloor}(-1)^{r+1}\binom{2g+2}{r+1}\binom{t(2g-2r-1)+2g-r-1}{2g}.$$

In particular

$$\deg(\Sigma_{2g}) = (2g+1)^{2g} + \sum_{r=0}^{\lfloor\frac{2g-1}{2}\rfloor}(-1)^{r+1}\binom{2g+2}{r+1}(2g-2r-1)^{2g}.$$

Furthermore, the automorphism group of Σ_m is isomorphic to the symmetric group on $n = m+3$ elements S_n for any $m \geq 2$.

Plan of the paper. The paper is organized as follows. In Section 1 we recall some well-known facts and prove some preliminary results on GIT quotient, moduli spaces of weighted pointed rational curves and we clarify the relations between them. In Section 2 we prove the analogue of the F-conjecture for Σ_m with $m = 2g$ even. In Section 3 we work out explicit formulas for the Hilbert polynomial a the degree of $\Sigma_m \subset \mathbb{P}^N$. Finally, in Section 4 we compute the automorphism groups of Σ_m.

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1. GIT quotients of $(\mathbb{P}^1)^n$

The main characters of the paper are the GIT quotients $(\mathbb{P}^1)^n /\!\!/ PGL(2)$, that we review now very quickly. For details there are plenty of very good references on this subject [MFK94, Do03, Do12, HMSV09, Bo11, DO88].

Let us consider the diagonal $PGL(2)$-action on $(\mathbb{P}^1)^n$. An ample line bundle L endowed with a linearization for the $PGL(2)$-action is called a polarization. Such a polarization on $(\mathbb{P}^1)^n$ is completely determined by an n-tuple $b = (b_1, ..., b_n)$ of positive integers:

$$L = \boxtimes_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(b_i).$$

Now let us set $|b| = b_1 + ... + b_n$. A point $x \in (\mathbb{P}^1)^n$ is said to be b-semistable if for some $k > 0$, there exists a $PGL(2)$-invariant section $s \in H^0((\mathbb{P}^1)^n, L^{\otimes k})^{PGL(2)}$ such that $X_s := \{y \in (\mathbb{P}^1)^n : s(y) \neq 0\}$ is affine and contains x. A semistable point $x \in (\mathbb{P}^1)^n$ is stable if its stabilizer under the $PGL(2)$ action is finite and all the orbits of $PGL(2)$ in X_s are closed. A categorical quotient of the open set $((\mathbb{P}^1)^n)^{ss}(b)$ of semistable points exists, and this is what we normally denote by $X(b) /\!\!/ PGL(2)$. We will omit to specify the polarization when it is $(1, ..., 1)$. If b is odd, then $H^0((\mathbb{P}^1)^n, L^{\otimes k})^{PGL(2)} = 0$ for odd k, and

$$(\mathbb{P}^1)^n(b) /\!\!/ PGL(2) = (\mathbb{P}^1)^n(2b) /\!\!/ PGL(2). \tag{1.0}$$

Therefore, by replacing b by $2b$ if necessary, we assume that b is even.

1.1. **Linear systems on $\mathbb{P}^n$.** Throughout the paper we will denote by $\mathcal{L}_{n,d}(m_1, ..., m_s)$ the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ passing through s general points $p_1, ..., p_s \in \mathbb{P}^n$ with multiplicities respectively $m_1, ..., m_s$. It was pointed out by Kumar [Ku03, Section 3.3] that the GIT quotients $(\mathbb{P}^1)^n(b) /\!\!/ PGL(2)$ can be obtained as the images of certain rational polynomial maps defined on $\mathbb{P}^{n-3}$.

Theorem 1.2. [Ku03, Theorem 3.4] *Let us assume that $b_i < \sum_{j\neq i} b_j$ for any $i = 1, ..., n$, and let $p_1, ..., p_{n-1} \in \mathbb{P}^{n-3}$ be general points.*

- *If $|b| = 2\overline{b}$ is even let*

$$\mathcal{L} = \mathcal{L}_{n-3,\overline{b}-b_n}(\overline{b} - b_n - b_1, ..., \overline{b} - b_i - b_n, ..., \overline{b} - b_n - b_n)$$

be the linear system of degree $\overline{b} - b_n$ hypersurfaces in $\mathbb{P}^{n-3}$ with multiplicity $\overline{b} - b_i - b_n$ at p_i.

- *If $|b|$ is odd let*

$$\mathcal{L} = \mathcal{L}_{n-3,b-2b_n}(b-2b_n-2b_1,...,b-2b_i-2b_n,...,b-2b_n-2b_n)$$

be the linear system of degree $b-2b_n$ hypersurfaces in $\mathbb{P}^{n-3}$ with multiplicity $b-2b_i-2b_n$ at p_i.

Finally, let $\phi_{\mathcal{L}} : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}(H^0(\mathbb{P}^{n-3},\mathcal{L})^*)$ be the rational map induced by $\mathcal{L}$. Then $\phi_{\mathcal{L}}$ maps birationally $\mathbb{P}^{n-3}$ onto $(\mathbb{P}^1)^n(b)/\!\!/PGL(2)$. In other words $(\mathbb{P}^1)^n(b)/\!\!/PGL(2)$ may be realized as the closure of the image $\phi_{\mathcal{L}}$ in $\mathbb{P}(H^0(\mathbb{P}^{n-3},\mathcal{L})^*)$.

Example 1.3. For instance, if $n=6$ and $b_1=...=b_6=1$ then $\overline{b}=3$, and $\mathcal{L}=\mathcal{L}_{3,2}(1,...,1)$. In this case the rational map $\phi_{\mathcal{L}}$ is given by the quadrics in $\mathbb{P}^3$ passing through five general points, and $(\mathbb{P}^1)^n(b)/\!\!/PGL(2) \subset \mathbb{P}^4$ is the Segre cubic 3-fold. This is a very well-known classical object, see [Do15, AB15, BB12] for some historic perspective and applications.

If $n=5$ and $b_1=...=b_5=1$ then $\mathcal{L}=\mathcal{L}_{2,3}(1,...,1)$ that is the linear system of plane cubics through four general points. In this case the quotient is a del Pezzo surface of degree five.

1.4. **Moduli of weighted pointed curves.** In [Has03], Hassett introduced moduli spaces of weighted pointed curves. Given $g \geq 0$ and rational weight data $A[n]=(a_1,...,a_n)$, $0 < a_i \leq 1$, satisfying $2g-2+\sum_{i=1}^n a_i > 0$, the moduli space $\overline{\mathcal{M}}_{g,A[n]}$ parametrizes genus g nodal n-pointed curves $\{C,(x_1,...,x_n)\}$ subject to the following stability conditions:

- each x_i is a smooth point of C, and the points $x_{i_1},\dots,x_{i_k}$ are allowed to coincide only if $\sum_{j=1}^k a_{i_j} \leq 1$,
- the twisted dualizing sheaf $\omega_C(a_1x_1+\dots+a_nx_n)$ is ample.

In particular, $\overline{\mathcal{M}}_{g,A[n]}$ is one of the compactifications of the moduli space $\overline{\mathcal{M}}_{g,n}$ of genus g smooth n-pointed curves.

1.5. For fixed g,n, consider two collections of weight data $A[n],B[n]$ such that $a_i \geq b_i$ for any $i=1,...,n$. Then there exists a birational *reduction morphism*

$$\rho_{B[n],A[n]} : \overline{\mathcal{M}}_{g,A[n]} \to \overline{\mathcal{M}}_{g,B[n]}$$

associating to a curve $[C,s_1,...,s_n] \in \overline{\mathcal{M}}_{g,A[n]}$ the curve $\rho_{B[n],A[n]}([C,s_1,...,s_n])$ obtained by collapsing components of C along which $\omega_C(b_1s_1+...+b_ns_n)$ fails to be ample, where ω_C denotes the dualizing sheaf of C.

1.6. Furthermore, for any g, consider a collection of weight data $A[n]=(a_1,...,a_n)$ and a subset $A[r]:=(a_{i_1},...,a_{i_r}) \subset A[n]$ such that $2g-2+a_{i_1}+...+a_{i_r} > 0$. Then there exists a *forgetful morphism*

$$\pi_{A[n],A[r]} : \overline{\mathcal{M}}_{g,A[n]} \to \overline{\mathcal{M}}_{g,A[r]}$$

associating to a curve $[C,s_1,...,s_n] \in \overline{\mathcal{M}}_{g,A[n]}$ the curve $\pi_{A[n],A[r]}([C,s_1,...,s_n])$ obtained by collapsing components of C along which $\omega_C(a_{i_1}s_{i_1}+...+a_{i_r}s_{i_r})$ fails to be ample.

One of the most elegant aspects of the theory of rational pointed curves is the relation with rational normal curves and their projective geometry. This has been outlined by Kapranov in [Ka93]. Here below we briefly recall this, and his construction of $\overline{\mathcal{M}}_{0,n}$ as an iterated blow-up of $\mathbb{P}^{n-3}$.

Kapranov's blow-up construction. We follow [Ka93]. Let $(C, x_1, ..., x_n)$ be a genus zero n-pointed stable curve. The dualizing sheaf ω_C of C is invertible, see [Kn83]. By [Kn83, Corollaries 1.10 and 1.11] the sheaf $\omega_C(x_1+...+x_n)$ is very ample and has $n-1$ independent sections. Then it defines an embedding $\phi : C \rightarrow \mathbb{P}^{n-2}$. In particular, if $C \cong \mathbb{P}^1$ then $\deg(\omega_C(x_1 + ... + x_n)) = n-2$, $\omega_C(x_1 + ... + x_n) \cong \phi^*\mathcal{O}_{\mathbb{P}^{n-2}}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n-2)$, and $\phi(C)$ is a degree $n-2$ rational normal curve in $\mathbb{P}^{n-2}$. By [Ka93, Lemma 1.4] if $(C, x_1, ..., x_n)$ is stable the points $p_i = \phi(x_i)$ are in linear general position in $\mathbb{P}^{n-2}$.

This fact combined with a careful analysis of limits in $\overline{\mathcal{M}}_{0,n}$ of 1-parameter families contained in $\mathcal{M}_{0,n}$ are the key for the proof of the following theorem [Ka93, Theorem 0.1].

Theorem 1.8. *Let $p_1, ..., p_n \in \mathbb{P}^{n-2}$ be points in linear general position, and let $V_0(p_1, ..., p_n)$ be the scheme parametrizing rational normal curves through $p_1, ..., p_n$. Consider $V_0(p_1, ..., p_n)$ as a subscheme of the Hilbert scheme $\mathcal{H}$ parametrizing subschemes of $\mathbb{P}^{n-2}$. Then*

- $V_0(p_1, ..., p_n) \cong \mathcal{M}_{0,n}$.
- *Let $V(p_1, ..., p_n)$ be the closure of $V_0(p_1, ..., p_n)$ in $\mathcal{H}$. Then $V(p_1, ..., p_n) \cong \overline{\mathcal{M}}_{0,n}$.*

Kapranov's construction allows to translate many questions about $\overline{\mathcal{M}}_{0,n}$ into statements on linear systems on $\mathbb{P}^{n-3}$. Consider a general line $L_i \subset \mathbb{P}^{n-2}$ through p_i. There exists a unique rational normal curve C_{L_i} through $p_1, ..., p_n$, and with tangent direction L_i in p_i. Let $[C, x_1, ..., x_n] \in \overline{\mathcal{M}}_{0,n}$ be a stable curve, and let $\Gamma \in V_0(p_1, ..., p_n)$ be the corresponding rational normal curve. Since $p_i \in \Gamma$ is a smooth point, by considering the tangent line $T_{p_i}\Gamma$ we get a morphism

$$f_i : \begin{array}{ccc} \overline{\mathcal{M}}_{0,n} & \longrightarrow & \mathbb{P}^{n-3} \\ [C, x_1, ..., x_n] & \longmapsto & T_{p_i}\Gamma \end{array} \tag{1.9}$$

Furthermore, f_i is birational and defines an isomorphism on $\mathcal{M}_{0,n}$. The birational maps $f_j \circ f_i^{-1}$

$$\begin{array}{ccccc} & & \overline{\mathcal{M}}_{0,n} & & \\ & \stackrel{f_i}{\swarrow} & & \stackrel{f_j}{\searrow} & \\ \mathbb{P}^{n-3} & & \stackrel{f_j \circ f_i^{-1}}{\dashrightarrow} & & \mathbb{P}^{n-3} \end{array}$$

are standard Cremona transformations of $\mathbb{P}^{n-3}$ [Ka93, Proposition 2.12]. For any $i = 1, ..., n$ the class Ψ_i is the line bundle on $\overline{\mathcal{M}}_{0,n}$ whose fiber on $[C, x_1, ..., x_n]$ is the tangent line $T_{p_i}C$. From the previous description we see that the line bundle Ψ_i induces the birational morphism $f_i : \overline{\mathcal{M}}_{0,n} \rightarrow \mathbb{P}^{n-3}$, that is $\Psi_i = f_i^*\mathcal{O}_{\mathbb{P}^{n-3}}(1)$. In [Ka93] Kapranov proved that Ψ_i is big and globally generated, and that the birational morphism f_i is an iterated blow-up of the projections from p_i of the points $p_1, ..., \hat{p_i}, ...p_n$ and of all strict transforms of the linear spaces that they generate, in order of increasing dimension.

Construction 1.10. Fix $(n-1)$-points $p_1, ..., p_{n-1} \in \mathbb{P}^{n-3}$ in linear general position:

(1) Blow-up the points $p_1, ..., p_{n-1}$,
(2) Blow-up the strict transforms of the lines $\langle p_{i_1}, p_{i_2}\rangle$, $i_1, i_2 = 1, ..., n-1$,
$\vdots$
(k) Blow-up the strict transforms of the $(k-1)$-planes $\langle p_{i_1}, ..., p_{i_k}\rangle$, $i_1, ..., i_k = 1, ..., n-1$,
$\vdots$

$(n-4)$ Blow-up the strict transforms of the $(n-5)$-planes $\langle p_{i_1},...,p_{i_{n-4}}\rangle$, $i_1,...,i_{n-4} = 1,...,n-1$.

Now, consider the moduli spaces of weighted pointed curves $X_k[n] := \overline{\mathcal{M}}_{0,A[n]}$ for $k = 1,...,n-4$, such that

- $a_i + a_n > 1$ for $i = 1,...,n-1$,
- $a_{i_1} + ... + a_{i_r} \leq 1$ for each $\{i_1,...,i_r\} \subset \{1,...,n-1\}$ with $r \leq n-k-2$,
- $a_{i_1} + ... + a_{i_r} > 1$ for each $\{i_1,...,i_r\} \subset \{1,...,n-1\}$ with $r > n-k-2$.

While apologizing for the new notation, we try to justify it by remarking that $X_k[n]$ is isomorphic to the variety obtained at the k^{th} step of the blow-up construction. The composition of these blow-up morphism here above is the morphism $f_n : \overline{\mathcal{M}}_{0,n} \rightarrow \mathbb{P}^{n-3}$ induced by the psi-class Ψ_n. Identifying $\overline{\mathcal{M}}_{0,n}$ with $V(p_1,...,p_n)$, and fixing a general $(n-3)$-plane $H \subset \mathbb{P}^{n-2}$, the morphism f_n associates to a curve $C \in V(p_1,...,p_n)$ the point $T_{p_n}C \cap H$.

In [Has03, Section 2.1.2] Hassett considers a natural variation of the moduli problem of weighted pointed rational stable curves by considering weights of the type $\widetilde{A}[n] = (a_1,...,a_n)$ such that $a_i \in \mathbb{Q}$, $0 < a_i \leq 1$ for any $i = 1,...,n$, and $\sum_{i=1}^{n} a_i = 2$.

By [Has03, Section 2.1.2] we may construct an explicit family of such weighted curves $\mathcal{C}(\widetilde{A}) \rightarrow \overline{\mathcal{M}}_{0,n}$ over $\overline{\mathcal{M}}_{0,n}$ as an explicit blow-down of the universal curve over $\overline{\mathcal{M}}_{0,n}$. Furthermore, if $a_i < 1$ for any $i = 1,...,n$ we may interpret the geometric invariant theory quotient $(\mathbb{P}^1)^n /\!\!/ PGL(2)$ with respect to the linearization $\mathcal{O}(a_1,...,a_n)$ as the moduli space $\overline{\mathcal{M}}_{0,\widetilde{A}[n]}$ associated to the family $\mathcal{C}(\widetilde{A})$.

