# On the Motion of Several Disks in an Unbounded Viscous Incompressible Fluid

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**Abstract**

In this paper, we study the time evolution of a finite number of homogeneous rigid disks within a viscous homogeneous incompressible fluid in the whole domain $\mathbb{R}^2$. The motion of the fluid is governed by Navier-Stokes equations, whereas the movement of each rigid body is described by the standard conservation laws of linear and angular momentum. The motion of the rigid bodies inside the fluid makes the fluid domain time dependent and unknown *a priori*. At first, we prove the local existence and uniqueness of strong solutions of the considered problem and then by careful analysis of how elliptic estimates for the Stokes operator depend on the geometry of the fluid domain, we extend these solutions up to collision. Finally, we prove contact between rigid bodies could not occur for almost arbitrary configurations.

**Keywords.** Navier–Stokes equations, rigid bodies, strong solutions, contact problem.

## 1 Introduction

We consider a finite number of homogeneous rigid bodies – each being represented by a closed disk $B_i(t) \subset \mathbb{R}^2$– moving in a viscous homogeneous incompressible fluid which occupies a domain $\Omega_F(t)$ at time $t$, where $\Omega_F(t) = \mathbb{R}^2 \backslash \bigcup_{i=1}^{k} B_i(t)$, with $k \in \mathbb{N}^*$ denoting the number of rigid bodies.

We suppose that the fluid is of viscosity $\nu > 0$, pressure $p$, velocity field $u$ and for simplicity, of density one. The motion of the fluid is governed by the Navier-Stokes equations for incompressible fluids:

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \quad x \in \Omega_F(t), \quad t \in (0,T), \tag{1.1}$$

$$\nabla \cdot u = 0, \quad x \in \Omega_F(t), \quad t \in (0,T), \tag{1.2}$$

where $f \in L^2((0,T) \times \mathbb{R}^2)$ denotes an external body force.

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For each rigid body, we define the density $\overline{\rho}_i$, the center of mass $h_i(t)$, the angular velocity $\omega_i(t)$ and the inertia matrix $J_i$ related to the center of mass of the $i$-th body by

$$\overline{\rho}_i = \frac{m_i}{|B_i(0)|}, \quad h_i(t) = \frac{1}{|B_i(0)|}\int_{B_i(t)} x\,dx, \quad J_i(t) = \int_{B_i(t)} \overline{\rho}_i |x - h_i(t)|^2 dx = \int_{B_i(0)} \overline{\rho}_i |y|^2 dy,$$

where $m_i$ denotes the mass of the $i$-th body.

The motion of the $i$-th body is governed by the balance equations for linear and angular momentum (Newton's Laws):

$$m_i h_i^{''}(t) = -\int_{\partial B_i(t)} \sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i(t)} f(t)dx, \qquad t \in (0,T), \tag{1.3}$$

$$J_i \omega_i^{'}(t) = -\int_{\partial B_i(t)} (x - h_i(t))^{\perp} \cdot \sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i(t)} (x - h_i(t))^{\perp} \cdot f(t)dx, \qquad t \in (0,T). \tag{1.4}$$

In the above equations, the matrix $\sigma$ denotes the Cauchy stress tensor in the fluid and is given by

$$\sigma(u,p) = -pI + 2\nu D[u],$$

where $I$ is the identity matrix and $D[u]$ denotes the rate of deformation tensor defined as follows

$$D[u] = \frac{1}{2}(\nabla u + \nabla u^T).$$

We denote by $(x_1, x_2)^{\perp} = (-x_2, x_1)$ the orthogonal vector of $(x_1, x_2)$ and we use the notation $\partial B_i(t)$ to denote the boundary of the $i$-th body at time $t$. The symbol $\nu_i(x,t)$ stands for the unit normal vector directed toward the interior of the $i$-th body. For simplicity, $\Omega_F(0)$ and $B_i(0)$ will be denoted later on by $\Omega_F$ and $B_i$ respectively.

We impose the no-slip boundary conditions at the fluid/rigid body interfaces

$$u(x,t) = h_i^{'}(t) + \omega_i(t)(x - h_i(t))^{\perp},\ x \in \partial B_i(t),\ t \in [0,T], \quad i \in \{1,\ldots,k\}. \tag{1.5}$$

To complete the system, we impose initial conditions at $t = 0$ :

$$u|_{\Omega_F} = u_0,\ h_i(0) = h_i^0,\ h_i'(0) = h_i^1,\ \omega_i(0) = \omega_i^0,\ \forall i \in \{1,\ldots,k\}. \tag{1.6}$$

Throughout this paper, we assume that there is no contact initially between the rigid bodies; that is

$$\gamma = \min_{1 \le i,j \le k} \{d(B_i(0), B_j(0)) : i \ne j\} > 0. \tag{1.7}$$

The problem of existence of weak solutions of problem (1.1)-(1.6) has been the subject of intensive studies of many authors. We mention here Desjardins and Esteban [4] and [5]; Conca, San Martin, and Tucsnak [2]; Gunzburger, Lee, and Seregin [13]; Hoffmann and Starovoitov [16]; San Martin, Starovoitov, and Tucsnak [21]; Serre [22], Judakov [18], and Silvestre [23]. Most of the above references deal with the case of a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ and all of them - except [4] and [5] - discuss the case of a single rigid body of arbitrary shape. Recently, Glass and Sueur investigated the problem of uniqueness of weak solutions in dimension 2 in [11] as long as no collision occurs. In dimension 3, the question of existence of global weak solutions was answered by Feireisl in [7]. However, the uniqueness of such solutions is still an open question even considering pre-collisional times. After contact, it is likely uniqueness does not hold as we miss some entropy condition describing the post-collisional dynamics.

As far as we know, the problem of existence of strong solutions for problem (1.1)-(1.6) in the case of single moving rigid body of arbitrary shape in a cavity is investigated in several studies. A local-in-time existence result of strong solutions in this case was proved in Grandmont and Maday [12], provided that the inertia of the rigid body is large enough with respect to the inertia of the fluid. Further development in this domain is the work of Takahashi in [24]. The author proves the existence and uniqueness of global strong solution without taking in consideration the assumption in [12]. The first *no collision* result for strong solutions was provided by T. Hesla [14] and M. Hillairet [15]. In [15], the author shows that any strong solution is global under the absence of external forces in the case of a moving disk in the half space $\mathbb{R}^2_+$. Thereafter, it has been studied the roughness-induced effect of the rigid body and the boundary of the domain on the collision process [10].

However, much less is known in the case of the fluid-rigid-body system filling the whole space. One of the available results in this case is due Takahashi and Tucsnak [25], where the authors prove the existence and uniqueness of strong solutions for an infinite cylinder in dimension 2. A similar result has been proved in Silvestre and Galdi [8] for a rigid body having an arbitrary form. Lately, Cumsile and Takahashi improved the result in [25]. They establish the existence and uniqueness of strong solution globally in dimension 2 and also in dimension 3 if the data are small enough [3]. A one-dimensional version of the problem of several rigid bodies is studied in Vázquez and Zuazua [26] where the asymptotic behavior of solutions is also investigated. Another approach of studying this problem is developed in [9] where the authors prove the existence of a unique, local, strong solution in the $L^p$ setting.

In this paper, we aim to generalize the local existence result of Takahashi in [24] and that of Cumsille and Takahashi in [3] to the case of several rigid bodies. In this respect, we establish the following theorem:

**Theorem 1.1** *Suppose that* $f \in L^2(0,T;\mathbf{L}^2(\mathbb{R}^2))$, $\gamma > 0$, $h_i^0 \in \mathbb{R}^2$, $h_i^1 \in \mathbb{R}^2$, $\omega_i^0 \in \mathbb{R}$, $u_0 \in \mathbf{H}^1(\mathbb{R}^2)$, *and that*

$$\begin{aligned} \nabla \cdot u_0 &= 0, && \text{in } \Omega_{\mathrm{F}}, \\ u_0(x) &= h_i^1 + \omega_i^0 (x - h_i^0)^\perp, && x \in \partial B_i,\ \forall i \in \{1,\dots,k\}. \end{aligned}$$

*Then there exists* $T_0 > 0$ *depending on* $\|u_0\|_{\mathbf{H}^1(\mathbb{R}^2)}$ *and* $\gamma$ *such that problem (1.1)-(1.6) admits a unique strong solution*

$$(u, p, (h_i, \omega_i)_{i\in\{1,\dots,k\}}) \in \mathcal{U}(0,T_1;\Omega_F(t)) \times L^2(0,T_1;\mathbf{H}^1(\Omega_F(t))) \times \left(H^2(0,T_1;\mathbb{R}^2) \times H^1(0,T_1;\mathbb{R})\right)^k,$$

*where*

$$\mathcal{U}(0,T_1;\Omega_F(t)) = L^2(0,T_1;\mathbf{H}^2(\Omega_F(t))) \cap \mathcal{C}([0,T_1],\mathbf{H}^1(\Omega_F(t))) \cap H^1(0,T_1;\mathbf{L}^2(\Omega_F(t)))$$

*on* $[0,T_1]$ *such that* $T_1 < T_0$.