Remark 1.11. Note that we may interpret the GIT quotient $(\mathbb{P}^1)^n(b) /\!\!/ PGL(2)$ as a moduli space $\overline{\mathcal{M}}_{0,\widetilde{A}[n]}$ by taking the weights $a_i = \frac{2}{|b|}b_i$. Conversely, given the space $\overline{\mathcal{M}}_{0,\widetilde{A}[n]}$ with $(a_1,...,a_n) = (\frac{\alpha_1}{\beta_1},...,\frac{\alpha_n}{\beta_n})$ such that $\sum_{i=1}^{n} a_i = 2$ we may consider the GIT quotient $(\mathbb{P}^1)^n(b) /\!\!/ PGL(2)$ with $b_i = a_i M$, where $M = LCM(\beta_i)$.

Remark 1.12. Let $\overline{\mathcal{M}}_{0,\widetilde{A}[n]}$ be a moduli space with weights $\widetilde{a}_i$ summing up to two, and let $\overline{\mathcal{M}}_{0,A[n]}$ be a moduli space with weights $a_i \geq \widetilde{a}_i$ for any $i = 1,...,n$. By [Has03, Theorem 8.3] there exists a reduction morphism $\rho_{\widetilde{A}[n],A[n]} : \overline{\mathcal{M}}_{0,A[n]} \rightarrow \overline{\mathcal{M}}_{0,\widetilde{A}[n]}$ operating as the standard reduction morphisms in 1.5.

Proposition 1.13. *Let $\phi_{\mathcal{L}} : \mathbb{P}^{n-3} \dashrightarrow (\mathbb{P}^1)^n(b) /\!\!/ PGL(2) \subset \mathbb{P}(H^0(\mathbb{P}^{n-3},\mathcal{L})^*)$ be the rational map in Theorem 1.2, and let $f_i : \overline{\mathcal{M}}_{0,n} \rightarrow \mathbb{P}^{n-3}$ in (1.9). Then there exits a reduction morphism $\rho : \overline{\mathcal{M}}_{0,n} \rightarrow (\mathbb{P}^1)^n(b) /\!\!/ PGL(2)$ making the following diagram*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n} & \overset{\rho}{\searrow} & \\ {\scriptstyle f_n}\downarrow & & \\ \mathbb{P}^{n-3} & \overset{\phi_{\mathcal{L}}}{\dashrightarrow} & (\mathbb{P}^1)^n(b) /\!\!/ PGL(2) \end{array}$$

commutative.

Proof. As observed in [Ku03] via the theory of associated points, each point $x \in \mathbb{P}^{n-3}$ which is linearly general with respect to the $n-1$ fixed points in $\mathbb{P}^{n-3}$ defines a configuration of points on the unique rational normal curve of degree $n-3$ passing through the $(n-1)+1 = n$ points. Moreover, this configuration is the image of x in $(\mathbb{P}^1)^n(b)/\!\!/PGL(2)$ via $\phi_{\mathcal{L}}$. Via the identification $V_0(p_1, \dots, p_n) \cong \mathcal{M}_{0,n}$ of Theorem 1.8, one easily obtains the claim. □

From the next section on, we will always omit the vector b of the polarization since it will always be either $(1, \dots, 1)$ or $(2, \dots, 2)$ when we consider an odd number of points, according to (1.0).

2. Birational geometry of GIT quotients of $(\mathbb{P}^1)^n$ and the F-conjecture

In this section we will study some birational aspects of the geometry of the GIT quotients we introduced. In particular, we will describe their Mori cone and show that its extremal rays are generated by 1-dimensional strata of the boundary.

Let us now fix a suitable notation for the GIT quotients. We will denote by Σ_m the GIT quotient $(\mathbb{P}^1)^{m+3}/\!\!/PGL(2)$.

2.1. **The odd dimensional case.**

We start by studying the case where the GIT quotients parametrize an even number of points, that is Σ_{2g-1}, for $g \geq 2$. This reveals to be less complicated than the even dimensional one, that we will consider eventually. In any case it is worth looking at it. In fact it will allow us to prove another interesting result in Theorem 2.7 that, as far as we know, does not seem to have appeared in the literature. First of all, we need a preliminary result.

Let us define $\mathcal{L}_{2g-1} := \mathcal{L}_{2g-1,g}(g-1, ..., g-1)$ as the linear system of degree g forms on $\mathbb{P}^{2g-1}$ vanishing with multiplicity g at $2g+1$ general points $p_1, ..., p_{2g+1} \in \mathbb{P}^{2g-1}$. In [Ku00, Theorem 4.1] Kumar proved that $\mathcal{L}_{2g-1}$ induces a birational map

$$\sigma_g : \mathbb{P}^{2g-1} \dashrightarrow \mathbb{P}(H^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1})^*) \tag{2.2}$$

and that the GIT quotient Σ_{2g-1}, that is the Segre g-variety, in Kumar's paper [Ku00], is obtained as the closure of the image of σ_g in $\mathbb{P}(H^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1})^*)$.

Proposition 2.3. *Let $p_1, ..., p_{2g+1} \in \mathbb{P}^{2g-1}$ be points in general position, and let X^{2g-1}_{g-2} be the variety obtained the step g of Construction 1.10. Then we have the following commutative diagram*

$$\begin{array}{ccc} X^{2g-1}_{g-2} & & \\ \downarrow f & \searrow^{\widetilde{\sigma}_g} & \\ \mathbb{P}^{2g-1} & \overset{\sigma_g}{\dashrightarrow} & \Sigma_{2g-1} \subset \mathbb{P}^N. \end{array}$$

That is, the blow-up morphism $f : X^{2g-1}_{g-2} \rightarrow \mathbb{P}^{2g-1}$ resolves the rational map σ_g.

Proof. By Construction 1.10 the blow-up X^{2g-1}_{g-2} may be interpreted as the moduli space $\overline{\mathcal{M}}_{0,A[2g+2]}$ with $A[2g+2] = \left(\frac{1}{g}, ..., \frac{1}{g}, 1\right)$.

Furthermore, by Remark 1.11 Σ_{2g-1} is the singular moduli space $\overline{\mathcal{M}}_{0,\widetilde{A}[2g+2]}$ with weights $A[2g+2] = \left(\frac{1}{g+1}, ..., \frac{1}{g+1}, \frac{1}{g+1}\right)$, and by Proposition 1.13 the morphism $\widetilde{\sigma}_g : X^{2g-1}_{g-2} \rightarrow \Sigma_g$ is exactly the reduction morphism $\rho_{\widetilde{A}[2g+2],A[2g+2]} : \overline{\mathcal{M}}_{0,A[2g+2]} \rightarrow \overline{\mathcal{M}}_{0,\widetilde{A}[2g+2]}$ defined just by lowering the weights. □

This fairly simple result has some interesting consequences, that we will illustrate in the rest of this section. Anyway, first we need a technical, and probably well-known, lemma.

Lemma 2.4. *Let X be a normal projective variety and $f: X \dashrightarrow Y$ a birational map of projective varieties not contracting any divisor. Then $K_X \sim f_*^{-1}K_Y$.*

Proof. Since X is normal f is defined in codimension one. Let $U \subseteq X$ be a dense open subset whose complementary set has codimension at least two where f is defined. Since f does not contract any divisor then its exceptional set has at least codimension two, hence we may assume that $f_{|U}$ is an isomorphism onto its image. Therefore $K_{X|U} \sim (f_*^{-1}K_Y)_{|U}$, and since U is at least of codimension two we get the statement. $\Box$

The first consequence of Proposition 2.3 is the following.

Lemma 2.5. *The canonical sheaf of $\Sigma_{2g-1} \subset \mathbb{P}^N$ is $K_{\Sigma_{2g-1}} \cong \mathcal{O}_{\Sigma_{2g-1}}(-2)$.*

Proof. Let X_1^{2g-1} be the blow-up of $\mathbb{P}^{2g-1}$ at $2g+1$ general points $p_1, ..., p_{2g+1}$, and let $f_g : X_1^{2g-1} \dashrightarrow \Sigma_{2g-1}$ be the birational map induced by the morphism $\widetilde{\sigma}_g : X_{g-2}^{2g-1} \to \Sigma_{2g-1}$ in Proposition 2.3. We will denote by H the pull-back in X_1^{2g-1} of the hyperplane section of $\mathbb{P}^{2g-1}$, and by $E_1, ..., E_{2g+1}$ the exceptional divisors.

The interpretation of $\widetilde{\sigma}_g$ as a reduction morphism in the proof of Proposition 2.3 yields that f_g does not contract any divisor. Indeed f_g contracts just the strict transforms of the $(g-1)$-planes generated by g of the blown-up points. Since σ_g is induced by the linear system $\mathcal{L}_{2g-1}$, the pull-back via f_g of a hyperplane section of Σ_{2g-1} is the strict transform of a hypersurface of degree g in $\mathbb{P}^{2g-1}$ having multiplicity $g-1$ at $p_1, ..., p_{2g+1}$. Since

$$-K_{X_1^{2g-1}} \sim 2gH - (2g-2)\sum_{i=1}^{2g+1} E_i = 2\left(gH - (g-1)\sum_{i=1}^{2g+1} E_i\right)$$

we have that $-K_{X_1^{2g-1}} \sim f_g^*\mathcal{O}_{\Sigma_{2g-1}}(2)$. Now, in order to conclude it is enough to apply Lemma 2.4 to the birational map $f_g : X_1^{2g-1} \dashrightarrow \Sigma_{2g-1}$. $\Box$

Proposition 2.6. *The divisor class group of Σ_{2g-1} is $\mathrm{Cl}(\Sigma_{2g-1}) \cong \mathbb{Z}^{2g+2}$. Furthermore $\mathrm{Pic}(\Sigma_{2g-1})$ is torsion free.*

Proof. The classical case of the Segre cubic $\Sigma_3 \subset \mathbb{P}^4$ has been treated in [Hu96, Section 3.2.2]. Hence we may assume that $g \geq 2$.

Let $Y = H \cap \Sigma_{2g-1}$ be a general hyperplane section of Σ_{2g-1}. Since $\dim(\mathrm{Sing}(\Sigma_{2g-1})) = 0$ by Bertini's theorem, see [Har77, Corollary 10.9] and [Har77, Remark 10.9.2], Y is smooth. Note that $X = (\widetilde{\sigma}_g)^{-1}(Y) \subset X_{g-2}^{2g-1}$ is the strict transform via f of a general element of the linear system $\mathcal{L}_{2g-1}$ inducing σ_g. Therefore, X is smooth and $\widetilde{\sigma}_{g|X} : X \to Y$ is a divisorial contraction between smooth varieties.

Since $\dim(X) > 2$, the Grothendieck-Lefschetz theorem (see for instance [Ba78, Theorem A]) yields that the natural restriction morphism $\mathrm{Pic}(X_{g-2}^{2g-1}) \to \mathrm{Pic}(X)$ is an isomorphism. Therefore $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_{g-2}^{2g-1}) \cong \mathbb{Z}^h$, with $h = 1 + (2g+1) + \binom{2g+1}{2} + ... + \binom{2g+1}{g}$. By the interpretation of $\widetilde{\sigma}_g$ as a reduction morphism in Proposition 2.3 we know that the codimension one part of the exceptional locus of $\widetilde{\sigma}_g$ consists of the exceptional divisors the blown-up positive dimensional linear subspaces of $\mathbb{P}^{2g-1}$. Therefore, $\mathrm{Pic}(Y) \cong \mathbb{Z}^{2g+2}$. Indeed $\mathrm{Pic}(Y)$ is generated by the images via $\widetilde{\sigma}_{g|X}$ of the pull-back of the hyperplane section of $\mathbb{P}^{2g-1}$ and of the exceptional divisors over the $2g+1$ blown-up points. However, since Σ_{2g-1} is not smooth we can not conclude by the Grothendieck-Lefschetz theorem that its Picard group is isomorphic to $\mathbb{Z}^{2g+2}$ as well.

On the other hand, thanks to a version for normal varieties of the Grothendieck-Lefschetz theorem [RS06, Theorem 1], we get that $\mathrm{Cl}(\Sigma_{2g-1}) \cong \mathrm{Cl}(Y)$, and since Y is smooth we have $\mathrm{Cl}(Y) \cong \mathrm{Pic}(Y)$.

Finally, by [Kl66, Corollary 2] we have that, even when the ambient variety is singular, the restriction morphism in the Grothendieck-Lefschetz theorem is injective. Therefore, we have an injective morphism $\mathrm{Pic}(\Sigma_{2g-1}) \hookrightarrow \mathrm{Pic}(Y) \cong \mathbb{Z}^{2g+2}$, and hence $\mathrm{Pic}(\Sigma_{2g-1})$ is torsion free. $\square$

This allows us to compute the Picard group of Σ_{2g-1}. By different methods, this computation was also carried out in [MS16].

Proposition 2.7. *The Picard group of the GIT quotient* $\Sigma_{2g-1} \subset \mathbb{P}^N$ *is* $\mathrm{Pic}(\Sigma_{2g-1}) \cong \mathbb{Z}\langle H\rangle$, *where* H *is the hyperplane class. In particular, we have that* $\mathrm{Nef}(\Sigma_{2g-1}) \cong \mathbb{R}^{\geq 0}$.

Proof. Let $\widetilde{\sigma}_g : X_{g-2}^{2g-1} \to \Sigma_{2g-1} \subset \mathbb{P}^N$ be the resolution of the birational map $\sigma_g : \mathbb{P}^{2g+1} \dashrightarrow \Sigma_{2g-1}$ in Proposition 2.3. Note that $\mathrm{Pic}(X_{g-2}^{2g-1}) \cong \mathbb{Z}^{\rho(X_{g-2}^{2g-1})}$, where $\rho(X_{g-2}^{2g-1}) = 1 + (2g+1) + \binom{2g+1}{2} + ... + \binom{2g+1}{g-1}$, and that $\widetilde{\sigma}_g$ contracts all the exceptional divisors of the blow-up $f : X_{g-2}^{2g-1} \to \mathbb{P}^{2g-1}$ over positive dimensional linear subspaces. Now, let $E_i \subset X_0^{2g-1}$ be the exceptional divisor over the point $p_i \in \mathbb{P}^{2g-1}$, and let $\widetilde{E}_i$ its strict transform in X_{g-2}^{2g-1}. Furthermore, denote by e_i the class of a general line in $E_i \subset X_0^{2g-1}$.

By Proposition 2.3 we know that, besides the divisors over the positive dimensional linear subspaces, $\widetilde{\sigma}_g$ also contracts the strict transforms $S_1, ..., S_r$, with $r = \binom{2g+1}{g}$, in X_{g-2}^{2g-1} of the $(g-1)$-planes $\langle p_{i_1}, ..., p_{i_g}\rangle$. Note that the contraction of S_i is given by the contraction of the strict transforms of the degree $g-1$ rational normal curves in $\langle p_{i_1}, ..., p_{i_g}\rangle$ passing through $p_{i_1}, ..., p_{i_g}$. In fact, any degree $g-1$ rational normal curve through $p_{i_1}, ..., p_{i_g}$ is contained in $\langle p_{i_1}, ..., p_{i_g}\rangle$ and a morphism contracts S_i to a point if and only if it contracts all these curves to a point. We may write the class in the cone $N_1(X_{g-2}^{2g-1})_{\mathbb{R}} \cong \mathbb{R}^{\rho(X_{g-2}^{2g-1})}$ of the strict transform of such a rational normal curve as

$$(g-1)l - e_{i_1} - ... - e_{i_g} \tag{2.8}$$

where l is the pull-back of a general line in $\mathbb{P}^{2g-1}$. Note that the classes in (2.8) generate the hyperplane

$$\left\{ gl + \sum_{j=1}^{2g+1} (g-1)e_i = 0 \right\}$$

in $N_1(X_1^{2g-1})_{\mathbb{R}} \cong \mathbb{R}^{2g+2}$, where X_1^{2g-1} is the blow-up of $\mathbb{P}^{2g-1}$ at $p_1, ..., p_{2g+1}$. Therefore, the birational morphism $\widetilde{\sigma}_g : X_{g-2}^{2g-1} \to \Sigma_{2g-1}$ contracts the locus spanned by classes of curves generating a subspace of dimension $\binom{2g+1}{2} + ... + \binom{2g+1}{g-1} + 2g+1$ of $N_1(X_{g-2}^{2g-1})_{\mathbb{R}} \cong \mathbb{R}^{\rho(X_{g-2}^{2g-1})}$, and then

$$\rho(\Sigma_{2g-1}) = \rho(X_{g-2}^{2g-1}) - \left(\binom{2g+1}{2} + ... + \binom{2g+1}{g-1} + 2g+1 \right) = 1.$$

Since by Proposition 2.6 the Picard group of Σ_{2g-1} is torsion free and $\Sigma_{2g-1} \subset \mathbb{P}^N$ contains lines we conclude that $\mathrm{Pic}(\Sigma_{2g-1}) = \mathbb{Z}\langle H \rangle$, where H is the hyperplane class. □

2.9. **The even dimensional case.** In this section we investigate the geometry of the counterpart of Σ_{2g-1}, that is the GIT quotient $(\mathbb{P}^1)^n(b)/\!\!/PGL(2)$ with $b_i = 1$ for $i = 1, ..., n$ and n odd. The quotient here has even dimension $2g$, for a positive integer g, and hence we will denote it by Σ_{2g}. Of course, we have $n = 2g + 3$. Note that in this case all the semistable points are indeed stable, and then Σ_{2g} is smooth. Recall that, by Theorem 1.2, $\Sigma_{2g} \subset \mathbb{P}(H^0(\mathbb{P}^{2g}, \mathcal{L}_{2g})^*)$ is the closure of the image of the rational map induced by the linear system $\mathcal{L}_{2g}$, where we define $\mathcal{L}_{2g}$ as given by degree $2g + 1$ hypersurfaces in $\mathbb{P}^{2g}$ with multiplicity $2g - 1$ at p_i for $i = 1, ..., 2g + 2$. We will denote by $\mu_g : \mathbb{P}^{2g} \dashrightarrow \Sigma_{2g} \subset \mathbb{P}(H^0(\mathbb{P}^{2g}, \mathcal{L}_{2g})^*) = \mathbb{P}^N$ this rational map.

This case is far more complicated and interesting than the odd dimensional one, and we will need a good amount of preliminary results.

Proposition 2.10. *Let X^{2g}_{g-1} be the variety obtained at the step g of Construction 1.10. Then there exists a morphism $\widetilde{\mu}_g : X^{2g}_{g-1} \to X(b)/\!\!/PGL(2)$ making the following diagram*

$$\begin{array}{ccc} X^{2g}_{g-1} & & \\ {\scriptstyle f}\downarrow & \searrow^{\widetilde{\mu}_g} & \\ \mathbb{P}^{2g} & \overset{\mu_g}{\dashrightarrow} & \Sigma_{2g} \subset \mathbb{P}^N \end{array}$$

commute, where $f : X^{2g}_{g-1} \to \mathbb{P}^{2g}$ is the blow-up morphism.

Proof. By Construction 1.10 $X^{2g}_{g-1} \cong \overline{\mathcal{M}}_{0,A[2g+3]}$ with $A[2g + 3] = \left(\frac{2}{2g+2}, ..., \frac{2}{2g+2}, 1\right)$, and by Remark 1.11 $\Sigma_{2g} \cong \overline{\mathcal{M}}_{0,\widetilde{A}[2g+3]}$ with $\widetilde{A}[2g+3] = \left(\frac{2}{2g+3}, ..., \frac{2}{2g+3}\right)$. Therefore we may take $\widetilde{\mu}_g = \rho_{\widetilde{A}[2g+3],A[2g+3]} : \overline{\mathcal{M}}_{0,A[2g+3]} \to \overline{\mathcal{M}}_{0,\widetilde{A}[2g+3]}$ and argue as in the proof of Proposition 2.3. □

Remark 2.11. Note that arguing as in the proof of Proposition 4.3 we see that the standard Cremona transformation of $\mathbb{P}^{2g}$ induces an automorphism of $\Sigma_{2g} \subset \mathbb{P}^N$. Indeed the automorphism induced by the Cremona and the group S_{2g+2} permuting the points $p_1, ..., p_{2g+2} \in \mathbb{P}^{2g}$ generates the symmetric group S_{2g+3} acting on Σ_{2g} by permuting the marked points.

2.12. **Linear subspaces of dimension g in $\Sigma_{2g} \subset \mathbb{P}^N$.** Let $H_I = H_{i_1,...,i_{g+1}}$ be the g-plane generated by $p_{i_1}, ..., p_{i_{g+1}} \in \mathbb{P}^{2g}$. The linear system $\mathcal{L}_{2g|H_I}$ is given by the hypersurfaces of degree $2g + 1$ in $H_I \cong \mathbb{P}^g$ with multiplicity $2g - 1$ at $p_{i_1}, ..., p_{i_{2g+2}}$. Now, let $H_J \subset H_I$ be a $(g - 1)$-plane generated by the points indexed by a subset $J \subset I$ with $|J| = g$. Then the general element of $\mathcal{L}_{2g|H_I}$ must contain H_J with multiplicity $g(2g-1)-(g-1)(2g+1) = 1$. This means that the divisor D equals $\bigcup_{\{J \subset I, \,|\, |J|=g\}} H_J$. Note that $\deg(D) = g + 1$ and $\mathrm{mult}_{p_{i_j}} D = g$ for any $i_j \in I$. Therefore, $\mathcal{L}_{2g|H_I}$ is the linear system of hypersurfaces of degree $2g + 1 - (g + 1) = g$ in $H_I \cong \mathbb{P}^g$ having multiplicity $2g - 1 - g = g - 1$ at p_{i_j} for any $i_j \in I$. This is the linear system of the standard Cremona transformation of $\mathbb{P}^g$. Therefore, $\mu_{g|H_I}(H_I) \subset \Sigma_{2g} \subset \mathbb{P}^N$ is a linear subspace of dimension g.

Now, let E_I^{g-1} be the exceptional divisor over the strict transform of a $(g-1)$-plane of $\mathbb{P}^{2g}$ generated by g of the p_i's. Note that the reduction morphism $\widetilde{\mu}_g : X_{g-1}^{2g} \cong \overline{\mathcal{M}}_{0,A[2g+3]} \rightarrow \Sigma_{2g} \cong \overline{\mathcal{M}}_{0,\widetilde{A}[2g+3]}$ in Proposition 2.10 contracts E_I^{g-1} to a g-plane $\widetilde{\mu}_g(E_I^{g-1}) \subset \Sigma_{2g} \subset \mathbb{P}^N$.

2.13. We found $\binom{2g+2}{g+1} + \binom{2g+2}{g}$ linear subspaces of dimension g in $\Sigma_{2g} \subset \mathbb{P}^N$. We will denote by

$$\mathbf{C} = \{\gamma_1, ..., \gamma_c\},\ \mathbf{D} = \{\delta_1, ..., \delta_d\}$$

where $c = \binom{2g+2}{g+1}$ and $d = \binom{2g+2}{g}$, the families of the g-planes coming from the H_I's and the E_I^{g-1}'s respectively. Note that:

- on any δ_i we have $g+2$ distinguished points determined by the intersections with the $g-1$ of the γ_j's coming from g-planes in $\mathbb{P}^{2g}$ containing the $(g-1)$-plane associated to δ_i,
- on any γ_i, say coming from H_I, we have $g+2$ distinguished points as well: $g+1$ of them coming from the exceptional divisors E_J^{g-1} with $J \subset I$, $|J| = g$, and another one determined as the image of the point $H_I \cap H_{I^c}$, where $I_c = \{1, ..., 2g+2\} \setminus I$.

Note that the $g+2$ distinguished points on γ_i and δ_j are in linear general position. This is clear for the γ_i's. In order to see that it is true for the δ_j's as well, notice that way map any δ_j to any γ_i just by acting with a suitable permutation in S_{2g+3} involving the standard Cremona transformation as in Remark 2.11.

Finally note that on any γ_i we have $g+2$ distinguished points and one of them is the intersection point of γ_i with a γ_j. We call such a γ_j the *complementary* of γ_i, and we denote it by $\gamma_j = \gamma_{i^c}$. Therefore γ_i and γ_{i^c} determine $2(g+2)-1$ distinguished points for any $i = 1, ..., \frac{1}{2}\binom{2g+2}{g+1}$. Summing up we have

$$\frac{1}{2}\binom{2g+2}{g+1}(2(g+2)-1) = \frac{(2g+3)!}{2((g+1)!)^2} \tag{2.14}$$

distinguished points in the configuration of g-planes $\mathbf{C} \cup \mathbf{D}$.

Remark 2.15. Arguing as in 4.9 and using the description of the sections of $\mathcal{L}_{2g}$ in [Ku03, Section 3.3] we get that in $\Sigma_{2g} \subset \mathbb{P}(H^0(\mathbb{P}^{2g}, \mathcal{L}_{2g})^*) = \mathbb{P}^N$ there are $N+2$ of the distinguished points described in 2.13 that are in linear general position in $\mathbb{P}^N$.

2.16. Let us consider the following diagram:

$$\begin{array}{ccc} & X_{g-1}^{2g} & \\ {\scriptstyle h}\swarrow & & \searrow{\scriptstyle \widetilde{\mu}_g} \\ X_1^{2g} & \overset{\psi}{\dashrightarrow} & \Sigma_{2g} \end{array}$$

where $h : X_{g-1}^{2g} \rightarrow X_1^{2g}$ is the composition of blow-ups in Construction 1.10. Note that, by interpreting the varieties appearing in the diagram as moduli spaces of weighted pointed curves, and h and $\widetilde{\mu}_g$ as reduction morphisms as in the proof of Proposition 2.10, we see that the rational map $\psi : X_1^{2g} \dashrightarrow \Sigma_{2g}$ is a composition of flips of strict transforms of linear subspaces generated by subsets of $\{p_1, ..., p_{2g+2}\}$ up to dimension $g-1$. In particular, ψ is an isomorphism in codimension one and $\mathrm{Pic}(\Sigma_{2g}) \cong \mathbb{Z}^{2g+3}$ is the free abelian group generated by the strict transforms via ψ of $H, E_1, ..., E_{2g+2}$. The Picard groups was also computed in [MS16]. Our computation has the advantage of showing explicit generators.

As Han-Bom Moon has observed in a personal communication, Σ_{2g-1} is not $\mathbb{Q}$-factorial, and many boundary divisors are just Weil divisors. This is basically the reason why the rank of $\mathrm{Pic}(\Sigma_{2g-1})$ drops so dramatically from the rank of $\mathrm{Pic}(\Sigma_{2g})$.

Lemma 2.17. *The effective cone* $\mathrm{Eff}(X_1^{2g}) \subset \mathbb{R}^{2g+3}$ *of* X_1^{2g} *is the polyhedral cone generated by the classes of the exceptional divisors* E_i *and of the strict transforms* $H - \sum_{i\in I} E_i$*,* $I \subset \{1, ..., 2g+2\}$*, with* $|I| = 2g$*, of the hyperplanes generated by* $2g$ *of the* p_i*'s.*

Proof. The faces of $\mathrm{Eff}(X_1^{2g})$ are described in [CT06, Lemma 4.24], and in [BDP16, Corollary 2.5]. It is straightforward to compute the extremal rays of $\mathrm{Eff}(X_1^{2g})$ by intersecting its faces. □

Lemma 2.18. *The canonical sheaf of* $\Sigma_{2g} \subset \mathbb{P}^N$ *is isomorphic to* $\mathcal{O}_{\Sigma_{2g}}(-1)$*.*

Proof. Let X_1^{2g} be the blow-up of $\mathbb{P}^{2g}$ at $2g+2$ general points $p_1, ..., p_{2g+2}$, and let $h_g : X_1^{2g} \dashrightarrow \Sigma_{2g}$ be the birational map induced by the morphism $\widetilde{\mu}_g : X_{g-1}^{2g} \to \Sigma_{2g}$ in Proposition 2.10. As usual we will denote by H the pull-back to X_1^{2g} of the hyperplane section of $\mathbb{P}^{2g}$, and by $E_1, ..., E_{2g+2}$ the exceptional divisors. In order to conclude, it is enough to note that

$$-K_{X_1^{2g}} \sim (2g+1)H - (2g-1)\sum_{i=1}^{2g+2} E_i$$

is an element of the linear system $\mathcal{L}_{2g}$ inducing $\mu_g : \mathbb{P}^{2g} \dashrightarrow \Sigma_{2g}$, and to argue as in the proof of Lemma 2.5. □

As we have seen in 2.16,t $\mathrm{Pic}(\Sigma_{2g}) \cong \mathbb{Z}^{2g+3}$. In the next section our aim will be to describe the cones of curves and divisors of Σ_{2g}.

2.19. **Mori Dream Spaces and chamber decomposition.** Let X be a normal projective variety. We denote by $N^1(X)$ the real vector space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence. The *nef cone* of X is the closed convex cone $\mathrm{Nef}(X) \subset N^1(X)$ generated by classes of nef divisors. The *movable cone* of X is the convex cone $\mathrm{Mov}(X) \subset N^1(X)$ generated by classes of *movable divisors*. These are Cartier divisors whose stable base locus has codimension at least two in X. The *effective cone* of X is the convex cone $\mathrm{Eff}(X) \subset N^1(X)$ generated by classes of *effective divisors*. We have inclusions:

$$\mathrm{Nef}(X) \ \subset\ \overline{\mathrm{Mov}(X)} \ \subset\ \overline{\mathrm{Eff}(X)}.$$

We say that a birational map $f : X \dashrightarrow X'$ into a normal projective variety X' is a *birational contraction* if its inverse does not contract any divisor. We say that it is a *small* $\mathbb{Q}$*-factorial modification* if X' is $\mathbb{Q}$-factorial and f is an isomorphism in codimension one. If $f : X \dashrightarrow X'$ is a small $\mathbb{Q}$-factorial modification, then the natural pullback map $f^* : N^1(X') \to N^1(X)$ sends $\mathrm{Mov}(X')$ and $\mathrm{Eff}(X')$ isomorphically onto $\mathrm{Mov}(X)$ and $\mathrm{Eff}(X)$, respectively. In particular, we have $f^*(\mathrm{Nef}(X')) \subset \overline{\mathrm{Mov}(X)}$.

Definition 2.20. A normal projective $\mathbb{Q}$-factorial variety X is called a *Mori dream space* if the following conditions hold:

- Pic (X) is finitely generated,
- Nef (X) is generated by the classes of finitely many semi-ample divisors,
- there is a finite collection of small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$, such that each X_i satisfies the second condition above, and

$$\mathrm{Mov}\,(X) \ = \ \bigcup_i f_i^*(\mathrm{Nef}\,(X_i)).$$

The collection of all faces of all cones $f_i^*(\mathrm{Nef}\,(X_i))$ above forms a fan which is supported on $\mathrm{Mov}(X)$. If two maximal cones of this fan, say $f_i^*(\mathrm{Nef}\,(X_i))$ and $f_j^*(\mathrm{Nef}\,(X_j))$, meet along a facet, then there exists a commutative diagram:

$$\begin{array}{ccc} X_i & \overset{\varphi}{\dashrightarrow} & X_j \\ & {\scriptstyle h_i}\searrow \quad \swarrow{\scriptstyle h_j} & \\ & Y & \end{array}$$

where Y is a normal projective variety, φ is a small modification, and h_i and h_j are small birational morphisms of relative Picard number one. The fan structure on $\mathrm{Mov}(X)$ can be extended to a fan supported on $\mathrm{Eff}(X)$ as follows.

Definition 2.21. Let X be a Mori dream space. We describe a fan structure on the effective cone $\mathrm{Eff}(X)$, called the *Mori chamber decomposition*. We refer to [HK00, Proposition 1.11] and [Ok16, Section 2.2] for details. There are finitely many birational contractions from X to Mori dream spaces, denoted by $g_i : X \to Y_i$. The set $\mathrm{Exc}(g_i)$ of exceptional prime divisors of g_i has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones $\mathcal{C}$ of the Mori chamber decomposition of $\mathrm{Eff}(X)$ are of the form:

$$\mathcal{C}_i \ = \mathrm{Cone}\ \Big(\ g_i^*\big(\mathrm{Nef}(Y_i)\big)\ ,\ \mathrm{Exc}(g_i)\ \Big).$$

We call $\mathcal{C}_i$ or its interior $\mathcal{C}_i^\circ$ a *maximal chamber* of $\mathrm{Eff}(X)$.