*Moreover, one of the following alternatives holds true:*

1. $T_0 = +\infty$,
2. $\limsup_{t\to T_0} \|u(t)\|_{\boldsymbol{H}^1(\mathbb{R}^2)} + \dfrac{1}{\min_{i\neq j} d(B_i(t),B_j(t))} = +\infty.$

Then we adapt the method of Gérard-Varet and Hillairet in [10] to our case and we arrive to the following result:

**Theorem 1.2** *Assume that the hypotheses of Theorem 1.1 hold true and that*

$$\textit{the fluid domain is connected at any time.} \tag{H1}$$

*Then problem (1.1)-(1.6) admits a unique global strong solution.*

First, we prove that the $H^1$ norm of the velocity field $u$ does not explode in finite time as long as the rigid bodies are not in contact. Then we show that collision for almost arbitrary configurations could not take place in finite time. The proof is based on a combined fluid-body weak formulation of the equation of motion:

$$\int_0^t \int_{\mathbb{R}^2} \big(\rho u\cdot\partial_t v + \rho u\otimes u : D[v] - 2\nu D[u] : D[v] + \rho f\cdot v\big) dxds + \int_{\mathbb{R}^2} \rho_0 u_0 \cdot v(0) dx = \Big(\int_{\mathbb{R}^2} \rho u\cdot v dx\Big)(t),\ \forall v\in\mathcal{H}, \tag{1.8}$$

where

$$\mathcal{H} = \{v\in H^1((0,T)\times\mathbb{R}^2) :\ \nabla\cdot v = 0 \text{ in } \Omega_F(t),\ D[v]=0 \text{ on } B_i(t),\ 1\le i\le k\},$$

with $u$ and $\rho$ denote respectively the global velocity and density.

We act by contradiction and we assume that collision can occur in finite time. The idea is to construct a divergence free vector field $v$ and use it as a test function in (1.8). The test function $v$ is constructed in two steps: first we construct it locally on the bridge connecting the bodies close to contact point (see Figure 2) and then we extend it outside the bridges by a regular vector field. When two disks approach each other, the viscous term dominates the acceleration term leading to a differential inequality which can be integrated to obtain the *no collision* result.

The novelty of our work is that we prove the existence and uniqueness of global strong solutions in the case of several disks and the full system fills the whole domain $\mathbb{R}^2$. The main restriction of the global-in-time existence result in Theorem 1.2 is that we need the fluid domain to be connected at any time. However, this assumption is always valid in the case when we have just two moving bodies and that many body contacts are really unlikely if we start from a sufficiently dilute suspension of bodies.

The main difficulty to handle the case of more than two rigid bodies is that collision could possibly divide the fluid domain into several connected components. On such situation, each connecting bridge between the colliding particles has two connected components inside the fluid domain. Unfortunately, the flux does not vanish on each of the connected components even if their sum does. This prevents us from extending the vector field $v$ into the fluid domain outside the bridges by a divergence free vector field.

The plan of this paper is as follows: in Section 2, we write problem (1.1)-(1.6) in cylindrical domain by introducing a change of variable $X$ and then we prove the existence and uniqueness of local-in-time strong solution for our problem. The remaining two sections are left to prove that the unique strong solution is global under the assumption (H1).

# 2 Local existence for solution

In this section, we introduce the mapping $X$ that enables to transform the free boundary value problem (1.1)-(1.6) into a problem in cylindrical domain. We follow the same approach used in [3] and [24]. This approach is characterized by a non-linear, local change of coordinates in a neighborhood of the rigid body. We recall that such transform $X$ is initially introduced by Inoue and Wakimoto [17].

We fix k functions $h_i : t \mapsto h_i(t)$ such that for $i \in \{1, \dots, k\}$, we have $h_i \in H^2(0, T; \mathbb{R}^2)$. Moreover, from now on we fix $\varepsilon$ such that $0 < \varepsilon < \gamma$.

For $i \in \{1, \dots, k\}$, we define the cut-off function $\psi_i \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ with compact support contained in $B(h_i(0), r_i + \frac{\gamma}{2})$ and equal 1 in $\overline{B(h_i(0), r_i + \frac{\delta-\varepsilon}{2})}$, where $r_i$ denotes the radius of the $i$-th disk.
Also, we define the functions $w_i : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$ by

$$w_i(x_1, x_2, t) = h'_{i,2}(t)x_1 - h'_{i,1}(t)x_2, \;\; i \in \{1, \dots, k\}. \tag{2.1}$$

We define the mapping $\Lambda : \mathbb{R}^2 \times [0, T] \to \mathbb{R}^2$ by

$$\Lambda(x_1, x_2, t) = \sum_{i=1}^{k} \nabla^\perp (w_i \psi_i). \tag{2.2}$$

The mapping $X$ is defined as a solution of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial X}{\partial t}(y,t) & = \Lambda(X(y,t),t), \;\; t \in ]0,T], \\ X(y,0) & = y \in \mathbb{R}^2. \end{cases} \tag{2.3}$$

Finally, We define the functions $U$, $P$, and $F$ using the transform $X$ as follows:

$$U(y,t) = J_Y(X(y,t),t)u(X(y,t),t), \;\; P(y,t) = p(X(y,t),t) \text{ and } F(y,t) = J_Y(X(y,t),t)f(X(y,t),t), \tag{2.4}$$

where the diffeomorphism $Y(.,t)$ denotes the inverse mapping of $X(.,t)$ and $J_Y$ is the Jacobian matrix of the diffeomorphism $Y(.,t)$.
We state in the following proposition the system satisfied by $(U, P, (h_i, \omega_i)_{i=1,\dots,k})$:

**Proposition 2.1** *Suppose that for all $i \in \{1, \dots, k\}$, we have $h_i \in H^2(0, T; \mathbb{R}^2)$ is such that*

$$B_i(t) \subset B(h_i(0), r_i + \frac{\delta - \varepsilon}{2}), \;\; \forall t \in [0, T].$$

*Then*

$$(u,p,(h_i,\omega_i)_{i=1,\dots,k}) \in \mathcal{U}(0,T,\Omega_F(t)) \times L^2(0,T,\dot{H}^1(\Omega_F(t))) \times \left(H^2(0,T;\mathbb{R}^2)\times H^1(0,T;\mathbb{R})\right)^k$$

*satisfies problem (1.1)-(1.6) if and only if*

$$(U,P,(h_i,\omega_i)_{i=1,\dots,k}) \in \mathcal{U}(0,T,\Omega_F) \times L^2(0,T,\dot{H}^1(\Omega_F)) \times \left(H^2(0,T;\mathbb{R}^2)\times H^1(0,T;\mathbb{R})\right)^k$$