Let $m > 1$ be an integer. Let X^m_{m+2} be the blow-up of $\mathbb{P}^m$ at $m+2$ general points. It is well-known that X^m_{m+2} is a Mori Dream Space [CT06, Theorem 1.3], [AM16, Theorem 1.3]. In what follows we will describe the cones of divisors of X^m_{m+2}, as well as its Mori chamber decomposition. Once this will be done, we will concentrate on the case where $m = 2g$ in order to use these results to compute the cone of curves of Σ_{2g}. We will consider the standard bases of $\mathrm{Pic}(X^m_{m+2})_{\mathbb{R}} \cong \mathbb{R}^{m+3}$ given by the pull-back H of the hyperplane section and the exceptional divisors $E_1, ..., E_{m+2}$.

Theorem 2.22. *Let X^m_{m+2} be the blow-up of $\mathbb{P}^m$ at $m+2$, and write the class of a general divisor $D \in \mathrm{Pic}(X^m_{m+2})_{\mathbb{R}}$ as $D = yH + \sum_{i=1}^{m+2} x_iE_i$. Then*

- *The effective cone $\mathrm{Eff}(X^m_{m+2})$ is defined by*

$$\left\{ \begin{array}{ll} y + x_i \geq 0 & i \in \{1, ..., m+2\}, \\ my + \sum_{i=1}^{m+2} x_i \geq 0. & \end{array} \right.$$

- *The Mori chamber decomposition of $\mathrm{Eff}(X^m_{m+2})$ is defined by the hyperplane arrangement*

$$(2.23)\quad \left\{\begin{array}{ll} (2-k)y-\sum_{i\in I}x_i=0 & if\ I\subseteq\{1,...,m+2\},\ |I|=k-1,\\ (m-k+1)y-\sum_{i\in I}x_i+\sum_{i=1}^{m+2}x_i=0 & if\ I\subseteq\{1,...,m+2\},\ |I|=k,\end{array}\right.$$

with $2\leq k\leq\frac{m+3}{2}$.

- *The movable cone* $\operatorname{Mov}(X^m_{m+2})$ *is given by*

$$\left\{\begin{array}{ll} (m-1)y-\sum_{i\in I}x_i+\sum_{i=1}^{m+2}x_i\geq 0 & if\ I\subseteq\{1,...,m+2\},\ |I|=2,\\ x_i\leq 0 & i\in\{1,...,m+2\},\\ y+x_i\geq 0 & i\in\{1,...,m+2\},\\ my+\sum_{i=1}^{m+2}x_i\geq 0. & \end{array}\right.$$

- *All small* $\mathbb{Q}$*-factorial modifications of* X *are smooth. Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be two adjacent chambers of* $\operatorname{Mov}(X)$*, corresponding to small* $\mathbb{Q}$*-factorial modifications of* X*,* $f: X\dashrightarrow\widetilde{X}$ *and* $f': X\dashrightarrow\widetilde{X}'$*, respectively. These chambers are separated by a hyperplane* H_I *in (2.23), with* $3\leq k\leq\frac{m+3}{2}$ *and* $|I|\in\{k-1,k\}$*. Assume that* $\varphi(\mathcal{C})\subset(H_I\leq k)$ *and* $\varphi(\mathcal{C}')\subset(H_I\geq k)$*. Then the birational map* $f'\circ f^{-1}:\widetilde{X}\dashrightarrow\widetilde{X}'$ *flips a* $\mathbb{P}^{k-2}$ *into a* $\mathbb{P}^{m+1-k}$*.*
- *Let* $\mathcal{C}$ *be a chamber of* $\operatorname{Mov}(X)$*, corresponding to small* $\mathbb{Q}$*-factorial modification* $\widetilde{X}$ *of* X*. Let* $\sigma\subset\partial\mathcal{C}$ *be a wall such that* $\sigma\subset\partial\operatorname{Mov}(X)$*, and let* $f:\widetilde{X}\rightarrow Y$ *be the corresponding elementary contraction. Then either* σ *is supported on a hyperplane of the form* $(y+x_i=0)$ *or* $(my-\sum_{i=1}^{m+2}x_i=0)$ *and* $f:\widetilde{X}\rightarrow Y$ *is a* $\mathbb{P}^1$*-bundle, or* σ *is supported on a hyperplane of the form* $(x_i=0)$ *or* $(m-1)y-\sum_{i\in I}x_i+\sum_{i=1}^{m+2}x_i=0$*,* $I\subseteq\{1,...,m+2\}$*,* $|I|=2$*, and* $f:\widetilde{X}\rightarrow Y$ *is the blow-up of a smooth point, and the exceptional divisor of* f *is the image in* $\widetilde{X}$ *of either an exceptional divisor* E_i *or a divisor of the form* $H-\sum_{i\in I}E_i$ *with* $I\subset\{1,...,m+2\}$*,* $|I|=m$*.*

Proof. The statement about the effective cone is in Lemma 2.17. By [Ok16, Theorem 1.2] the inequalities for the movable cones and for the Mori chamber decomposition follow by removing from the analogous systems of inequalities for the blow-up of $\mathbb{P}^m$ in $m+3$ points in [AM16, Theorem 1.3] the inequalities involving the exceptional divisor E_{m+3}. Similarly, by [Ok16, Theorem 1.2] the last two claims follow easily from the last two items in [AM16, Theorem 1.3]. □

Remark 2.24. Theorem 2.22 allows us to find explicit inequalities defining the cones $\operatorname{Eff}(X^m_{m+2})$, $\operatorname{Mov}(X^m_{m+2})$, and $\operatorname{Nef}(\widetilde{X})$, for any small $\mathbb{Q}$-factorial modification $\widetilde{X}$ of X^m_{m+2}. For instance, $\operatorname{Nef}(X^m_{m+2})$ is given by

$$\left\{\begin{array}{ll} x_i\leq 0 & i\in\{1,...,m+2\},\\ y+x_i+x_j\geq 0 & i,j\in\{1,...,m+2\},\ i\neq j.\end{array}\right.$$

2.25. **The Fano model of X^m_{m+2}.** First, let us consider the case when $m=2g$ is even. Then the anti-canonical divisor

$$-K_{X^{2g}_{2g+2}}\sim(2g+1)H-(2g-1)\sum_{i=1}^{2g+2}E_i$$

lies in the interior of the chamber $\mathcal{C}_{Fano}$ defined by

$$(2.26) \qquad \begin{cases} (1-g)y - \sum_{i\in I} x_i = 0 & if\ I \subseteq \{1,...,m+2\},\ |I| = g, \\ gy - \sum_{i\in I} x_i + \sum_{i=1}^{m+2} x_i = 0 & if\ I \subseteq \{1,...,m+2\},\ |I| = g+1, \end{cases}$$

that is the chamber in (2.23) of Theorem 2.22 for $k = g+1$. By Theorem 2.22 the chamber $\mathcal{C}_{Fano}$ corresponds to a smooth small $\mathbb{Q}$-factorial modification X^{2g}_{Fano} of X^{2g}_{2g+2} whose nef cone is given by (2.26). By Lemma 2.18 we get that

$$(2.27) \qquad X^{2g}_{Fano} \cong \Sigma_{2g}.$$

If $m = 2g-1$ is odd the class

$$-K_{X^{2g-1}_{2g+1}} \sim 2gH - (2g-2)\sum_{i=1}^{2g+1} E_i$$

lies in the intersection of the hyperplanes

$$(2.28) \qquad \begin{cases} (1-g)y - \sum_{i\in I} x_i = 0 & I \subseteq \{1,...,2g+1\},\ |I| = g, \\ (g-1)y - \sum_{i\in I} x_i + \sum_{i=1}^{2g+1} x_i = 0 & I \subseteq \{1,...,2g+1\},\ |I| = g+1. \end{cases}$$

The variety corresponding to the intersection (2.28) is a small non $\mathbb{Q}$-factorial modification of X^{2g-1}_{2g+1}, and by Lemma 2.5 we can identify this variety with the GIT quotient Σ_{2g-1}.

Corollary 2.29. *The Mori cone* $\mathrm{NE}(X^{2g}_{Fano})$ *of* X^{2g}_{Fano} *has* $\binom{2g+2}{g} + \binom{2g+2}{g+1} = \binom{2g+3}{g+1}$ *extremal rays.*

Proof. Since X^{2g}_{Fano} is a Mori Dream Space, $\mathrm{NE}(X^{2g}_{Fano})$ is polyhedral and finitely generated. Furthermore, $\mathrm{NE}(X^{2g}_{Fano})$ is dual to $\mathrm{Nef}(X^{2g}_{Fano})$. Therefore, the number of extremal rays of $\mathrm{NE}(X^{2g}_{Fano})$ is equal to the number of faces of $\mathrm{Nef}(X^{2g}_{Fano})$. Now, the statement follows from (2.26). □

2.30. **Fibrations on X^{2g}_{Fano}.** Let us stick to the situation where $m = 2g$ is the dimension of the GIT quotient. By the isomorphism (2.27) we may identify the Fano model X^{2g}_{Fano} with the GIT quotient Σ_{2g}, which in turn, by Remark (1.11) is isomorphic to the moduli space $\overline{\mathcal{M}}_{0,\widetilde{A}[2g+3]}$ with weights $\widetilde{A}[2g+3] = \left(\frac{2}{2g+3},...,\frac{2}{2g+3}\right)$. Now, let us consider the moduli space $\overline{\mathcal{M}}_{0,\widetilde{B}[2g+3]}$ with weights $\widetilde{B}[2g+3] = \left(\frac{2}{2g+2},...,\frac{2}{2g+2}\right)$, and the reduction morphism $\rho_{\widetilde{A}[2g+3],\widetilde{B}[2g+3]} : \overline{\mathcal{M}}_{0,\widetilde{B}[2g+3]} \to \overline{\mathcal{M}}_{0,\widetilde{A}[2g+3]}$. Note that $k\frac{2}{2g+2} > 1$ if and only if $k \geq g+1$, and $k \geq g+2$ implies that $k\frac{2}{2g+3} > 1$. Therefore, [Has03, Corollary 4.7] yields that $\rho_{\widetilde{A}[2g+3],\widetilde{B}[2g+3]}$ is an isomorphism, and we may identify X^{2g}_{Fano} with the moduli space $\overline{\mathcal{M}}_{0,\widetilde{B}[2g+3]}$.

Recall that by Remark 1.11 we may interpret Σ_{2g-1} as the Hassett space $\overline{\mathcal{M}}_{0,\widetilde{B}[2g+2]}$ with $\widetilde{B}[2g+2] = \left(\frac{2}{2g+2},...,\frac{2}{2g+2}\right)$. Therefore, for any $i = 1,...,2g+3$ we have a forgetful morphism

$$\pi_i : \overline{\mathcal{M}}_{0,\widetilde{B}[2g+3]} \cong X^{2g}_{Fano} \to \overline{\mathcal{M}}_{0,\widetilde{B}[2g+2]} \cong \Sigma_{2g-1}.$$

Now, consider the g-planes in X^{2g}_{Fano} described in (2.13). Note that in $\mathbf{C}\cup\mathbf{D}$ we have

$$\binom{2g+2}{g+1}+\binom{2g+2}{g}=\binom{2g+3}{g+1} \tag{2.31}$$

g-planes. From the modular point of view these g-planes parametrize configurations of $2g+3$ points $(\mathbb{P}^1, x_1,...,x_{2g+3})$ in $\mathbb{P}^1$ with $g+1$ points coinciding. Let us fix a marked point, say x_{2g+3}. For any choice of $g+1$ points in $(x_1,...,x_{2g+2})$ we get a g-plane H in $\mathbf{C}\cup\mathbf{D}$ and its complementary g-plane H^c. For instance, if H is given by $(x_1=...=x_{g+1})$ then H^c is defined by $(x_{g+2}=...=x_{2g+2})$. Note that H and H^c intersects a point representing the configuration $(x_1=...=x_{g+1}, x_{g+2}=...=x_{2g+2}, x_{2g+3})$, and that the morphism $\pi_{2g}:\overline{\mathcal{M}}_{0,\widetilde{B}[2g+3]}\cong X^{2g}_{Fano}\rightarrow\overline{\mathcal{M}}_{0,\widetilde{B}[2g+2]}\cong\Sigma_{2g-1}$ contracts $H\cup H^c$ to the singular point of Σ_{2g-1} representing the configuration $(x_1=...=x_{g+1},x_{g+2},...,x_{2g+2}) = (x_1,...,x_{g+1},x_{g+2}=...=x_{2g+2},x_{2g+3})$.

Thanks to the modular interpretation of π_{2g+3} we see that $\pi^{-1}_{2g+3}(p)\cong\mathbb{P}^1$ for any $p\in\Sigma_{2g-1}\setminus\mathrm{Sing}(\Sigma_{2g-1})$, while $\pi^{-1}_{2g+3}(p_j)$ is the union of two g-planes in $\mathbf{C}\cup\mathbf{D}$ intersecting in one point for any $p_j\in\mathrm{Sing}(\Sigma_{2g-1})$. Indeed

$$\frac{1}{2}\binom{2g+2}{g+1}=\binom{2g+1}{g}$$

is exactly the number of singular points of Σ_{2g-1}. More generally for any $i=1,...,2g+3$ we may consider the set $\{x_1,...,x_{i-1},x_{i+1},...,x_{2g+3}\}$, and for any subset I of cardinality $g+1$ of $\{x_1,...,x_{i-1},x_{i+1},...,x_{2g+3}\}$ we have a g-plane H_I defined by requiring the points marked by I to coincide, and the complementary g-plane H_{I^c} defined as the locus parametrizing configurations where the marked points in I^c coincide. Then the morphism $\pi_i:\overline{\mathcal{M}}_{0,\widetilde{B}[2g+3]}\cong X^{2g}_{Fano}\rightarrow\overline{\mathcal{M}}_{0,\widetilde{B}[2g+2]}\cong\Sigma_{2g-1}$ contracts the unions $H\cup H^c$ to the singular points of Σ_{2g-1}.

For any g-plane H_I in $\mathbf{C}\cup\mathbf{D}$ we will denote by L_I the class of a line in H_I. Note that L_I is the class of 1-dimensional boundary stratum of Σ_{2g} corresponding to configurations where the points in I coincide, other g points coincide as well, and the remaining two points are different.

Proposition 2.32. *The classes L_I, L_{I^c} of lines in $H_I, H_{I^c}\cong\mathbb{P}^g$ described above generate the extremal rays of* $\mathrm{NE}(X^{2g}_{Fano})$.

Proof. From the above discussion we have that for any L_I there is a complementary class L_{I^c}, and a forgetful morphism $\pi_i:\overline{\mathcal{M}}_{0,\widetilde{B}[2g+3]}\cong X^{2g}_{Fano}\rightarrow\overline{\mathcal{M}}_{0,\widetilde{B}[2g+2]}\cong\Sigma_{2g-1}$, such that the fibers of π_i over $\Sigma_{2g-1}\setminus\mathrm{Sing}(\Sigma_{2g-1})$ are isomorphic to $\mathbb{P}^1$, and $H_I\cup H_{I^c}$ is contracted by π_i onto a singular point of Σ_{2g-1}. Our aim is to compute the relative Mori cone $\mathrm{NE}(\pi_i)$ of the morphism π_i. Let $C\subset X^{2g}_{Fano}$ be a curve contracted by π_i. Then C is either a fiber of π_i over a smooth point of Σ_{2g-1} or a curve in $H_I\cup H_{I^c}$. In the latter case the class of C may be written as a combination with non-negative coefficients of the classes L_I and L_{I^c}.

Now, for simplicity of notation assume that $i=2g+3$, and consider the surface $S\subseteq X^{2g}_{Fano}$ parametrizing configurations of points of the type $(x_1=...=x_g,x_{g+1},x_{g+2}=...=x_{2g+1},x_{2g+2},x_{2g+3})$. Then the image of $\pi_{2g+3|S}:S\rightarrow\Sigma_{2g-1}$ is the line Γ passing through a singular point $p\in\Sigma_{2g-1}$ parametrizing the configurations $(x_1=...=x_g,x_{g+1},x_{g+2}=...=$

$x_{2g+1}, x_{2g+2}, x_{2g+3})$. Furthermore, $\pi^{-1}_{2g+3|S}(q) \cong \mathbb{P}^1$ for any $q \in \Gamma \setminus \{p\}$, and $\pi^{-1}_{2g+3|S}(q) = L_I \cup L_{I^c}$. Therefore, the general fiber of π_{2g+3} is numerically equivalent to $L_I + L_{I^c}$.