*satisfies the following equations:*

$$\frac{\partial U}{\partial t} - \nu[LU] + [MU] + [NU] + [GP] \qquad = F, \qquad \text{in } \Omega_\mathrm{F}\times]0,\mathrm{T}[, \tag{2.5}$$

$$\nabla\cdot U \qquad = 0, \qquad \text{in } \Omega_\mathrm{F}\times]0,\mathrm{T}[, \tag{2.6}$$

$$U(y,0) \quad = u_0(y), \quad y\in\Omega_F, \tag{2.7}$$

*and for all $i\in\{1,\dots,k\}$, we have:*

$$m_i h_i^{''}(t) = -\int_{\partial B_i}\Sigma\nu_i d\Gamma_i + \overline{\rho}_i\int_{\partial B_i}F(t)dy, \qquad t\in]0,T[, \tag{2.8}$$

$$J_i\omega_i^{'}(t) = -\int_{\partial B_i}(y-h_i(0))^\perp\cdot\Sigma\nu_i d\Gamma_i + \overline{\rho}_i\int_{\partial B_i}(y-h_i(0))^\perp\cdot F(t)dy, \qquad t\in]0,T[, \tag{2.9}$$

$$U(y,t) = h_i'(t) + \omega_i(t)(y-h_i(0))^\perp, \qquad \text{in } \partial\mathrm{B_i}\times[0,\mathrm{T}[, \tag{2.10}$$

$$h_i(0)=h_i^0,\;\; h_i'(0)=h_i^1,\;\; \omega_i(0)=\omega_i^0, \qquad \forall i\in\{1,\dots,k\}, \tag{2.11}$$

*where $\Sigma(U,P)$ is the Cauchy stress tensor field associated to $U$ and $P$. The operators $[LU],[MU],[NU]$ and $[GP]$ that appear in the left hand side of (2.5) are defined as follows:*

$$[LU]_i = \sum_{j,k=1}^2\frac{\partial}{\partial y_j}(g^{jk}\frac{\partial U_i}{\partial y_k}) + 2\sum_{j,k,\ell=1}^2 g^{k\ell}\Gamma^i_{j,k}\frac{\partial U_j}{\partial y_\ell} + \sum_{j,k,\ell=1}^2\left\{\frac{\partial}{\partial y_k}(g^{k\ell}\Gamma^i_{j,\ell}) + \sum_{m=1}^2 g^{k\ell}\Gamma^m_{j,\ell}\Gamma^i_{k,m}\right\}U_j, \tag{2.12}$$

$$[MU]_i = \sum_{j=1}^2\frac{\partial Y_j}{\partial t}\frac{\partial U_i}{\partial y_j} + \sum_{j,k=1}^2\left\{\Gamma^i_{j,k}\frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k}\frac{\partial^2 X_k}{\partial t\partial y_j}\right\}U_j, \tag{2.13}$$

$$[NU]_i = \sum_{j=1}^2 U_j\frac{\partial U_i}{\partial y_j} + \sum_{j,k=1}^2\Gamma^i_{j,k}U_jU_k, \tag{2.14}$$

$$[GP]_i = \sum_{j=1}^2 g^{ij}\frac{\partial P}{\partial y_j}, \tag{2.15}$$

*where for all $i,j,k\in\{1,2\}$, we have denoted*

$$g^{ij} = \sum_{k=1}^2\frac{\partial Y_i}{\partial x_k}\frac{\partial Y_j}{\partial x_k}, \tag{2.16}$$

$$\Gamma^k_{i,j} = \frac{1}{2}\sum_{\ell=1}^2 g^{k\ell}\left\{\frac{\partial g_{i\ell}}{\partial y_j} + \frac{\partial g_{j\ell}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_\ell}\right\}, \tag{2.17}$$

$$g_{ij} = \sum_{k=1}^2\frac{\partial X_k}{\partial y_i}\frac{\partial X_k}{\partial y_j}. \tag{2.18}$$

For a proof of this Proposition, we refer the reader to [17] and [24].

Following the same approach of [24], we get that for $T$ small enough, problem (2.5)-(2.11) admits a unique strong solution

$$(U, P, (h_i, \omega_i)_{i=1,\dots,k}) \in \mathcal{U}(0,T;\Omega_F) \times L^2(0,T;\mathbf{H}^1(\Omega_F)) \times \left(H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R})\right)^k.$$

Theorem 1.1 can be deduced then by using the inverse transform $Y$. For more details, we refer the reader to [20].

# 3 Estimating the $\mathbf{H}^1$-norm of $u(t)$

In the previous section, we prove that there exists a time $T > 0$ such that the problem (1.1)-(1.6) admits a unique strong solution $(u, p, (h_i, \omega_i)_{i=1,\dots k})$ in $[0, T]$. Moreover, if we define $T_0$ as the maximal time of existence of strong solutions, that is

$$T_0 := \sup \{T \in \mathbb{R}_+^* : \text{ problem } (1.1) - (1.6) \text{ admits a unique strong solution in } [0,T]\},$$

then one of the following alternatives holds true:

1. $T_0 = +\infty$,
2. $\limsup_{t \to T_0} \|u(t)\|_{\mathbf{H}^1(\mathbb{R}^2)} + \dfrac{1}{\min_{i \neq j} d(B_i(t), B_j(t))} = +\infty.$

In the present section, we aim to prove that the $H^1$ norm of the solution $u$ does not explode in finite time as long as there is no contact between the rigid bodies. More precisely, we state the following proposition:

**Proposition 3.1** *If* $T_0 < +\infty$ *and* $\min_{i \neq j} d(B_i(t), B_j(t)) > \varepsilon > 0$ *on* $[0, T_0]$*, then the mapping*

$$t \to \|u(t)\|_{\mathbf{H}^1(\mathbb{R}^2)}$$

*is bounded on* $[0, T_0)$ *by a constant that depends on* $\varepsilon, \gamma$ *and the initial data.*

The following lemma shows that the mapping $t \to \|u(t)\|_{\mathbf{L}^2(\mathbb{R}^2)}$ is bounded on $[0, T_0)$.

**Lemma 3.1** *Let* $(u, p, (h_i, \omega_i)_{i=1,\dots k})$ *be the strong solution associated to problem (1.1)-(1.6) on* $[0, T]$*. If* $T_0 < \infty$*, then there exists a positive constant* $M = M(T_0, (\overline{\rho}_i, B_i)_{i=1,\dots,k})$*, such that*

$$\sup_{[0,T_0)} \left(||u(t)||^2_{\mathbf{L}^2(\Omega_F(t))} + \sum_{i=1}^k (|h_i'(t)|^2 + |\omega_i(t)|^2)\right) + 2\nu \int_0^{T_0} ||\nabla u(t)||^2_{[L^2(\mathbb{R}^2)]^4} dt$$
$$\leq M\left(||f||^2_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))} + ||u_0||^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k (|h_i^1|^2 + |\omega_i^0|^2)\right). \quad (3.1)$$

Proof. By taking the inner product of equation (1.1) with $u$, and integrating over $\Omega_F(t)$, we get that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_F(t)} |u(t)|^2 dx + 2\nu \int_{\Omega_F(t)} |D[u(t)]|^2 dx = \sum_{i=1}^k \int_{\partial B_i(t)} \left(\sigma(u(t), p(t))u(t)\right) \cdot \nu_i d\Gamma_i + \int_{\Omega_F(t)} f(t) \cdot u(t) dx. \quad (3.2)$$

Taking now the inner product of (1.3) with $h_i'(t)$ and that of (1.4) by $\omega_i(t)$, and noting the no-slip condition (1.5), we obtain

$$\frac{m_i}{2}\frac{d}{dt}|h_i'(t)|^2 + \frac{J_i}{2}\frac{d}{dt}|\omega_i(t)|^2 = -\int_{B_i(t)} u(t)\cdot\sigma\nu_i d\Gamma_i$$
$$+\,\overline{\rho}_i\int_{B_i(t)} (h_i'(t)+\omega_i(t)(x-h_i(t))^{\perp})\cdot f(t)dx,\ \forall i\in\{1,\dots,k\}. \quad (3.3)$$

Combining (3.2) with the $k$ equations in (3.3) noting that the Cauchy stress tensor field $\sigma$ is symmetric yields

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_F(t)}|u(t)|^2dx + \sum_{i=1}^{k}(m_i|h_i'(t)|^2 + J_i|\omega_i(t)|^2)\right) + 2\nu\int_{\Omega_F(t)}|D[u(t)]|^2dx$$
$$= \int_{\Omega_F(t)} f(t)\cdot u(t)dx + \overline{\rho}_i\int_{B_i(t)}(h_i'(t)+\omega_i(t)(x-h_i(t))^{\perp})\cdot f(t)dx. \quad (3.4)$$

We denote by $(.,.)$ the inner product in $\mathbf{L}^2(\mathbb{R}^2)$ defined by

$$(\phi,\psi) = \int_{\Omega_F}\phi\cdot\psi\,dy + \sum_{i=1}^{k}\int_{B_i}\overline{\rho}_i\phi\cdot\psi\,dy,$$

and its associated norm $\|.\|_{\mathcal{L}^2(\mathbb{R}^2)}$. This latter norm is equivalent to the usual norm of $\mathbf{L}^2(\mathbb{R}^2)$.
For $\phi$ and $\psi$ in $\mathbb{H}$ where

$$\mathbb{H} = \{\phi\in\mathbf{L}^2(\mathbb{R}^2) : \nabla\cdot\phi = 0 \text{ in } \mathbb{R}^2,\ D[\phi]=0 \text{ in } B_i,\ \forall i\in\{1,\dots,k\}\},$$

we have:

$$(\phi,\psi) = \int_{\Omega_F}\phi\cdot\psi\,dy + \sum_{i=1}^{k} m_iV_{\phi,i}\cdot V_{\psi,i} + J_i\omega_{\phi,i}\,\omega_{\psi,i}. \quad (3.5)$$

By Lemma 4.1 in [25], we have

$$\|D[u]\|_{[\mathbf{L}^2(\mathbb{R}^2)]^4} = \|\nabla u\|_{[\mathbf{L}^2(\mathbb{R}^2)]^4},$$

Using the above inner product $(.,.)$ and combining (3.4) together with the above relation, we get that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\mathcal{L}^2(\mathbb{R}^2)} + \nu\int_{\mathbb{R}^2}|\nabla u(t)|^2dx = (f(t),u(t)). \quad (3.6)$$

Thus for almost $t$ in $[0,T_0)$, we have

$$||u(t)||^2_{\mathcal{L}^2(\mathbb{R}^2)} + 2\nu\int_0^t\int_{\mathbb{R}^2}|\nabla u(t)|^2dx \le \int_0^t ||u(s)||^2_{\mathcal{L}^2(\mathbb{R}^2)}ds + \int_0^{T_0}\|f(s)\|^2_{\mathcal{L}^2(\mathbb{R}^2)}ds + ||u_0||^2_{\mathbf{L}^2(\mathbb{R}^2)}. \quad (3.7)$$

Gronwall lemma implies that

$$||u(t)||^2_{\mathcal{L}^2(\mathbb{R}^2)} \le e^{T_0}\Big(\int_0^{T_0}\|f(s)\|^2_{\mathcal{L}^2(\mathbb{R}^2)}ds + ||u_0||^2_{\mathcal{L}^2(\mathbb{R}^2)}\Big),\ \text{a.e on } [0,T_0). \quad (3.8)$$

Combining the above inequality with that in (3.7), it follows that

$$2\nu\int_0^t\int_{\mathbb{R}^2}|\nabla u(t)|^2dx \le (1+T_0e^{T_0})\Big(\int_0^{T_0}\|f(s)\|^2_{\mathcal{L}^2(\mathbb{R}^2)}ds + ||u_0||^2_{\mathcal{L}^2(\mathbb{R}^2)})\Big),\ \text{a.e on } [0,T_0). \quad (3.9)$$

□

In the rest of the work, we keep the constant $M$ as it is defined in the above Lemma and we define $K_1$ such that

$$K_1 = \Big( ||f||^2_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))} + ||u_0||^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^{k} (|h_i^1|^2 + |\omega_i^0|^2) \Big)^{\frac{1}{2}}.$$

In order to prove Proposition 3.1, it remains to bound the norm of $\nabla u$ in $[\mathbf{L}^2(\mathbb{R}^2)]^4$. To do this, we follow the method of Cumsille and Takahashi in [3] and we start with defining some auxiliary functions.

We consider a family of smooth functions $\{\zeta_i\}_{i=1,\dots,k}$ ; each of compact support contained in $B(h_i(0), r_i + \frac{\delta}{2})$ and equal 1 on $B_i$. For a fixed $i$ in $\{1,\dots,k\}$, we set $\hat{\psi}_i(x,t) = \zeta_i(x - h_i(t) + h_i(0))$ and we define the mapping $\hat{\Lambda} : \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2$ by

$$\hat{\Lambda}(x_1, x_2, t) = \sum_{i=1}^{k} \nabla^{\perp}(w_i \hat{\psi}_i). \tag{3.10}$$

Let $\hat{X}$ be the solution of the initial value problem

$$\begin{cases} \dfrac{\partial \hat{X}}{\partial t}(y,t) & = \hat{\Lambda}(\hat{X}(y,t),t), \ \ t \in ]0,T], \\ \hat{X}(y,0) & = y \in \mathbb{R}^2. \end{cases} \tag{3.11}$$

Then for $y \in B_i$, we have

$$\hat{X}(y,t) = y + h_i(t) - h_i(0).$$

It is easy to see that

$$\exists C > 0 \text{ such that } ||\hat{\Lambda}||_{\mathbf{W}^{2,\infty}(\Omega_F(t))} \leq C \sum_{i=1}^{k} |h_i'(t)|.$$

By the previous lemma, we get

$$||\hat{\Lambda}||_{\mathbf{W}^{2,\infty}(\Omega_F(t))} \leq C M^{\frac{1}{2}} K_1.$$

Also, we define for $(y,t) \in \mathbb{R}^2 \times [0,T]$ and $i \in \{1,\dots,k\}$, the mapping:

$$\overline{w}_i(y,t) = h'_{i,2}(t) y_1 - h'_{i,1}(t) y_2 + \frac{\omega_i(t)}{2} |y - h_i(0)|^2.$$

Finally, we define the mapping $\overline{\Lambda} : \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2$ by

$$\overline{\Lambda}(x_1, x_2, t) = \sum_{i=1}^{k} \nabla^{\perp}(\overline{w}_i \zeta_i). \tag{3.12}$$

We note here that

$$\overline{\Lambda}(y,t) = h_i'(t) + \omega_i(t)(y - h_i(0))^{\perp}, \ \ \forall y \in B_i,$$

and

$$||\overline{\Lambda}(t)||_{\mathbf{H}^2(\Omega_F(t))} \leq C \sum_{i=1}^{k} (|h_i'(t)| + |\omega_i(t)|), \ \ \forall t \in [0, T_0).$$

Again by Lemma 3.1, we have

$$||\overline{\Lambda}||_{\mathbf{H}^2(\Omega_F(t))} \leq C M^{\frac{1}{2}} K_1.$$

Next, we state the following lemma without proof as its demonstration is similar to that of Lemma 4.3 in [3].

**Lemma 3.2** *Let $(u, p, (h_i, \omega_i)_{i=1,\dots k})$ be the strong solution associated to problem (1.1)-(1.6). Then for almost all $t \in (0, T_0)$, we have*

$$-\int_{\Omega_F(t)} [\nabla \cdot \sigma(u,p)] \cdot (\frac{\partial u}{\partial t} + (\hat{\Lambda} \cdot \nabla)u - (u \cdot \nabla)\hat{\Lambda})dx$$
$$= \nu \frac{d}{dt} \int_{\Omega_F(t)} |D[u]|^2 dx + \sum_{i=1}^{k} \Big( m_i |h_i^{''}(t)|^2 + J_i |\omega_i^{'}(t)|^2 - \int_{B_i(t)} \overline{\rho}_i f(x,t) \cdot (h_i^{''}(t) + \omega_i^{'}(t)(x - h_i(t))^{\perp} \Big)$$
$$+ 2\nu \int_{\Omega_F(t)} D[u] : (\nabla u \nabla \hat{\Lambda})dx - 2\nu \int_{\Omega_F(t)} D[u] : D[(u \cdot \nabla)\hat{\Lambda}])dx. \quad (3.13)$$

Proposition 3.1 will be deduced from the following proposition.