This means that $\mathrm{NE}(\pi_i)$ is generated by the classes L_I, L_{I^c} for $\{x_1, ..., x_{i-1}, x_{i+1}, ..., x_{2g+3}\}$ with $|I| = g+1$. Finally, since by [De01, Proposition 1.14] the subcone $\mathrm{NE}(\pi_i)$ of $\mathrm{NE}(X^{2g}_{Fano})$ is extremal, we get that L_I, L_{I^c} generate extremal rays of $\mathrm{NE}(X^{2g}_{Fano})$. □

We are now in the position to claim and prove the F-conjecture for Σ_{2g}.

Theorem 2.33. *The Mori cone of the GIT quotient Σ_{2g} is generated by the classes L_I, L_{I^c} of 1-dimensional boundary strata.*

Proof. By Proposition 2.32 the classes L_I, L_{I^c} generate extremal rays of $\mathrm{NE}(\Sigma_{2g})$. By (2.31) these are $\binom{2g+3}{g+1}$ rays which by Corollary 2.29 is exactly the number of extremal rays of $\mathrm{NE}(\Sigma_{2g})$. □

Remark 2.34. Consider the Mori chamber decomposition in Theorem 2.22. Note that in order to go from the chamber corresponding to $\mathrm{Nef}(X^{2g}_{2g+2})$ to the chamber corresponding to $\mathrm{Nef}(X^{2g}_{Fano})$ we must cross the walls in (2.23) for any $3 \leq k \leq 2g+1$. Indeed by the modular description of the small modification $\psi : X^{2g}_{2g+2} \dashrightarrow X^{2g}_{Fano}$ we see that it factors as:

$$X_0 = X^{2g}_{2g+2} \overset{\psi_1}{\dashrightarrow} X_1 \overset{\psi_2}{\dashrightarrow} X_2 \overset{\psi_3}{\dashrightarrow} ... \overset{\psi_{g-1}}{\dashrightarrow} X_{g-1} = X^{2g}_{Fano}$$

where $\psi_i : X_{i-1} \dashrightarrow X_i$ is the flip of the strict transform in X_{i-1} of the i-planes in $\mathbb{P}^{2g}$ generated by $i+1$ among the blown-up points. These strict transforms are disjoint i-planes in X_{i-1}, while the flipped locus in X_i is a disjoint union of $(2g-1-i)$-planes. In particular the $\binom{2g+2}{g+1} + \binom{2g+2}{g} = \binom{2g+3}{g+1}$ extremal rays in (2.31) correspond to the $\binom{2g+2}{g+1}$ g-planes coming as strict transforms of the g-planes in $\mathbb{P}^m$ generated by $g+1$ of the marked points, plus the g-planes that are the flipped loci of the $(g-1)$-planes in $\mathbb{P}^n$ generated by g of the marked points.

2.35. **The cone of moving curves of Σ_{2g}.** In this section we describe extremal rays of the cone of moving curves $\mathrm{Mov}_1(\Sigma_{2g})$ of Σ_{2g}. Recall that an irreducible curve C on a projective variety X is called a moving curve if C is a member of an algebraic family of curves covering a dense subset of X. By [BDPP13, Theorems 2.2 and 2.4] the cone of moving curves is dual to the cone of pseudoeffective divisor classes which is spanned by classes that appear as limits of sequences of effective $\mathbb{Q}$-divisors.

By Theorem 2.22 we have that $\mathrm{Eff}(\Sigma_{2g})$ is closed and therefore $\mathrm{Mov}_1(\Sigma_{2g})$ is the dual cone of $\mathrm{Eff}(\Sigma_{2g})$. In particular, by the description of $\mathrm{Eff}(X^{2g}_{2g+2})$ in Theorem 2.22 we get that $\mathrm{Mov}_1(\Sigma_{2g})$ has exactly $2g+3$ extremal rays.

Theorem 2.36. *The cone of moving curves $\mathrm{Mov}_1(\Sigma_{2g})$ of Σ_{2g} is generated by the the classes of the fiber of the forgetful morphisms $\pi_i : \Sigma_{2g} \to \Sigma_{2g-1}$ for $i = 1, ..., 2g+3$.*

Proof. By the description of the faces of $\mathrm{Eff}(X^{2g}_{2g+2})$ in Theorem 2.22 we see that the $2g+3$ extremal rays of $\mathrm{Mov}_1(X^{2g}_{2g+2}) = \mathrm{Eff}(X^{2g}_{2g+2})^\vee$ are generated by the class of the strict transform L_i of a line through the blown-up point $p_i \in \mathbb{P}^{2g}$ for $i = 1, ..., 2g+2$, and by the class of the strict transform of a degree $2g$ rational normal curve C through $p_1, ..., p_{2g+2}$.

Now, consider the following diagram

where $\psi : X^{2g}_{2g+2} \dashrightarrow \Sigma_{2g}$ is the sequence of flips in Remark 2.34, $\rho : \overline{\mathcal{M}}_{0,2g+3} \to \Sigma_{2g}$ is the reduction morphism, $f : \overline{\mathcal{M}}_{0,2g+3} \to X^{2g}_{2g+2}$ is the Kapranov's blow-up morphism in Construction 1.10, $\pi_i : \Sigma_{2g} \to \Sigma_{2g-1}$ is a forgetful morphism, and $\widetilde{\pi}_i : \overline{\mathcal{M}}_{0,2g+3} \to \overline{\mathcal{M}}_{0,2g+2}$ is the corresponding forgetful morphism on $\overline{\mathcal{M}}_{0,2g+3}$. Now, by [Ka93] the fibers of the forgetful morphism $\widetilde{\pi}_i : \overline{\mathcal{M}}_{0,2g+3} \to \overline{\mathcal{M}}_{0,2g+2}$ are either the strict transforms of the lines through a point p_i or the strict transforms of the degree $2g$ rational normal curves through $p_1, ..., p_{2g+2}$. Now, note that ρ, and hence ψ, map a general fiber of $\widetilde{\pi}_i$ onto a general fiber of π_i.

Finally, to conclude it is enough to observe that since the L_i's and C generate extremal rays of $\mathrm{Mov}_1(X^{2g}_{2g+2})$ and $\psi : X^{2g}_{2g+2} \dashrightarrow \Sigma_{2g}$ is a sequence of flips of small elementary contractions [Bar08, Proposition 3.14] yields that the fibers of the forgetful morphisms $\pi_i : \Sigma_{2g} \to \Sigma_{2g-1}$ generates extremal rays of $\mathrm{Mov}_1(\Sigma_{2g})$. $\square$

2.37. **Analogy with the geometric invariant theory of moduli of weighted pointed curves.** Finally, we would like to stress another link between the GIT quotients we are studying and moduli of weighted curves. Consider the map:

$$\begin{array}{rccc} \phi : & \mathrm{Eff}(X^{2g}_{2g+2}) \subset \mathrm{Pic}(X^{2g}_{2g+2})_{\mathbb{Q}} & \longrightarrow & \mathbb{Q}^{2g+3} \\ & (y, x_1, ..., x_{2g+2}) & \longmapsto & (a_1, ..., a_{2g+3}), \end{array} \tag{2.38}$$

where

$$a_j = \frac{y + x_j}{(2g+1)y + \sum_{i=1}^{2g+2} x_i}, \text{ for } j = 1, ..., 2g+2,$$

and

$$a_{2g+3} = 2 - \sum_{i=1}^{2g+2} a_i.$$

Note that ϕ maps $\mathrm{Eff}(X^{2g}_{2g+2})$ onto the hypercube $[0,1]^{2g+3} \subset \mathbb{R}^{2g+3}$. Furthermore, via ϕ the walls of the Mori chamber decomposition in (2.23) translate into $\sum_{i\in I} a_i = 1$ for $I \subset \{1, ..., 2g+3\}$, with $|I| \in \{k-1, k\}$ and $2 \leq k \leq \frac{2g+3}{2}$. These walls are exactly the ones used in [Has03, Section 8] to describe variations of some GIT quotients of products of $\mathbb{P}^1$ in terms of moduli of weighted curves. Therefore, interpreting the rational numbers $(a_1, ..., a_{2g+3})$ as the weights of pointed curves, we have that the map ϕ in (2.38) translates the Mori chamber decomposition of Theorem 2.22 into the GIT chamber decomposition from [Has03, Section 8].

For instance, $\phi(-K_{X^3_5}) = \phi(4, -2, ..., -2) = \left(\frac{1}{3}, ..., \frac{1}{3}\right)$ and by Remark 1.11 these are the weights of the Segre cubic threefold. Similarly, taking $D = 3H - E_1 - ... - E_5$, which is ample on X^3_5, we have that $\phi(D) = \phi(3, -1, ..., -1) = \left(\frac{2}{7}, ..., \frac{2}{7}, \frac{4}{7}\right)$, and by Construction 1.10 the moduli space with these weights is isomorphic to X^3_5 itself.

3. Degrees of projections of Veronese varieties

In this section, we use part of the preliminary results developed in the preceding ones to describe some projective geometry of the GIT quotients. In particular we make massive use once again of the linear systems on the projective space of Theorem 1.2, in order to compute the Hilbert polynomial and the degree of Σ_{2g-1} and Σ_{2g} in their natural embeddings in the spaces of invariants. A formula for the degree of the GIT quotients already appeared in

[HKL16]. Our formula has the advantage of not being recursive, as it is that in [HKL16]. Similar methods allow us to show that $\overline{\mathcal{M}}_{0,6}$ is a weak Fano variety.

For any integer $0 \leq r \leq s-1$ and for any multi-index $I(r) = \{i_1, ..., i_{r+1}\} \subseteq \{1, ..., s\}$ set

$$k_{I(r)} = \max\{m_{i_1} + ... + m_{i_{r+1}} - rd, 0\}. \tag{3.1}$$

In [BDP15, Definition 3.2] the authors define the *linear virtual dimension* of the linear system $\mathcal{L}_{n,d}(m_1, ..., m_s)$ on $\mathbb{P}^n$ as the number

$$\binom{n+d}{d} + \sum_{r=0}^{s-1} \sum_{I[r]\subseteq\{1,\dots,s\}} (-1)^{r+1} \binom{n + k_{I[r]} - r - 1}{n} \tag{3.2}$$

Furthermore, they define the *linear expected dimension* of $\mathcal{L}_{n,d}(m_1, ..., m_s)$, denoted by $\mathrm{ldim}(\mathcal{L})$, as follows: if the linear system $\mathcal{L}_{n,d}(m_1, ..., m_s)$ is contained in a linear system whose linear virtual dimension is negative then we set $\mathrm{ldim}(\mathcal{L}) = -1$; otherwise we define $\mathrm{ldim}(\mathcal{L})$ to be the maximum between the linear virtual dimension of $\mathcal{L}_{n,d}(m_1, ..., m_s)$ and -1.

Proposition 3.3. *Let $\phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{L})^*)$ be the rational map induced by the linear system $\mathcal{L} := \mathcal{L}_{n,d}(m_1, ..., m_s)$. Assume that $\phi_{\mathcal{L}}$ is birational, and let $X_{\mathcal{L}} = \overline{\phi_{\mathcal{L}}(\mathbb{P}^n)}$. If $s \leq n+2$ then the Hilbert polynomial of $X_{\mathcal{L}}$ is given by*

$$h_{X_{\mathcal{L}}}(t) = \binom{dt+n}{n} + \sum_{r=0}^{s-1} \sum_{I[r]\subseteq\{1,\dots,s\}} (-1)^{r+1} \binom{n + tk_{I[r]} - r - 1}{n}.$$

In particular

$$\deg(X_{\mathcal{L}}) = d^n + \sum_{r=0}^{s-1} \sum_{I[r]\subseteq\{1,\dots,s\}} (-1)^{r+1} k_{I[r]}^n.$$

Proof. Polynomials of degree $t \in \mathbb{N}$ on $\mathbb{P}(H^0(\mathbb{P}^n, \mathcal{L})^*)$ correspond to degree td polynomials on $\mathbb{P}^n$ vanishing with multiplicity m_i at p_i. Therefore, the Hilbert polynomial of $X_{\mathcal{L}}$ is given by

$$h_{X_{\mathcal{L}}}(t) = h^0(\mathbb{P}^n, t\mathcal{L}).$$

Now, since $t\mathcal{L}$ is effective for any $t \geq 0$, and $s \leq n+1$ [BDP15, Theorem 4.6] yields that

$$h^0(\mathbb{P}^n, t\mathcal{L}) = \mathrm{ldim}(t\mathcal{L})$$

where $\mathrm{ldim}(t\mathcal{L})$ is the linear expected dimension of $t\mathcal{L}$ defined by (3.2). To get the formula for the Hilbert polynomial it is enough to observe that, in the notation of (3.1), for the linear system $t\mathcal{L}_g$ we have

$$k_{I[r]}(t\mathcal{L}) = \max\{t(m_{i_1} + ... + m_{i_{r+1}} - rd), 0\} = tk_{I[r]}(\mathcal{L}).$$

Finally, note that $h_{X_{\mathcal{L}}}(t)$ may be written as

$$h_{X_{\mathcal{L}}}(t) = \frac{d^n + \sum_{r=0}^{s-1} \sum_{I[r]\subseteq\{1,\dots,s\}} k_{I[r]}^n}{n!} t^n + P(t)$$

where $P(t)$ is a polynomial in t of degree $\deg(P) \leq n-1$. Therefore, the volume of the big linear system $\mathcal{L}$ is given by

$$\mathrm{Vol}(\mathcal{L}) = \limsup_{t\mapsto+\infty} \frac{h_{X_{\mathcal{L}}}(t)}{t^n/n!} = d^n + \sum_{r=0}^{s-1} \sum_{I[r]\subseteq\{1,\dots,s\}} k_{I[r]}^n$$

and $\mathrm{Vol}(\mathcal{L})$ is exactly the degree $X_{\mathcal{L}} \subseteq \mathbb{P}(H^0(\mathbb{P}^n,\mathcal{L})^*)$. □

Recall from Section 2.1 the we define $\mathcal{L}_{2g-1} = \mathcal{L}_{2g-1,g}(g-1,\dots,g-1)$ as the linear system of degree g forms on $\mathbb{P}^{2g-1}$ vanishing with multiplicity g at $2g+1$ general points $p_1,\dots,p_{2g+1} \in \mathbb{P}^{2g-1}$ and that it induces a birational map $\sigma_g : \mathbb{P}^{2g-1} \dashrightarrow \mathbb{P}(H^0(\mathbb{P}^{2g-1},\mathcal{L}_{2g-1})^*)$. The GIT quotient Σ_{2g-1} is the closure of the image of σ_g in $\mathbb{P}(H^0(\mathbb{P}^{2g-1},\mathcal{L}_{2g-1})^*)$.

Corollary 3.4. *The Hilbert polynomial of* $\Sigma_{2g-1} \subseteq \mathbb{P}(H^0(\mathbb{P}^{2g-1},\mathcal{L}_{2g-1})^*)$ *is given by*

$$h_{\Sigma_{2g-1}}(t) = \binom{gt+2g-1}{2g-1} + \sum_{r=0}^{g-2}(-1)^{r+1}\binom{2g+1}{r+1}\binom{t(g-r-1)+2g-1-r-1}{2g-1}.$$

In particular

$$h^0(\mathbb{P}^{2g-1},\mathcal{L}_{2g-1}) = \binom{3g-1}{2g-1} + \sum_{r=0}^{g-2}(-1)^{r+1}\binom{2g+1}{r+1}\binom{3g-2r-3}{2g-1}$$

and

$$\deg(\Sigma_{2g-1}) = g^{2g-1} + \sum_{r=0}^{g-2}(-1)^{r+1}\binom{2g+1}{r+1}(g-r-1)^{2g-1}.$$

Proof. By Theorem 1.2 $\Sigma_{2g-1} = \overline{\sigma_g(\mathbb{P}^{2g-1})}$, where σ_g is the map in (2.2). Then polynomials of degree $t \in \mathbb{N}$ on $\mathbb{P}(H^0(\mathbb{P}^{2g-1},\mathcal{L}_{2g-1})^*)$ correspond to degree tg polynomials on $\mathbb{P}^{2g-1}$ vanishing with multiplicity $t(g-1)$ at $p_1,\dots,p_{2g+1}$. Note that in the notation of (3.1) for the linear system $t\mathcal{L}_{2g-1}$ we have

$$k_{I[r]} = \begin{cases} t(g-r-1) & \text{if} \quad r \leq g-2, \\ 0 & \text{if} \quad r \geq g-1. \end{cases}$$

Furthermore, note that in (3.2) we have

$$\binom{n+k_{I[r]}-r-1}{n} = \binom{2h-1+t(g-r-1)-r-1}{2g-1} \neq 0$$

for $t \gg 0$ if and only if $r \leq g-2$.