**Proposition 3.2** *Let $(u, p, (h_i, \omega_i)_{i=1,\dots,k})$ be the strong solution in $[0, T_1]$, where $T_1 < T_0$ is small enough and depends on $\nu$, $M$ and the initial data. Then there exists $K > 1$ such that*

$$\sup_{t \in [0,T_1]} \|\nabla u(t)\|^2_{[L^2(\mathbb{R}^2)]^4} \leq K\big(\|\nabla u_0\|^2_{[L^2(\mathbb{R}^2)]^4} + 1\big), \quad (3.14)$$

*and*

$$\int_0^{T_1} \Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(s))} ds + \sum_{i=1}^{k} \Big( \int_0^{T_1} |h_i^{''}(s)|^2 ds + \int_0^{T_1} |\omega_i^{'}(s)|^2 ds \Big) \leq K\big(\|\nabla u_0\|^2_{[L^2(\mathbb{R}^2)]^4} + 1\big)^2. \quad (3.15)$$

*where the constant $K$ depends on $\Omega_F, B_i, \nu, \overline{\rho}_i, T_0, \|u_0\|_{\mathbf{L}^2(\Omega_F)}, |h_i^1|, |\omega_i^0|$ and $\|f\|_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))}$.*

**Remark 3.1** *As the system is autonomous and dissipative, then for all $t \geq 0$ the above proposition is still valid on any interval $[t, t+T_1] \subset [0, T_0[$.*

Before giving the proof of Proposition 3.2, let us see how it implies Proposition 3.1.
Proposition 3.2 implies that the mapping $t \mapsto \|\nabla u\|_{[L^2(\mathbb{R}^2)]^4}$ is bounded on $[0, T_1]$ for $T_1$ is small enough. We can choose $T_1$ such that $T_0 = NT_1$, for some $N \in \mathbb{N}^*$. This implies that

$$\|\nabla u(t)\|^2_{[L^2(\mathbb{R}^2)]^4} \leq K\|\nabla u((n-1)T_1)\|^2_{[L^2(\mathbb{R}^2)]^4} + K, \text{ a.e on } [(n-1)T_1, nT_1[, \ n = 1, \dots, N.$$

By induction, we get that

$$\|\nabla u(t)\|^2_{[L^2(\mathbb{R}^2)]^4} \leq K^n \|\nabla u(0)\|^2_{[L^2(\mathbb{R}^2)]^4} + \frac{K^{n+1} - K}{K - 1}, \text{ a.e on } [(n-1)T_1, nT_1[, \ n = 1, \dots, N,$$

and thus

$$\sup_{t \in [0,T_0[} \|\nabla u(t)\|^2_{[L^2(\mathbb{R}^2)]^4} \leq K^N \|\nabla u(0)\|^2_{[L^2(\mathbb{R}^2)]^4} + \frac{K^{N+1} - K}{K - 1}.$$

Combining this result with Lemma 3.1, we get that for $T_0 < +\infty$, the mapping $t \to \|u\|_{\mathbf{H}^1(\mathbb{R}^2)}$ is bounded on $[0, T_0)$ whenever there is no contact between the rigid bodies.

## 3.1 Proof of Proposition 3.2

Taking the inner product of equation (1.1) with $\partial_t u + (\hat{\Lambda} \cdot \nabla)u - (u \cdot \nabla)\hat{\Lambda}$ yields to

$$\int_{\Omega_F(t)} \Big|\frac{\partial u}{\partial t}\Big|^2 dx + \int_{\Omega_F(t)} \frac{\partial u}{\partial t}\cdot\Big((\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx - \int_{\Omega_F(t)} \nabla\cdot\sigma(u,p)\cdot\Big(\frac{\partial u}{\partial t}+(\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx$$
$$= -\int_{\Omega_F(t)} [(u\cdot\nabla)u]\cdot\Big(\frac{\partial u}{\partial t}+(\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx + \int_{\Omega_F(t)} f\cdot\Big(\frac{\partial u}{\partial t}+(\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx.$$

Combining the above equation with the formula in (3.13), we obtain

$$\begin{aligned}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))} &+\nu\frac{d}{dt}\int_{\Omega_F(t)}|D[u]|^2dx+\sum_{i=1}^k\Big(m_i|h_i^{''}(t)|^2+J_i|\omega_i^{'}(t)|^2\Big)\\ &=2\nu\int_{\Omega_F(t)}\Big(D[u]:D[(u\cdot\nabla)\hat\Lambda]-D[u]:(\nabla u\nabla\hat\Lambda)\Big)dx\\ &+\sum_{i=1}^k\int_{B_i(t)}\overline{\rho}_i f(x,t)\cdot\Big(h_i^{''}(t)+\omega_i^{'}(t)(x-h_i(t))^\perp\Big)\\ &-\int_{\Omega_F(t)}\frac{\partial u}{\partial t}\cdot\Big((\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx\\ &-\int_{\Omega_F(t)}[(u\cdot\nabla)u]\cdot\Big(\frac{\partial u}{\partial t}+(\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx\\ &+\int_{\Omega_F(t)} f\cdot\Big(\frac{\partial u}{\partial t}+(\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx,\quad \text{a.e in }(0,T_1).\end{aligned}\tag{3.16}$$

By using the estimate in (3.1), there exists a real constant $C_1 = C_1(T_0,\nu,(\overline{\rho}_i,B_i)_{i=1,\dots,k})$, such that the following holds true for almost $t\in(0,T_1)$

$$\Big|2\nu\int_{\Omega_F(t)} D[u]:D[(u\cdot\nabla)\hat\Lambda]-D[u]:(\nabla u\nabla\hat\Lambda)dx\Big| \le C_1\Big((1+K_1^2)\|\nabla u\|^2_{[L^2(\Omega_F(t))]^4}+K_1^4\Big),$$
$$\Big|\int_{B_i(t)}\overline{\rho}_i f(x,t)\cdot\Big(h_i^{''}(t)+\omega_i^{'}(t)(x-h_i(t))^\perp\Big)\Big| \le \frac{\overline{\rho}_i}{2}\|f\|^2_{\mathbf{L}^2(B_i(t))}+\frac{J_i}{2}|\omega_i^{'}(t)|^2+\frac{m_i}{2}|h_i^{''}(t)|^2,$$
$$\Big|\int_{\Omega_F(t)}\frac{\partial u}{\partial t}\cdot\Big((\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx\Big| \le \frac14\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}+C_1K_1^2\|\nabla u\|^2_{[L^2(\Omega_F(t))]^4}+C_1K_1^4,$$
$$\begin{aligned}\Big|\int_{\Omega_F(t)}[(u\cdot\nabla)u]\cdot\Big(\frac{\partial u}{\partial t}+(\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx\Big| &\le \frac18\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}+3\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))}\\ &+C_1K_1^2\|\nabla u\|^2_{[L^2(\Omega_F(t))]^4}+C_1K_1^4,\end{aligned}$$
$$\Big|\int_{\Omega_F(t)} f\cdot\Big(\frac{\partial u}{\partial t}+(\hat\Lambda\cdot\nabla)u-(u\cdot\nabla)\hat\Lambda\Big)dx\Big| \le \frac18\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}+C_1K_1^2\|\nabla u\|^2_{[L^2(\Omega_F(t))]^4}+\frac52\|f\|^2_{\mathbf{L}^2(\Omega_F(t))}+C_1K_1^4.$$

Combining the above estimates with (3.16), we get that

$$\begin{aligned}\frac12\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}&+\nu\frac{d}{dt}\int_{\Omega_F(t)}|D[u]|^2dx+\frac12\sum_{i=1}^k\Big(m_i|h_i^{''}(t)|^2+J_i|\omega_i^{'}(t)|^2\Big)\\ &\le C_1\Big(K_1^4+(K_1^2+1)\|\nabla u\|^2_{[L^2(\Omega_F(t))]^4}+\|f\|^2_{\mathbf{L}^2(\mathbb{R}^2)}\Big)+3\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))}.\end{aligned}\tag{3.17}$$

We are now in position to estimate the term $(u\cdot\nabla)u$ in terms of the left hand side of inequality (3.17). To do this, we need the following lemmas and we postpone their proof to the Appendix.