Now, in order to conclude it is enough to observe that for any $r = 0,\dots,g-2$ we have $\binom{2g+1}{r+1}$ subsets of $\{1,\dots,2g+1\}$ of the form $I[r]$. Then the formulas for the Hilbert polynomial and the degree of Σ_{2g-1} follow from Proposition 3.3. In particular, the dimension of the linear system $\mathcal{L}_{2g-1}$ is then given by $h^0(\mathbb{P}^{2g-1},\mathcal{L}_{2g-1}) = h_{\Sigma_{2g-1}}(1)$. □

Corollary 3.5. *Let us consider the GIT quotient* Σ_{2g} *and let* $n = 2g+3$ *be the number of points on* $\mathbb{P}^1$ *that it parametrizes.The Hilbert polynomial of* Σ_{2g} *is given by*

$$h_{\Sigma_{2g}}(t) = \binom{(2g+1)t+2g}{2g} + \sum_{r=0}^{\lfloor\frac{2g-1}{2}\rfloor}(-1)^{r+1}\binom{2g+2}{r+1}\binom{t(2g-2r-1)+2g-r-1}{2g}.$$

In particular

$$\deg(\Sigma_{2g}) = (2g+1)^{2g} + \sum_{r=0}^{\lfloor\frac{2g-1}{2}\rfloor}(-1)^{r+1}\binom{2g+2}{r+1}(2g-2r-1)^{2g}.$$

Proof. By Theorem 1.2 $\Sigma_{2g} \subset \mathbb{P}(H^0(\mathbb{P}^{2g}, \mathcal{L}_{2g})^*)$ is the closure of the image of the rational map induced by the linear system $\mathcal{L}_{2g}$ of degree $2g+1$ hypersurfaces in $\mathbb{P}^{2g}$ with multiplicity $2g-1$ at p_i for $i = 1, ..., 2g+2$. Now, to conclude it is enough to observe that for the linear system $t\mathcal{L}_{2g}$ we have

$$k_{I[r]} = \begin{cases} t(2g-1-2r) & \text{if} \quad r \leq \lfloor\frac{2g-1}{2}\rfloor, \\ 0 & \text{if} \quad r > \lfloor\frac{2g-1}{2}\rfloor. \end{cases}$$

and to argue as in the proof of Corollary 3.4. □

3.6. $\overline{\mathcal{M}}_{0,6}$ **is weak Fano.** In this section we prove that $\overline{\mathcal{M}}_{0,6}$ is weak Fano that is $-K_{\overline{\mathcal{M}}_{0,6}}$ is nef and big.

Construction 3.7. For any effective divisor D in $\mathcal{L}_{n,d}(m_1, ..., m_s)$, we denote by D_h the strict transform of D in the space X_h^n obtained as the blow-up of of $\mathbb{P}^n$ along the linear base locus of D up to dimension h, with $h \leq n-1$. That is:

- X_0^n is the blow-up $\mathbb{P}^n$ at the points $p_1, ..., p_s$;
- X_1^n is the blow-up of X_0^n along the strict transforms of the lines $\langle p_{i_1}, p_{i_2}\rangle$;

  ⋮
- X_h^n is the blow-up of X_{h-1}^n along the strict transforms of the h-planes $\langle p_{i_1}, ..., p_{i_{h+1}}\rangle$.

Note that the number $k_{I[r]}$ defined in (3.1) is the multiplicity of a general element of $\mathcal{L}_{n,d}(m_1, ..., m_s)$ along an r-plane $\langle p_{i_1}, ..., p_{i_{r+1}}\rangle$. Therefore, the strict transform D_h of D in X_h^n may be written as:

$$D_h = dH - \sum_{r=1}^{h}\sum_{I[r]\subseteq\{1,...,s\}} k_{I[r]}E_{I[r]}$$

where H is the pull-back of the hyperplane section of $\mathbb{P}^n$, and $E_{I[r]}$, with $I[r] = \{i_1, ..., i_{r+1}\}$, is the exceptional divisor over the r-plane $\langle p_{i_1}, ..., p_{i_{r+1}}\rangle$. Finally, let $\widetilde{D} := D_{n-1}$.

Proposition 3.8. *Let us denote by $\phi_{\mathcal{L}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ the birational map induced by $\mathcal{L} := \mathcal{L}_{n,d}(m_1, ..., m_s)$ with image $X_{\mathcal{L}} = \overline{\phi_{\mathcal{L}}(\mathbb{P}^n)}$. If $\widetilde{D}$ is nef then*

$$\deg(X_{\mathcal{L}}) = d^n + \sum_{r=0}^{s-1}\sum_{I[r]\subseteq\{1,...,s\}}(-1)^{r+1}k_{I[r]}^n \tag{3.9}$$

In particular, if $m_1 = ... = m_s = m$ and $\overline{r}$ is the maximal dimension of a linear subspace contained in the base locus of $\mathcal{L}$ then

$$\deg(X_{\mathcal{L}}) = d^n + \sum_{r=0}^{\overline{r}}(-1)^{r+1}\binom{s}{r+1}((r+1)m - rd)^n \tag{3.10}$$

Proof. Polynomials of degree $t \in \mathbb{N}$ on $\mathbb{P}^N$ correspond to degree td polynomials on $\mathbb{P}^n$ vanishing with multiplicity tm_i at p_i for $i = 1, ..., s$. Therefore, the Hilbert polynomial of $X_{\mathcal{L}}$ is given by

$$h_{X_{\mathcal{L}}}(t) = h^0(\mathbb{P}^n, t\mathcal{L}).$$

By [DP14, Theorem 1.5] we have that

$$h^0(\mathbb{P}^n, t\mathcal{L}) = \operatorname{ldim}(tD) + \sum_{i=1}^{n}(-1)^i h^i(X_{n-1}, t\widetilde{D})$$

Now, since by hypothesis $\widetilde{D}$ is nef, the asymptotic Riemann-Roch theorem [La04, Theorem 1.4.40] yields that

$$h^i(X_{n-1}, t\widetilde{D}) = O(t^{n-i})$$

Now, we may compute the degree of $X_{\mathcal{L}}$ as the volume of the big linear system $\mathcal{L}$:

$$\operatorname{Vol}(\mathcal{L}) = \limsup_{t\mapsto+\infty} \frac{\operatorname{ldim}(tD) + \sum_{i=1}^{n} O(t^{n-i})}{t^n/n!} = \limsup_{t\mapsto+\infty} \frac{\operatorname{ldim}(tD)}{t^n/n!}$$

Now, in order to conclude is enough to note that by (3.2) we have

$$\limsup_{t\mapsto+\infty} \frac{\operatorname{ldim}(tD)}{t^n/n!} = d^n + \sum_{r=0}^{s-1} \sum_{I[r]\subseteq\{1,...,s\}} (-1)^{r+1} k_{I[r]}^n$$

Finally, note that if $m_1 = ... = m_s = m$ and $\overline{r}$ is the maximal dimension of a linear subspace contained in the base locus of $\mathcal{L}$ then

$$k_{I[r]} = \begin{cases} (r+1)m - rd & \text{if} \quad r \leq \overline{r}, \\ 0 & \text{if} \quad r > \overline{r}. \end{cases}$$

Now, to get formula (3.10) in the statement, it is enough to plug these values of $k_{I[r]}$ in formula (3.9), and to notice that we have exactly $\binom{s}{r+1}$ r-planes of type $\langle p_{i_1}, ..., p_{i_{r+1}} \rangle$ in the base locus of $\mathcal{L}$. □

In Proposition 3.8, when $\mathcal{L}$ does not have fixed components and $\widetilde{D}$ is base-point-free - so in particular $\widetilde{D}$ is nef - $\deg(X_{\mathcal{L}})$ may be also computed as the top self-intersection of $\widetilde{D}$. In the rest of this section we will work out the case $n = 3$, and while doing this we will get a simple and direct argument proving that $\overline{\mathcal{M}}_{0,6}$ is a weak Fano variety, that is the anti-canonical divisor $-K_{\overline{\mathcal{M}}_{0,6}}$ is nef and big.

Let us recall Construction 1.10 for $\overline{\mathcal{M}}_{0,6}$: let $p_1, ..., p_5 \in \mathbb{P}^3$ be points in linear general position, and consider

- $\pi_1 : X \to \mathbb{P}^3$ the blow-up of $p_1, ..., p_5$,
- $\pi_2 : Y \to X$ the blow-up of the strict transforms of the lines $\langle p_i, p_j \rangle$, $i, j = 1, ..., 5$,

Then $Y \cong \overline{\mathcal{M}}_{0,6}$, and the morphism $f_6 = \pi_1 \circ \pi_2 : \overline{\mathcal{M}}_{0,6} \to \mathbb{P}^3$ is induced by the psi-class Ψ_6 on $\overline{\mathcal{M}}_{0,6}$.

By [KMc96, Theorem 1.2] the Mori Cone $\operatorname{NE}(\overline{\mathcal{M}}_{0,6})$ of $\overline{\mathcal{M}}_{0,6}$ is generated by classes of vital curves. Let us denote by E_i and $E_{i,j}$ the exceptional divisors over p_i and the strict transform of $\langle p_i, p_j \rangle$ respectively.

In the first blow-up X the strict transforms of the lines $\langle p_i, p_j \rangle$ intersects the exceptional divisor E_i over p_i in four points q_j for $j \neq i$. Therefore, after blowing-up all the strict transforms of the lines, the divisor E_i in $\overline{\mathcal{M}}_{0,6}$ is isomorphic to the blow-up of $\mathbb{P}^2$ in four points. We denote by $L^i_{h,k}$ the strict transform in E_i of the line spanned by q_h and q_k, and by R^i_h the exceptional divisor over q_h. So, in any exceptional divisor, we get 10 vital curves: 6 of type $L^i_{h,k}$ and 4 of type R^i_h.

Now, for any line $\langle p_i, p_j\rangle \subset \mathbb{P}^3$ we have three planes $\langle p_i, p_j, p_k\rangle$ for $k \neq i,j$ containing this line. The strict transforms of the three planes intersects the exceptional divisor $E_{i,j}$ in three vital curves $\sigma^k_{i,j}$. Therefore, we have $\binom{5}{2}\cdot 3 = 30$ of them.

Now remark that $E_{i,j}$ is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$. Note moreover that R^i_j is numerically equivalent to R^j_i for any i,j because they are fibers of the same ruling of $E_{i,j}$. Furthermore, the $\sigma^k_{i,j}$'s for $k\neq i,j$ are all numerically equivalent because they are fibers of the other ruling of $E_{i,j}$. We conclude that $NE(\overline{\mathcal{M}}_{0,6})$ is a polyhedral cone generated by 50 extremal rays.

Lemma 3.11. *For any i we have $H^2E_i = HE_i^2 = 0$, $E_i^3 = 1$. Furthermore, $HE_{i,j}^2 = -1$, $H^2E_{i,j} = 0$ for any i,j, and*

$$E_iE^2_{h,k} = \begin{cases} -1 & \text{if } i \in \{h,k\},\\ 0 & \text{if } i\notin\{h,k\}.\end{cases}$$

Finally $E_i^2E_{h,k} = 0$ for any i,h,k, and $E^3_{i,j} = 2$.

Proof. We will denote by E_i both the exceptional divisor over p_i in X and its strict transform in Y. Let H_i be the strict transform of a general plane through p_i. Then $H_i = H - E_i$ and $H_i^3 = H^3 - 3H^2E_i + 3HE_i^2 - E_i^3$, $H_i^3 = H^2E_i = HE_i^2 = 0$ yield $E_i^3 = H^3 = 1$.

Now, let us consider the following diagram, where $L_{i,j}$ is the strict transform of the line $\langle p_i, p_j\rangle$.

$$\begin{array}{ccc} E_{i,j} & \xrightarrow{\;j\;} & Y \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi} \\ L_{i,j} & \xrightarrow{\;i\;} & X \end{array}$$

where $\pi_E = \pi_{|E_{i,j}}$. We have $(H-E_{i,j})^2 = H^3 - H^2(H-E_{i,j}) = 0$. Therefore,

$$\begin{aligned} HE^2_{i,j} &= \pi^*Hj_*E^2_{i,j} = j_*(E^2_{i,j}\pi_E^*i_*H) = -1,\\ H^2E_{i,j} &= \pi^*H^2j_*E_{i,j} = j_*(E_{i,j}\pi_E^*i^*H^2) = 0,\\ E_iE^2_{i,j} &= \pi^*E_iE^2_{i,j} = j_*(E^2_{i,j}\pi_E^*i^*E_i) = -1,\\ E_i^2E_{i,j} &= \pi^*E_i^2E_{i,j} = j_*(E_{i,j}\pi_E^*i^*E_i^2) = 0. \end{aligned}$$

Finally $(H-E_i-E_j-E_{i,j})^3 = H^3 - E_i^3 - E_j^3 - E_{ij}^3 + 3HE_{ij}^2 - 3E_iE_{ij}^2 - 3E_jE^2_{ij} = 0$ yields $E^3_{i,j} = 2$. □

Proposition 3.12. *The moduli space $\overline{\mathcal{M}}_{0,6}$ is weak Fano.*

Proof. The anti-canonical bundle is given by

$$-K_{\overline{\mathcal{M}}_{0,6}} = 4H - 2\sum_{i=1}^{5}E_i - \sum_{i,j=1}^{5}E_{i,j}.$$

First we consider the curves of type $L^i_{h,k}$. We have

$$L^i_{h,k}E_t = \begin{cases} -1 & \text{if } i = t, \\ 0 & \text{if } i \neq t. \end{cases}$$

Furthermore,

$$L^i_{h,k}E_{s,t} = \begin{cases} 1 & \text{if } s = i \text{ and } t \in \{h,k\}, \\ 0 & \text{otherwise}. \end{cases}$$

Finally, $L^i_{h,k}H = 0$, and $-K_{\overline{\mathcal{M}}_{0,6}}L^i_{h,k} = -2(-1) - (1+1) = 0$. Now, let us consider a curve of type R^i_j. Then $R^i_jH = R^i_jE_k = 0$ for any i, j, k, and

$$R^i_jE_{h,k} = \begin{cases} -1 & \text{if } \{i,j\} = \{h,k\}, \\ 0 & \text{otherwise}. \end{cases}$$

This yields $-K_{\overline{\mathcal{M}}_{0,6}}R^i_j = 1$. Finally, we consider a curve of type $\sigma_{i,j}$. Note that the normal bundle of the strict transform of a line $L_{i,j} = \langle p_i, p_j\rangle$ is $\mathcal{N}_{L_{i,j}} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore, $\mathcal{O}_{E_{i,j}}(E_{i,j}) = \mathcal{O}_{E_{i,j}}(-1,-1)$. This yields

$$\sigma_{i,j}E_{h,k} = \begin{cases} -1 & \text{if } \{i,j\} = \{h,k\}, \\ 0 & \text{otherwise}. \end{cases}$$

Furthermore $\sigma_{i,j}H = 1$ and

$$\sigma_{i,j}E_h = \begin{cases} 1 & \text{if } h \in \{i,j\}, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore $-K_{\overline{\mathcal{M}}_{0,6}}\sigma_{i,j} = 4 - 2(1+1) - (-1) = 1$. This means that $-K_{\overline{\mathcal{M}}_{0,6}}$ is nef. Now, by the formulas in Lemma 3.11 we get that $(-K_{\overline{\mathcal{M}}_{0,6}})^3 > 0$ which implies that $-K_{\overline{\mathcal{M}}_{0,6}}$ is big. □

We would like to stress that by [KMc96] $\overline{\mathcal{M}}_{0,n}$ is not even log Fano for $n \geq 7$.

Example 3.13. Under the hypothesis of Proposition 3.8 let us consider the case $n = 3$, $m_1 = ... = m_s = m$ then $k_{I[0]} = m$, $k_{I[1]} = 2m - d$ and

$$\widetilde{D} = dH - \sum_{i=1}^{s} mE_i - \sum_{i\neq j}(2m-d)E_{ij}$$

where E_i is the exceptional divisor over p_i, and E_{ij} is the exceptional divisor over the strict transform L_{ij} in X^3_0 of the line $\langle p_i, p_j\rangle \subset \mathbb{P}^3$. By Lemma 3.11 we have

$$\widetilde{D}^3 = d^3H^3 + 3d\sum_{i\neq j}(2m-d)^2HE_{ij}^2 - 3\sum_{i=1}^{s} m(2m-d)^2\sum_{i\neq j}E_iE_{ij}^2 - \sum_{i=1}^{s} m^3E_i^3 - \sum_{i\neq j}(2m-d)^3E_{ij}^3$$

and then $\widetilde{D}^3 = d^3 - sm^3 + \binom{s}{2}(2m-d)^3$, which is exactly formula (3.10) for $n = 3$ and $\overline{r} = 1$. For instance, if $d = 3$, $m = 2$ and $s = 4$ then $\widetilde{D}^3 = 1$. Indeed, in this case $\mathcal{L} = \mathcal{L}_{3,3}(2,2,2,2)$ is the linear system inducing the standard Cremona transformation of $\mathbb{P}^3$.

4. Symmetries of GIT quotients

In this section, in analogy with Section 2.12, we describe a special arrangement of linear spaces contained in Σ_{2g-1}, and exploiting these arrangements we manage to compute the automorphism group of the GIT quotients Σ_{2g} and Σ_{2g-1}. In several cases, automorphisms of moduli spaces tend to be modular, in the sense that they can be described in terms of the objects parametrized by the moduli spaces themselves. See for instance, [BMe13], [Ma14], [MaM14], [MaM16], [FM16], [Ma16], [FM17], [Lin04], [Lin11], [Ro71] for moduli spaces of pointed and weighted curves, [BGM13] for moduli spaces of vector bundles over a curve, and [BM16] for generalized quot schemes. We confirm this behavior also for the GIT quotients Σ_{2g} and Σ_{2g-1}. We would like to stress that while the results in the above cited paper relies on arguments coming from birational geometry and moduli theory, in this case we use fairly different techniques based on explicit projective geometry.

4.1. **The odd dimensional case.** Let $p_1,...,p_{n+1}\in\mathbb{P}^n$ be general points, and X^n_{n+1} be the blow-up of $\mathbb{P}^n$ at $p_1,...,p_{n+1}$. We may assume that $p_1=[1:0:...:0],...,p_{n+1}=[0:...:0:1]$. Let us consider the standard Cremona transformation:

$$\begin{array}{rcl}\psi_n:\mathbb{P}^n & \dashrightarrow & \mathbb{P}^n\\ [x_0:...:x_n] & \longmapsto & [\frac{1}{x_0}:...:\frac{1}{x_n}]\end{array}$$

Note that $\psi_n\circ\psi_n=Id_{\mathbb{P}^n}$, and $\psi_n^{-1}=\psi_n$. Let $H_1,...,H_{n+1}$ be the coordinate hyperplanes of $\mathbb{P}^n$. Then ψ_n is not defined on the locus $\bigcup_{1\leq i<j\leq n+1}H_i\cap H_j$. Furthermore, ψ_n is an isomorphism off of the union $\bigcup_{1\leq i\leq n+1}H_i$.

Now, ψ_n induces a birational transformation $\widetilde{\psi}_n: X^n_{n+1}\dashrightarrow X^n_{n+1}$ and we have the following commutative diagram:

$$\begin{array}{ccc}X^n_{n+1} & \overset{\widetilde{\psi}_n}{\dashrightarrow} & X^n_{n+1}\\ \downarrow & & \downarrow\\ \mathbb{P}^n & \overset{\psi_n}{\dashrightarrow} & \mathbb{P}^n\end{array}$$

Note that, since ψ_n contracts the hyperplane H_i spanned by the n points $p_1,...,\hat{p}_i,...,p_{n+1}$ onto the point p_i, the map $\widetilde{\psi}_n$ maps the strict transform of H_i onto the exceptional divisor E_i. Therefore $\widetilde{\psi}_n$ is an isomorphism in codimension one. Indeed, it is a composition of flops. In particular $\widetilde{\psi}_n$ induces an isomorphism $\mathrm{Pic}(X^n_{n+1})\rightarrow\mathrm{Pic}(X^n_{n+1})$.

Now, the linear system on $\mathbb{P}^n$ associated to the standard Cremona transformation ψ_n is $\mathcal{H}=\mathcal{O}_{\mathbb{P}^n}(n)\otimes\mathcal{I}_{(n-1)(p_1+...+p_{n+1})}$, that is $\mathcal{H}$ is the linear system of hypersurfaces in $\mathbb{P}^n$ of degree n having points of multiplicity at least $n-1$ in $p_1,...,p_{n+1}$. Therefore, the inverse image of a general hyperplane of $\mathbb{P}^n$ via ψ_n is an hypersurface of degree n with points of multiplicity $n-1$ in $p_1,...,p_{n+1}$, and $\widetilde{\psi}_n^*H=nH-(n-1)(E_1+...+E_{n+1})$.

Furthermore, since ψ_n contracts the hyperplane H_i spanned by the n points $p_1,...,\hat{p}_i,...,p_{n+1}$ onto the point p_i we have $\widetilde{\psi}_n^*E_i=H-E_1-...-\hat{E}_i-...-E_{n+1}$.

Lemma 4.2. *Let $D\subset\mathbb{P}^n$ be a hypersurface of degree d having points of multiplicities $m_1,...,m_{n+1}$ in $p_1,...,p_{n+1}$, and let $\psi_n:\mathbb{P}^n\dashrightarrow\mathbb{P}^n$ be the standard Cremona transformation* *of* $\mathbb{P}^n$*. Then*

$$\deg(\psi_n(D)) = dn - \sum_{i=1}^{n+1} m_i$$

and

$$\operatorname{mult}_{p_i} \psi_n(D) = d(n-1) - \sum_{j\neq i} m_j$$

for any $i = 1, ..., n+1$*.*

Proof. Let $X_{n+1}^n = Bl_{p_1,...,p_{n+1}}\mathbb{P}^n$, and $\widetilde{\psi}_n : X_{n+1}^n \dashrightarrow X_{n+1}^n$ be the birational map induced by ψ_n. The strict transform of D in $X_{n+1}^n \dashrightarrow X_{n+1}^n$ can be written as $\widetilde{D} \cong dH - \sum_{i=1}^{n+1} m_i E_i$. Now, since $\widetilde{\psi}_{n*}H = nH - \sum_{i=1}^{n+1}(n-1)E_i$, and $\widetilde{\psi}_{n*}E_i = H - \sum_{j\neq i} E_i$ we get the formula

$$\begin{aligned} \widetilde{\psi}_{n*}D = \ & d(nH - \textstyle\sum_{i=1}^{n+1} E_i) - \sum_{i=1}^{n+1} m_i(H - \sum_{j\neq i} E_j) = \\ & dnH - d\textstyle\sum_{i=1}^{n+1}(n-1)E_i - \sum_{i=1}^{n+1} H + \sum_{i=1}^{n+1} m_i \sum_{j\neq i} E_j = \\ & (dn - \textstyle\sum_{i=1}^{n+1} m_i)H - \sum_{i=1}^{n+1}(d(n-1) - \sum_{j\neq i} m_j)E_j. \end{aligned}$$

which gives exactly the statement. □

Proposition 4.3. *The standard Cremona transformation* $\psi_{2g-1} : \mathbb{P}^{2g-1} \dashrightarrow \mathbb{P}^{2g-1}$ *induces an automorphism of the odd dimensional GIT quotient* Σ_{2g-1}*.*

Proof. Recall that Σ_{2g-1} is closure of the image of the map $\sigma_g : \mathbb{P}^{2g-1} \dashrightarrow \mathbb{P}(H^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1})^*)$ induced by the linear system $\mathcal{L}_{2g-1}$ of degree g hypersurfaces having multiplicity $g-1$ at $p_1, ..., p_{2g}$ and $p_{2g+1} = [1 : ... : 1]$. Note that $\psi_{2g-1}(p_{2g+1}) = p_{2g+1}$.

Now, let $D \in H^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1})$ be a section. By Lemma 4.2 we get $\deg(\psi_{2g-1}(D)) = g(2g-1) - \sum_{i=1}^{2g} m_i = g(2g-1) - 2g(g-1) = g$ and $\operatorname{mult}_{p_i} \psi_{2g-1}(D) = g(2g-2) - \sum_{j\neq i} m_j = g(2g-2) - (2g-1)(g-1) = g-1$ for $i = 1, ..., 2g$.

Furthermore, since ψ_{2g-1} is an isomorphism in a neighborhood of p_{2g+1} we have that $\operatorname{mult}_{p_{2g+1}} \psi_{2g-1}(D) = g-1$ as well.

Therefore, ψ_{2g-1} acts on the sections of $\mathcal{L}_{2g-1}$, and hence it induces an automorphism $\widetilde{\psi}_{2g-1}$ of $\mathbb{P}(H^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1})^*)$ that keeps Σ_{2g-1} stable. □

Example 4.4. The famous Segre cubic 3-fold is the image of the map

$$\sigma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$$

given by

$$\sigma(x_0, x_1, x_2, x_3) = [x_0x_2 - x_0x_1 : x_0x_3 - x_0x_1 : x_1x_2 - x_0x_1 : x_1x_3 - x_0x_1 : x_2x_3 - x_0x_1]$$

and the standard Cremona transformation of $\mathbb{P}^3$ may be written as

$$\phi_3(x_0, x_1, x_2, x_3) = [x_0x_1x_2 : x_0x_1x_3 : x_0x_2x_3 : x_1x_2x_3]$$

Let $z_0, ..., z_4$ be the homogeneous coordinates on $\mathbb{P}^4$. Then the equation of the Segre cubic is

$$z_0z_1z_2 - z_0z_1z_3 - z_0z_2z_3 + z_1z_2z_3 - z_1z_2z_4 + z_0z_3z_4 = 0 \tag{4.5}$$

and

$$\begin{array}{ll} \phi_3(z_0) = \phi_3(x_0x_2 - x_0x_1) = & x_0x_1x_2x_3(x_0x_2 - x_0x_1) \\ \phi_3(z_1) = \phi_3(x_0x_3 - x_0x_1) = & x_0x_1x_2x_3(x_1x_2 - x_0x_1) \\ \phi_3(z_2) = \phi_3(x_1x_2 - x_0x_1) = & x_0x_1x_2x_3(x_0x_3 - x_0x_1) \\ \phi_3(z_3) = \phi_3(x_1x_3 - x_0x_1) = & x_0x_1x_2x_3(x_1x_3 - x_0x_1) \\ \phi_3(z_2) = \phi_3(x_2x_3 - x_0x_1) = & x_0x_1x_2x_3(x_2x_3 - x_0x_1) \end{array}$$

Therefore, ψ_3 induces the automorphism $\widetilde{\psi}_3(z_0,z_1,z_2,z_3,z_4) = [z_0 : z_2 : z_1 : z_3 : z_4]$ of $\mathbb{P}^4$ which clearly preserves Equation (4.5).

4.6. **Linear subspaces of dimension g in Σ_{2g-1}.** In this section we will study a particular configuration of g-planes contained in Σ_{2g-1}, and then we will exploit this configuration to compute the symmetries of Σ_{2g-1}.

4.7. Let $H_I = H_{i_1,...,i_{g+1}}$ be the g-plane in $\mathbb{P}^{2g-1}$, linear span of the points $p_{i_1},...,p_{i_{g+1}}$, and let $\mathcal{L}_{2g-1|H_I}$ be the restriction to H_I of the linear system $\mathcal{L}_{2g-1}$ inducing σ_g. Then $\mathcal{L}_{2g-1|H_I}$ is the linear system of degree g hypersurfaces in $H_I \cong \mathbb{P}^g$ having multiplicity $g-1$ at $p_{i_1},...,p_{i_{g+1}}$. This means that $\sigma_{g|H_I}$ is the standard Cremona transformation of $\mathbb{P}^g$. Therefore, $\sigma_g(H_I)$ is a g-plane in Σ_g passing through the singular points given by the contractions of the $(g-1)$-planes generated by subsets of cardinality g of $\{p_{i_1},...,p_{i_{g+1}}\}$. Now let Π_{I^c} the $(g-1)$-plane generated by the points in $\{p_1,...,p_{2g+1}\} \setminus \{p_{i_1},...,p_{i_{g+1}}\}$. Note that H_{I^c} intersects H_I in one point, hence $\sigma_g(H_I)$ passes through the singular point $\sigma_g(H_{I^c})$ as well. We conclude that there are $g+1+1 = g+2$ singular points of Σ_{2g-1} lying on the g-plane $\sigma_g(H_I)$.

Now, let $E_I^{g-2} \subset X_{g-2}^{2g-1}$ be the exceptional divisor over the $(g-2)$-plane H_I^{g-2} generated by the p_i's for $i \in I$. Then E_I^{g-2} is a $\mathbb{P}^g$-bundle over the strict transform of H_I^{g-2} in X_{g-3}^{2g-1}. For any $j \notin I$ let $H_{I\cup\{j\}}^{g-1}$ be the $(g-1)$-plane generated by the p_i's for $i \in I$ and p_j. Note that the strict transform of $H_{I\cup\{j\}}^{g-1}$ intersects E_I^{g-2} along a section s which is mapped by the blow-up morphism isomorphically onto H_I^{g-2}. Since the strict transform of $H_{I\cup\{j\}}^{g-1}$ is contracted to a point by $\tilde{\sigma}_g$, the section s must be contracted to a point as well. Therefore, $\tilde{\sigma}_g(E_I^{g-1})$ is a g-plane passing through $g+2$ singular points of Σ_{2g-1}.

So far we have found $\binom{2g+1}{g+1} + 2g + 1$ linear spaces of dimension g in Σ_{2g-1}, and each of them contains at least $g+2$ of the $\binom{2g+1}{g}$ singular points of Σ_{2g-1}. We will divide the g-planes inside Σ_{2g-1} passing through a singular point $p \in \Sigma_{2g-1}$ into two families according to their mutual intersection.

4.8. The singular locus of Σ_{2g-1} consists of $\binom{2g+1}{g}$ points corresponding to the $\binom{2g+1}{g}$ linear subspaces $\langle p_{i_1},...,p_{i_g}\rangle$ with $\{i_1,...,i_g\} \subset \{1,...,2g+1\}$ contracted by σ_g.

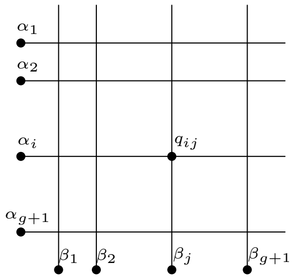
Now, let $p \in \Sigma_{2g-1}$ be a singular point. So far, we found $2g+2$ linear subspaces of dimension g in Σ_{2g-1} passing through $p \in \Sigma_g$. These g-planes may be divided in two families:

$$\mathbf{A}_p = \{\alpha_1,...,\alpha_{g+1}\},\ \mathbf{B}_p = \{\beta_1,...,\beta_{g+1}\}$$

with the following properties:

- $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \{p\}$ for any $i,j = 1,...,g+1$,
- $\alpha_i \cap \beta_j = \langle p, q_{ij}\rangle$ for any $i,j = 1,...,g+1$, where $q_{ij} \in \Sigma_{2g-1}$ is a singular point $q_{ij} \neq p$.

The configuration is summarized in the following picture:



where the black dots should all be interpreted as representing the singular point $p \in \Sigma_{2g-1}$.