**Lemma 3.3** *There exists a strong 2-extension operator $E$ for $\Omega_F(t)$. Moreover, there exists a positive constant $k = k(\varepsilon)$ such that for $u \in \mathbf{H}^2(\Omega_F(t))$, we have:*

$$\|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{L}^2(\Omega_F(t))}, \tag{3.18}$$

$$\|Eu\|_{\mathbf{H}^1(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^1(\Omega_F(t))}, \tag{3.19}$$

$$\|Eu\|_{\mathbf{H}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^2(\Omega_F(t))}. \tag{3.20}$$

**Lemma 3.4** *Let $u$ be the unique strong solution of problem (1.1)-(1.6). Then we have*

$$\|u\|_{\mathbf{H}^2(\Omega_F(t))} \leq K\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F(t))} + \|u\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|\nabla u\|^2_{[L^2(\Omega_F(t))]^4} + \|f\|_{\mathbf{L}^2(\Omega_F(t))} + \|\overline{\Lambda}\|_{\mathbf{H}^2(\mathbb{R}^2)} + 1\Big), \tag{3.21}$$

*where $K$ is a positive constant that depends on $\Omega_F, B_i, \overline{\rho}_i, \nu, T_0, \|u_0\|_{\mathbf{L}^2(\mathbb{R}^2)}$ and $\|f\|_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))}$.*

We return now to complete the proof of Proposition 3.2. Lemma 3.3 implies that

$$\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))} \leq \|(Eu\cdot\nabla)Eu\|^2_{\mathbf{L}^2(\mathbb{R}^2)} \leq \|Eu\|^2_{\mathbf{L}^4(\mathbb{R}^2)}\|\nabla Eu\|^2_{\mathbf{L}^4(\mathbb{R}^2)}.$$

Moreover, using the continuous embedding of $H^{1/2}(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$ and the interpolation inequality in Lions–Magenes [19], we have that

$$\|z\|_{L^4(\mathbb{R}^2)} \leq C_2\|z\|_{H^{1/2}(\mathbb{R}^2)} \leq C_2\|z\|^{1/2}_{L^2(\mathbb{R}^2)}\|z\|^{1/2}_{H^1(\mathbb{R}^2)}, \ \ \forall z \in H^1(\mathbb{R}^2),$$

where $C_2 = C_2(\mathbb{R}^2)$ is a positive real constant. Hence, we get

$$\begin{aligned}\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))} &\leq C_2\|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)}\|Eu\|_{\mathbf{H}^1(\mathbb{R}^2)}\|\nabla Eu\|_{\mathbf{L}^2(\mathbb{R}^2)}\|\nabla Eu\|_{\mathbf{H}^1(\mathbb{R}^2)}\\ &\leq C_2\|u\|_{\mathbf{L}^2(\Omega_F(t))}\|u\|_{\mathbf{H}^1(\Omega_F(t))}\|Eu\|_{\mathbf{H}^1(\mathbb{R}^2)}\|Eu\|_{\mathbf{H}^2(\mathbb{R}^2)}\\ &\leq C_2\|u\|_{\mathbf{L}^2(\Omega_F(t))}\|u\|^2_{\mathbf{H}^1(\Omega_F(t))}\|u\|_{\mathbf{H}^2(\Omega_F(t))}\\ &\leq C_2\|u\|_{\mathbf{L}^2(\Omega_F(t))}\big(\|u\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}\big)\|u\|_{\mathbf{H}^2(\Omega_F(t))}.\end{aligned} \tag{3.22}$$

Let $K > 1$ be a constant that depends on $\Omega_F, B_i, \overline{\rho}_i, \nu, T_0, \|u_0\|_{\mathbf{L}^2(\mathbb{R}^2)}$ and $\|f\|_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))}$ that may changes between lines.

Combining (3.22) with Lemma 3.4, we get that

$$\begin{aligned}\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))} \leq K\|u\|_{\mathbf{L}^2(\Omega_F(t))}\big(\|u\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|\nabla u\|^2_{[L^2(\Omega_F(t))]^4}\big)\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F(t))} + \|u\|^2_{\mathbf{L}^2(\Omega_F(t))}\\ + \|\nabla u\|^2_{[L^2(\Omega_F(t))]^4} + \|f\|_{\mathbf{L}^2(\Omega_F(t))} + \|\overline{\Lambda}\|_{\mathbf{H}^2(\mathbb{R}^2)} + 1\Big).\end{aligned} \tag{3.23}$$

By Young's inequality, we get for all $\epsilon > 0$

$$\begin{aligned}\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))} \leq \frac{K}{4\epsilon}\|u\|^2_{\mathbf{L}^2(\Omega_F(t))}\big(\|u\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|\nabla u\|^2_{[L^2(\Omega_F(t))]^4}\big)^2 + \epsilon\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|u\|^4_{\mathbf{L}^2(\Omega_F(t))}\\ + \|\nabla u\|^4_{[L^2(\Omega_F(t))]^4} + \|f\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|\overline{\Lambda}\|^2_{\mathbf{H}^1(\mathbb{R}^2)} + 1\Big),\end{aligned} \tag{3.24}$$

By combining the above inequality with (3.1), we get

$$\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))} \leq K\big(1 + \|\nabla u\|^4_{[L^2(\Omega_F(t))]^4}\big) + \epsilon\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F)} + \|f\|^2_{\mathbf{L}^2(\Omega_F)}\Big). \tag{3.25}$$

We set $\epsilon = \dfrac{1}{12}$ in (3.25), then we combine the resulting inequality with that in (3.17) and thus we get that for almost $t$ in $[0, T_1]$

$$\frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))} + \nu\frac{d}{dt}\int_{\Omega_F(t)}|D[u]|^2dx + \frac{1}{2}\sum_{i=1}^k\Big(m_i|h_i^{''}(t)|^2 + J_i|\omega_i^{'}(t)|^2\Big)$$
$$\leq K\Big(1 + \|\nabla u\|^4_{[L^2(\Omega_F(t))]^4} + \|f\|^2_{\mathbf{L}^2(\mathbb{R}^2)}\Big) + \frac{1}{4}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}. \quad (3.26)$$

Hence, for almost $t$ in $[0, T_1]$, we have

$$\frac{1}{4}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))} + \frac{\nu}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla u|^2dx + \frac{1}{2}\sum_{i=1}^k\Big(m_i|h_i^{''}(t)|^2 + J_i|\omega_i^{'}(t)|^2\Big)$$
$$\leq K\Big(1 + \|\nabla u\|^4_{[L^2(\Omega_F(t))]^4} + \|f\|^2_{ltr}\Big). \quad (3.27)$$

By integrating (3.27) with respect to $t$, we obtain for all $t$ in $[0, T_1]$

$$\frac{1}{4}\int_0^t\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(s))}ds + \frac{\nu}{2}\int_{\mathbb{R}^2}|\nabla u|^2dx + \frac{1}{2}\sum_{i=1}^k\Big(m_i\int_0^t|h_i^{''}(s)|^2ds + J_i\int_0^t|\omega_i^{'}(s)|^2ds\Big)$$
$$\leq \frac{\nu}{2}\|\nabla u_0\|^2_{[L^2(\mathbb{R}^2)]^4} + K\Big(1 + \int_0^t\|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4}ds + \int_0^t\|\nabla u\|^4_{[L^2(\mathbb{R}^2)]^4}ds\Big). \quad (3.28)$$

The above inequality implies that

$$\|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4} \leq \|\nabla u_0\|_{[L^2(\mathbb{R}^2)]^4} + K + K\int_0^t\|\nabla u\|^4_{L^2(\mathbb{R}^2)]^4}ds, \text{ a.e on } [0, T_1]. \quad (3.29)$$

Applying Gronwall's lemma to the above inequality yields to

$$\|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4} \leq \Big(\|\nabla u_0\|^2_{[L^2(\mathbb{R}^2)]^4} + K\Big)\exp\Big(K\int_0^{T_1}\|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4}ds\Big), \text{ a.e on } [0, T_1], \quad (3.30)$$

and thus by (3.1), we get that

$$\|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4} \leq K\Big(\|\nabla u_0\|^2_{[L^2(\mathbb{R}^2)]^4} + 1\Big). \text{ a.e on } [0, T_1]. \quad (3.31)$$

Moreover, by combining (3.28) with (3.31), we get for almost $t$ on $[0, T_1]$:

$$\frac{1}{4}\int_0^t\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(s))}ds + \frac{\nu}{2}\|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4} + \frac{1}{2}\sum_{i=1}^k\Big(m_i\int_0^t|h_i^{''}(s)|^2ds + J_i\int_0^t|\omega_i^{'}(s)|^2ds\Big)$$
$$\leq \frac{\nu}{2}\|\nabla u_0\|^2_{[L^2(\mathbb{R}^2)]^4} + K + KT_1\Big(\|\nabla u_0\|^2_{[L^2(\mathbb{R}^2)]^4} + 1\Big)^2. \quad (3.32)$$

□

# 4 Mechanism preventing from collision

This section is devoted to accomplish the proof of Theorem 1.2. We follow the approach used in [10] and [15]. We act by contradiction and we assume that collision could take place in finite time under the assumption (H1). The idea is to construct a proper candidate $v$ and use it in the weak formulation (4.26) leading to a differential equation which can be integrated so that we get the *no-collision* result.