4.9. Let $R = \{I \subset \{1, ..., 2g\} \,|\, |I| = g\}$, $S = \{J \subset \{1, ..., 2g\} \,|\, |J| = g-2\}$, and $x_I = x_{i_1}...x_{i_g}$ where $I = \{i_1, ..., i_g\}$, and the x_i's are homogeneous coordinates on $\mathbb{P}^{2g-1}$. By [Ku00, Theorem 4.1] we have that

$$H^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1}) = \left\{ \sum_{I \in R} a_I x_I \mid \sum_{J \subset I \in R} a_I = 0 \, \forall \, J \in S \right\}. \tag{4.10}$$

By Proposition 3.4 we have $h^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1}) = \binom{2g}{g} - \binom{2g}{g-2}$. Now, set $N = \binom{2g}{g} - \binom{2g}{g-2} - 1$, and consider the expressions $s_i = \sum_{I \in R} a_I^i x_I$ for $i = 0, ..., N$. Let $H_1, ..., H_{N+2}$ be $(g-1)$-planes in $\mathbb{P}^{2g-1}$ generated by subsets of cardinality g of $\{[1:0:...:0], ..., [0:...:0:1]\} \subset \mathbb{P}^{2g-1}$. Note that imposing $[s_1(H_i):...:s_N(H_i)] = [0:...:0:1:0:...:0]$, with the non-zero entry in the i-th position, for $i = 1, ..., N+1$ we get $N(N+1)$ equations. Furthermore, by setting $[s_1(H_{N+2}):...:s_N(H_{N+2})]$ equal to $[1:...:1]$ we get N more equations. Recall that by (4.10) for each $i = 0, ..., N$ there are $\binom{2g}{g-2}$ relations among the a_I^i's. Therefore, we get $\binom{2g}{g-2}(N+1)$ more constraints. Summing up we have a linear system of $N(N+1) + N + \binom{2g}{g-2}(N+1)$ homogeneous equations in the $\binom{2g}{g}(N+1)$ indeterminates a_I^i. Note that

$$\binom{2g}{g}(N+1) - \left(N(N+1) + N + \binom{2g}{g-2}(N+1)\right) = 1,$$

hence there exists a non-trivial solution. Let $s_0, ..., s_N$ be the sections of $H^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1})$ associated to such a solution. These sections yield an explicit realization of the map $\sigma_g : \mathbb{P}^{2g-1} \dashrightarrow \Sigma_{2g-1} \subset \mathbb{P}(H^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1})^*) = \mathbb{P}^N$, $\sigma_g(x) = [s_0(x):...:s_N(x)]$. By construction and by the description of the singular locus of Σ_{2g-1} in 4.8, we see that, with respect to this expression for σ_g, the points $[1:0:...:0], ..., [0:...:0:1], [1:...:1] \in \mathbb{P}^N$ are singular points of Σ_{2g-1}.

Now we need two technical lemmas.

Lemma 4.11. *Let $p \in \Sigma_{2g-1} \subset \mathbb{P}^N$ be a singular point. The tangent cone of Σ_{2g-1} at p is a cone with vertex p over the Segre product $\mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$. In particular, Σ_{2g-1} has an ordinary singularity of multiplicity $\frac{(2g-2)!}{((g-1)!)^2}$ at $p \in \Sigma_{2g-1}$.*

Proof. The statement follows from [HMSV09, Lemma 4.3]. □

Corollary 4.12. *Any automorphism of the GIT quotient $\Sigma_{2g-1} \subset \mathbb{P}^N$ is induced by an automorphism of $\mathbb{P}^N$.*

Proof. Recall that, by Proposition 2.6, $\mathrm{Pic}(\Sigma_{2g-1})$ is torsion free. Let ϕ be an automorphism of Σ_{2g-1}. Then $\phi^*K_{\Sigma_{2g-1}} \sim K_{\Sigma_{2g-1}}$. Lemma 2.5 yields that $\phi^*\mathcal{O}_{\Sigma_{2g-1}}(1) \sim \mathcal{O}_{\Sigma_{2g-1}}(1)$, that is ϕ is induced by an automorphism of $\mathbb{P}^N$. □

Theorem 4.13. *The automorphism group of the GIT quotient Σ_{2g-1} is the symmetric group on $2g+2$ elements:*

$$\mathrm{Aut}(\Sigma_g) \cong S_{2g+2}$$

for any $g \geq 2$.

Proof. Let $p \in \Sigma_{2g-1} \subset \mathbb{P}^N$ be a singular point, and H a g-plane inside Σ_{2g-1} passing through p. Then H is contained in the tangent cone of Σ_{2g-1} at p, and by Lemma 4.11 H must be a linear space generated by p and $(g-1)$-plane in the Segre embedding of $\mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$, and such a $(g-1)$-plane must be either of the form $\{pt\} \times \mathbb{P}^{g-1}$ or of the form $\mathbb{P}^{g-1} \times \{pt\}$.

Assume that H intersects the g-planes of the family $\mathbf{A}_p$ in the line $\langle p, q_{ij}\rangle$, where $q_{i,j} = \alpha_i \cap \beta_j$ is a singular point of Σ_{2g-1}, and the planes of the family $\mathbf{B}_p$ in p. Our aim is to prove that then H must be one of the β_j's.

By Proposition 2.3, the resolution $\widetilde{\sigma}_g$ has a modular interpretation as the reduction morphism $\rho_{\widetilde{A}[2g+2],A[2g+2]} : \overline{\mathcal{M}}_{0,A[2g+2]} \to \overline{\mathcal{M}}_{0,\widetilde{A}[2g+2]}$. The only g-planes in Σ_{2g-1} that are images of subvarieties contained in exceptional locus of the blow-up $f : \overline{\mathcal{M}}_{0,A[2g+2]} \to \mathbb{P}^{2g-1}$ are the $\widetilde{\sigma}_g(E_I^{g-1})$ described in Section 4.7. Therefore, we may assume that H is the closure of the image via σ_g of a g-dimensional variety $Z \subset \mathbb{P}^{2g-1}$. We may assume that $p = \sigma_g(\langle p_1, ..., p_g\rangle)$, and consider $\Pi = \sigma_g(\langle p_{g+1}, ..., p_{2g+1}\rangle)$. Let $\xi_i = \sigma_g(\langle p_{g+1}, ..., \hat{p}_i, ..., p_{2g+1}\rangle)$ be the other $g+1$ singular points of Σ_{2g-1} lying on Π, and denote by $\pi_p : \mathbb{TC}_p\Sigma_g \to \mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$ the projection from the tangent cone of Σ_{2g-1} at p onto its base, and let $\pi_i : \mathbb{P}^{g-1} \times \mathbb{P}^{g-1} \to \mathbb{P}^{g-1}$ be the projection onto the factors for $i = 1, 2$. Then H must be of the form $(\pi_1 \circ \pi_p)^{-1}(\xi_i)$ for some $i = 1, ..., g+1$. Therefore, Z is of the form $\langle p_1, ..., p_g, p_i\rangle$ for some $i = g+1, ..., 2g+1$. Hence $\sigma_{g|Z} : Z \dashrightarrow H$ is the standard Cremona transformation of $\mathbb{P}^g$ and H is one of the β_i's.

Now, assume that H is a g-plane inside Σ_{2g-1} through p intersecting the g-planes of the family $\mathbf{B}_p$ in the line $\langle p, q_{ij}\rangle$, where $q_{i,j} = \alpha_i \cap \beta_j$ is a singular point of Σ_{2g-1}, and that it also intersects the planes of the family $\mathbf{A}_p$ in p. By Proposition 4.3, the standard Cremona transformation of $\mathbb{P}^{2g-1}$ induces an automorphism $\phi_{Cr} : \Sigma_{2g-1} \to \Sigma_{2g-1}$, set $q = \phi_{Cr}(p)$. Note that ϕ_{Cr} maps the family $\mathbf{A}_p$ to the family $\mathbf{B}_q$, and the family $\mathbf{B}_p$ to the family $\mathbf{A}_q$. Since in our argument p is an arbitrary singular point of Σ_{2g-1}, proceeding as in the previous case we can show that H must be one of the α_i's.

Clearly, since $\Sigma_{2g-1} \cong \overline{\mathcal{M}}_{0,\widetilde{A}[2g+2]}$ with symmetric weights $A[2g+2] = \left(\frac{1}{g+1}, ..., \frac{1}{g+1}, \frac{1}{g+1}\right)$, the symmetric group S_{2g+2} acts on Σ_{2g-1} by permuting the marked points. Our aim is now to show that Σ_{2g-1} has at most $(2g+2)!$ automorphisms.

Now, let $\phi : \Sigma_{2g-1} \to \Sigma_{2g-1}$ be an automorphism. If $p \in \Sigma_{2g-1}$ is a singular point then $\phi(p)$ must be a singular point as well. Therefore, we have at most $|\mathrm{Sing}(\Sigma_{2g-1})| = \binom{2g+1}{g}$ choices for the image of p.

By Corollary 4.12 ϕ is induced by a linear automorphism of the ambient projective space. Then ϕ must map g-planes through p to g-planes through $\phi(p)$. In particular, since ϕ stabilizes $\mathrm{Sing}(\Sigma_{2g-1})$, it maps $A_p \cup B_p$ to $A_{\phi(p)} \cup B_{\phi(p)}$. In order to do this, we have the following $2((g+1)!)^2$ possibilities:

$$\left\{\begin{array}{l}\alpha_{i,p} \mapsto \alpha_{j,\phi(p)}\\ \beta_{i,p} \mapsto \beta_{k,\phi(p)}\end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l}\alpha_{i,p} \mapsto \beta_{j,\phi(p)}\\ \beta_{i,p} \mapsto \alpha_{k,\phi(p)}\end{array}\right.$$

Summing up, we have

$$2((g+1)!)^2\binom{2g+1}{g} = (2g+2)!$$

possibilities. Now, assume that $\phi(p) = p$, and that ϕ maps $\alpha_{i,p}$ to $\alpha_{i,p}$, and $\beta_{i,p}$ to $\beta_{i,p}$ for any $i = 1, ..., g+1$. Then ϕ must fix all the singular points q_{ij} determined by $\mathbf{A}_p \cup \mathbf{B}_p$.

Now, let us take into account one of these points, say $q_{g+1,g+1}$. Since ϕ fixes $g+2$ nodes in linear general position on α_{g+1}, and $g+2$ nodes in linear general position on β_{g+1} we have that ϕ is the identity on both α_{g+1} and β_{g+1}. On the other hand, α_{g+1} and β_{g+1} are elements of the configuration $\mathbf{A}_{q_{g+1}} \cup \mathbf{B}_{q_{g+1}}$, therefore all the singular points of Σ_{2g-1} determined by $\mathbf{A}_{q_{g+1}} \cup \mathbf{B}_{q_{g+1}}$ are fixed by ϕ as well. Proceeding recursively this way, we see that ϕ must then fix all the singular points of $\Sigma_{2g-1} \subset \mathbb{P}^N$.

Note that, with respect to the expression for σ_g given in 4.9, the points $[1:0...:0], [0:...:0:1], [1:...:1] \in \Sigma_{2g-1} \subset \mathbb{P}(H^0(\mathbb{P}^{2g-1}, \mathcal{L}_{2g-1}^*)) = \mathbb{P}^N$ are singular points.

Hence the automorphism of $\mathbb{P}^N$ inducing ϕ fixes $N+2$ points in linear general position. Therefore, it is the identity and then $\phi = Id_{\Sigma_{2g-1}}$. □

This completes the proof for Σ_{2g-1}, in the next section we will go through to case of Σ_{2g}.

4.14. **The even dimensional case.** For the reader's relief, in this section we will make large use of results that we have already proven in the preceding sections. Let us recall shortly the notation. Recall that Σ_{2g} parametrizes ordered configurations of $n = 2g+3$ points on $\mathbb{P}^1$, with the democratic polarization. By Theorem 1.2, $\Sigma_{2g} \subset \mathbb{P}(H^0(\mathbb{P}^{2g}, \mathcal{L}_{2g})^*)$ is the closure of the image of the rational map induced by the linear system $\mathcal{L}_{2g}$, see also Section 2.9. We denote by $\mu_g : \mathbb{P}^{2g} \dashrightarrow \Sigma_{2g} \subset \mathbb{P}(H^0(\mathbb{P}^{2g}, \mathcal{L}_{2g})^*) = \mathbb{P}^N$ the rational map induced by this linear system.

Lemma 4.15. *Any automorphism of $\Sigma_{2g} \subset \mathbb{P}^N$ is induced by an automorphism of $\mathbb{P}^N$.*

Proof. Recall that by 2.16 $\mathrm{Pic}(\Sigma_{2g}) \cong \mathbb{Z}^{2g+3}$. Then it is enough to argue as in the proof of Corollary 4.12 using Lemma 2.18 instead of Lemma 2.5. □

Theorem 4.16. *The automorphism group of the GIT quotient Σ_{2g} is the symmetric group on n elements:*

$$\mathrm{Aut}(\Sigma_{2g}) \cong S_{2g+3}$$

for any $g \geq 1$.

Proof. Let $\phi \in \mathrm{Aut}(\Sigma_{2g})$ be an automorphism. By the discussion in 2.16, ϕ induces a pseudo-automorphism, that is an automorphism in codimension two,

$$\theta := \psi^{-1} \circ \phi \circ \psi : X_1^{2g} \dashrightarrow X_1^{2g}.$$

The pseudo-automorphism θ must preserve the set of the extremal rays of $\mathrm{Eff}(X_1^{2g})$. Let $D \subset X_1^{2g}$ be the union of the exceptional divisors E_i and of the strict transforms of the hyperplanes generated by $2g$ of the p_i's. By Lemma 2.17 any irreducible component of D generates an extremal ray of $\mathrm{Eff}(X_1^{2g})$. Furthermore, any of these irreducible components is the unique element in its linear equivalence class. Therefore, θ must keep D stable.

Set $\widetilde{D} = \psi(D)$. Then the automorphism ϕ stabilizes $\widetilde{D}$. As we did in Section 2.12, we will now consider g-planes; check that section for the needed definitions. Note that any g-plane $\gamma_i \in \mathbf{C}$ is the intersection of $\binom{g+1}{g-1}$ divisors of type $\psi(H - \sum_{i\in I} E_i)$, and that any g-plane $\delta_i \in \mathbf{D}$ is the intersection of g divisors of type $\psi(E_i)$. Therefore ϕ stabilizes the configuration of g-planes $\mathbf{C} \cup \mathbf{D}$.

Now, let $p \in \Sigma_{2g}$ be one of the distinguished points in 2.14. Then $q = \phi(p)$ must be a distinguished point as well. Let us denote by H_1, H_2 the two g-planes intersecting in $\{p\}$, and by Π_1, Π_2 the two g-planes intersecting in $\{q\}$. Therefore, by 2.14 we have $\frac{(2g+3)!}{2((g+1)!)^2}$ choices for the image of p.

Let $p, p_1^1, ..., p_{g+1}^1$ and $p, p_1^2, ..., p_{g+1}^2$ be the distinguished points on H_1 and H_2 respectively. Similarly, we denote by $q, q_1^1, ..., q_{g+1}^1$ and by $q, q_1^2, ..., q_{g+1}^2$ the distinguished points on Π_1 and Π_2, respectively. Now, for the image of p_1^1 we have $2(g+1)$ possibilities, namely $q_1^1, ..., q_{g+1}^1, q_1^2, ..., q_{g+1}^2$. Once this choice is made, it is determined whether the image of H_1 via ϕ is either Π_1 or Π_2. Therefore, for the image of p_2^1 we have g possibilities, for p_3^1 we have $g-1$ possibilities, and so on until we are left with just one possibility for p_{g+1}^1. Summing up, for the images of $p_1^1, ..., p_{g+1}^1$ we have $2(g+1)!$ possibilities. Similarly, we have also $(g+1)!$ possibilities for the images of $p_1^2, ..., p_{g+1}^2$. Finally, we have

$$2(g+1)!(g+1)!\frac{(2g+3)!}{2((g+1)!)^2} = (2g+3)! = |S_{2g+3}|$$

possibilities. Now, assume that ϕ fixes the points $p, p_1^1, ..., p_{g+1}^1, p_1^2, ..., p_{g+1}^2$. Since by Corollary 4.12 ϕ is induced by an automorphism of $\mathbb{P}^N$ and the points $p, p_1^1, ..., p_{g+1}^1 \in H_1$, $p, p_1^2, ..., p_{g+1}^2 \in H_2$ are in linear general position, then ϕ restricts to the identity on both H_1 and H_2.

Then, by the description of the configuration $\mathbf{C} \cup \mathbf{D}$ in 2.13, ϕ must fix all the distinguished points in $\mathbf{C} \cup \mathbf{D}$, and then Remark 2.15 yields that the automorphism ϕ must be the identity. ☐

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