## 4.1 Construction of the test function

We suppose that $T_0 < +\infty$ and we start to prove that collision in pair - as that between the disks $B_1$ and $B_2$ or between $B_8$ and $B_7$ in Figure 1 - could not take place. Both cases can be summarized by the assumption that one disk has a collision with only one other disk. Up to renumbering, this assumption can be stated as follows:

$$d(B_1(T_0), B_2(T_0)) = 0, \text{ and } d(B_1(T_0), B_i(T_0)) > 0, \forall\, i = 3, \dots, k. \tag{H2}$$

Since the disks $B_1$ and $B_2$ collide at $T_0$, then we can choose an initial time $t_0 < T_0$ such that for all $t \geq t_0$ and all

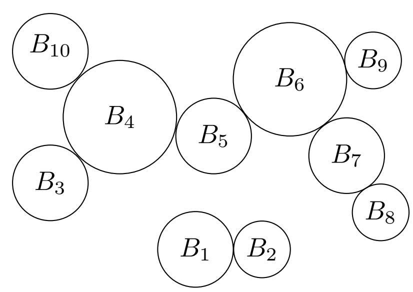


Figure 1: Example of collision at time $T_0$

$j \notin J$, we have $d(B_1(t), B_2(t)) < 2r_j$, where $J = \{1, 2\}$. In other words, we can choose initial time $t_0$ such that there is no possibility to find a disk separating the rigid bodies $B_1(t)$ and $B_2(t)$ for all $t \in [t_0, T_0]$. For all $i \in \{2, \dots, k\}$, we define $d_{1,i}(t) := d(B_1(t), B_i(t))$. Since $d_{1,i}(T_0)$ is positive as long as $i \notin J$, then $\beta := \inf\limits_{t_0 \leq t \leq T_0} \min\limits_{i \notin J} d_{1,i}(t) > 0$. Also contact at time $T_0$ can only occur at a single point between any pair of disks as the domains of the rigid bodies are convex.

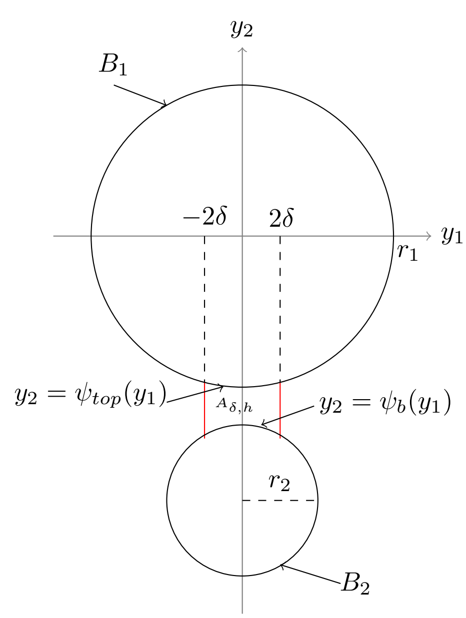


Figure 2: Geometry in the local coordinates

We will see later that the expression of the test function $v$ involves the boundary functions of the disks in the neighborhood of the contact point. This oblige us to define the vector field $v$ at first in the local coordinates $(h_1(t), e_1(t), e_2(t))$, where $e_2(t) = \dfrac{h_1(t) - h_2(t)}{|h_1(t) - h_2(t)|}$ and $e_1(t) = -e_2^{\perp}(t)$, so that we can always represent the boundaries of the disks close to the collision point by a suitable boundary functions of simple expressions: one is the lower boundary of the disk $B_1$ and the other is the upper boundary of the disk $B_2$.

We introduce the change of variable $Y$ defined as follows:

$$Y(t,x) = \left( -\frac{(x - h_1(t)) \cdot (h_1(t) - h_2(t))^{\perp}}{|h_1(t) - h_2(t)|}, \frac{(x - h_1(t)) \cdot (h_1(t) - h_2(t))}{|h_1(t) - h_2(t)|} \right). \tag{4.1}$$

In what follows, we fix $h \in (0, d_{\max})$ where $d_{\max} := \sup_{t_0 \le t \le T_0} d_{1,2}(t)$. In the new coordinates, $B_1$ is the disk of center $(0,0)$ and radius $r_1$ whereas $B_2$ is the disk of center $(0, -r_1 - r_2 - h)$ and radius $r_2$ (see Figure 2). Also, we fix $\delta > 0$ such that $2\delta < \min(r_1, r_2)$, and we define the bridge $A_{\delta,h}$ in the local coordinates by

$$A_{\delta,h} := \{y \in \mathbb{R}^2 : \ |y_1| < 2\delta, \ \psi_b(y_1) < y_2 < \psi_{top}(y_1)\},$$

where the boundary functions $\psi_{top}$ and $\psi_b$ of the disks $B_1$ and $B_2$ respectively are given by:

$$\begin{aligned} \psi_{top}(y_1) &:= -\sqrt{r_1^2 - y_1^2}, & \forall y_1 \in [-r_1, r_1], \\ \psi_b(y_1) &:= \sqrt{r_2^2 - y_1^2} - r_1 - r_2 - h, & \forall y_1 \in [-r_2, r_2]. \end{aligned}$$

Moreover, we choose $\delta$ such that

$$A_{\delta, d_{1,2}(t)} \cap B_j(t) = \emptyset, \ \forall t \in [t_0, T_0], \ \forall j \notin \{1,2\}.$$

Before we proceed, we mention some properties of the boundary functions $\psi_{top}$ and $\psi_b$ that will be useful later on. It is easy to see that for all $y \in A_{\delta,h}$, we have:

$$y_2 - \psi_b(y_1) \le \psi_{top}(y_1) - \psi_b(y_1) \quad \text{and} \quad h \le \psi_{top}(y_1) - \psi_b(y_1). \tag{4.2}$$

Moreover, there exists a constant $K = K(\delta, r_1, r_2)$ such that

$$|\psi'_{top}(y_1)| \le K|y_1|, \quad |\psi'_b(y_1)| \le K|y_1|, \ \forall y_1 \in [-2\delta, 2\delta], \tag{4.3}$$

$$|\psi''_{top}(y_1)| \le K, \quad |\psi''_{top}(y_1)| \le K, \quad \forall y_1 \in [-2\delta, 2\delta]. \tag{4.4}$$

Furthermore, the following inequality

$$\frac{t^2}{2} \le 1 - \sqrt{1 - t^2} \le t^2, \ \forall t \in [-1, 1],$$

implies that

$$h + a y_1^2 \le \psi_{top}(y_1) - \psi_b(y_1) \le h + 2a y_1^2, \ \forall y_1 \in [-2\delta, 2\delta] \tag{4.5}$$

with $a = \frac{1}{2r_1} + \frac{1}{2r_2}$.

We turn now to define the test function $v$. To describe $v$ in the neighborhood of $B_1$, we define a smooth function $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ with compact support included in $B(0, \alpha)$ such that $\phi \equiv 1$ on $B(0, \frac{r_1 + \alpha}{2})$, where

$$\alpha \le \min(r_1 + \beta, \sqrt{r_1^2 + \delta^2}).$$

Then we introduce a smooth function $\chi : \mathbb{R} \mapsto [0,1]$ such that

$$\chi(r) = \begin{cases} 1 & \text{if } |r| \leq \delta, \\ 0 & \text{if } |r| \geq 2\delta. \end{cases}$$

We set

$$\overline{v}_h := \nabla^{\perp} \tilde{g}_h, \tag{4.6}$$

where $\tilde{g}_h(y) = y_1 \varphi_h$ with $\varphi_h : \mathbb{R}^2 \to \mathbb{R}$ is defined as follows:

$$\begin{aligned}
&\varphi_h = \phi \ \text{ in } \mathbb{R}^2 \backslash \big(A_{\delta,h} \cup \big(B_2 \cap B(0,\alpha)\big)\big), \\
&\varphi_h = (1 - \chi(y_1))\phi(y) + \chi(y_1)\Big(\frac{y_2 - \psi_b(y_1)}{\psi_{top}(y_1) - \psi_b(y_1)}\Big)^2 \Big(3 - 2\frac{y_2 - \psi_b(y_1)}{\psi_{top}(y_1) - \psi_b(y_1)}\Big) \text{ in } A_{\delta,h}, \\
&\varphi_h = 0 \ \text{ in } B_2 \cap B(0,\alpha).
\end{aligned}$$

Finally, we define

$$v(t,x) = J_X(Y(x,t),t)\overline{v}(Y(x,t),t), \tag{4.7}$$

where the mapping $\overline{v}$ is defined from $\mathbb{R}^2 \times [0, T_0)$ into $\mathbb{R}^2$ by

$$\overline{v}(y,t) = \overline{v}_{d_{1,2}(t)}(y). \tag{4.8}$$

**Remark 4.1** *We note that $\varphi_h$ and hence $\overline{v}_h$ are regular up to $h = 0$ outside $A_{\delta,h}$, and singularities at $h = 0$ correspond to the term*

$$g_h(y) = y_1 \Big(\frac{y_2 - \psi_b(y_1)}{\psi_{top}(y_1) - \psi_b(y_1)}\Big)^2 \Big(3 - 2\frac{y_2 - \psi_b(y_1)}{\psi_{top}(y_1) - \psi_b(y_1)}\Big), \tag{4.9}$$

*as it involves the difference term between the boundary functions $\psi_{top}$ and $\psi_b$. Hence, all the Sobolev norms of $\overline{v}_h$ are dominated by a constant in $\Omega_{F,\mathbf{h}} \backslash A_{\delta,h}$, where $\Omega_{F,\mathbf{h}}$ denotes the fluid domain in the new geometry.*

We state some properties of $\overline{v}_h$ in the following lemma and in this respect, we refer the reader to [15].

**Lemma 4.1** *Let $h > 0$, then $\overline{v}_h \in \mathbf{H}^1(\mathbb{R}^2)$ and has a compact support. Moreover, we have:*

*i.* $\nabla \cdot \overline{v}_h = 0$ *in* $\mathbb{R}^2$,

*ii.* $\overline{v}_h = e_2$ *on* $B_1$.

*iii.* $\overline{v}_h = 0$ *on the other disks.*

## 4.2 No collision result

This subsection is dedicated to prove the following theorem from which we can deduce the proof of Theorem 1.2.

**Theorem 4.1** *Assume* (H2) *holds true, then we have $d(B_1, B_2)(T_0) > 0$.*

To prove the above theorem, we need some estimates on the test function $v$. The following lemma shows that we can perform such estimates on the vector field $\overline{v}$ instead of $v$.

**Lemma 4.2** *Let $u(t) \in \mathbf{H}^1(\mathbb{R}^2)$ and $v(t) \in \mathbf{L}^p(\mathbb{R}^2)$ be two vector fields with $p \geq 1$. We define $\overline{u}(y,t) = J_Y(X(y,t),t)u(X(y,t),t)$, where $X$ denotes the inverse of the diffeomorphism $Y$ defined in (4.1). Then we have:*

$$\|v(t)\|_{\mathbf{L}^p(\mathbb{R}^2)} = \|\overline{v}(t)\|_{\mathbf{L}^p(\mathbb{R}^2)},\ t \in [0,T_0),$$
$$D[u] : D[v] = D[\overline{u}] : D[\overline{v}], \quad \forall v \in \mathbf{H}^1(\mathbb{R}^2).$$

The above lemma is straightforward from the fact that the diffeomorphism $Y$ is an isometry.

Next, we state the following lemma which enables to estimate some terms in the weak formulation, such as the non-linear term and the source term.

**Lemma 4.3** *Let $h \in (0, d_{\max})$ and consider the vector field $\overline{v}_h$ defined in (4.6). Then there exists a constant $K_m = K_m(\delta, r_1, r_2, d_{\max})$ such that the vector field $\overline{v}_h \in \mathbf{L}^p(\mathbb{R}^2)$ for all $1 \leq p < 3$ and we have*

$$\|\overline{v}_h\|_{\mathbf{L}^p(\mathbb{R}^2)} \leq K_m. \tag{4.10}$$

Proof. By Remark 4.1, all the Sobolev norms of $\overline{v}_h$ in $\Omega_{F,\mathbf{h}} \backslash A_{\delta,h}$ are dominated by a constant. From the definition of $g_h$ in (4.9), we have

$$\overline{v}_h(y) = \nabla^\perp\Big(y_1\,(1-\chi(y))\phi(y)\Big) + g_h(y)\nabla^\perp\chi(y) + \chi(y)\nabla^\perp g_h(y),\ \ \forall y \in A_{\delta,h}.$$

Using the properties of the boundary functions $\psi_{top}$ and $\psi_b$ stated in the previous section, we get that there exists $K = K(\delta, r_1, r_2) > 0$ and $C > 0$ such that

$$|g_h(y)| \leq K, \tag{4.11}$$
$$\left|\frac{\partial g_h}{\partial y_1}(y)\right| \leq C + K\frac{y_1^2}{\psi_{top}(y_1) - \psi_b(y_1)}, \tag{4.12}$$
$$\left|\frac{\partial g_h}{\partial y_2}(y)\right| \leq C\frac{|y_1|}{\psi_{top}(y_1) - \psi_b(y_1)}. \tag{4.13}$$

This implies that for all $y \in A_{\delta,h}$, we have:

$$|\overline{v}_{h,1}(y)| \leq C\Big(1 + K + \frac{|y_1|}{\psi_{top}(y_1) - \psi_b(y_1)}\Big) \text{ and } |\overline{v}_{h,2}(y)| \leq C\Big(1 + K + \frac{Ky_1^2}{\psi_{top}(y_1) - \psi_b(y_1)}\Big).$$

Hence, $\overline{v}_h \in \mathbf{L}^1(A_{\delta,h})$ and thus it is in $\mathbf{L}^1(\mathbb{R}^2)$. For $1 < p < 3$, there exists a positive constant $K_m = K_m(\delta, r_1, r_2, d_{\max})$ such that

$$\|\overline{v}_h(y)\|^p_{\mathbf{L}^p(A_{\delta,h})} \leq K_m\Big(1 + \int_0^{2\delta} \frac{y_1^p}{\big(\psi_{top}(y_1) - \psi_b(y_1)\big)^{p-1}}dy_1\Big).$$

Using the inequality (4.5), we obtain

$$\int_0^{2\delta} \frac{y_1^p}{\big(\psi_{top}(y_1) - \psi_b(y_1)\big)^{p-1}}dy_1 \leq \int_0^{2\delta} \frac{y_1^p}{\big(h + ay_1^2\big)^{p-1}}dy_1,$$

and thus

$$\|\overline{v}_h(y)\|^p_{\mathbf{L}^p(A_{\delta,h})} \leq K_m\big(1 + \int_0^{2\delta} \frac{dy_1}{y_1^{p-2}}\big).$$

The integral in the right hand side of the above inequality is finite as $|p-2| < 1$, Therefore (4.10) holds.

□

To estimate the term that contains $\partial_t v$ in the weak formulation, we need the following lemma:

**Lemma 4.4** *Let $h \in (0, d_{\max})$. Then there exists a positive constant $K_m = K_m(\delta, r_1, r_2, d_{\max})$ such that*

$$\left\|\partial_h \tilde{g}_h\right\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq K_m. \tag{4.14}$$

Proof. From the definition of $\tilde{g}_h$ in (4.6) and by standard calculations, we have for all $y \in A_{\delta,h}$:

$$\partial_h \tilde{g}_h(y)(y) = 6y_1\chi(y_1)\Big(\frac{(y_2-\psi_b(y_1))^3}{(\psi_{top}(y_1)-\psi_b(y_1))^4} - 2\frac{(y_2-\psi_b(y_1))^2}{(\psi_{top}(y_1)-\psi_b(y_1))^3} + \frac{y_2-\psi_b(y_1)}{(\psi_{top}(y_1)-\psi_b(y_1))^2}\Big).$$

Hence, there exists some $C > 0$ such that

$$|\partial_h \tilde{g}_h(y)| \leq C\frac{|y_1|}{\psi_{top}(y_1)-\psi_b(y_1)}, \ \ \forall y \in A_{\delta,h}.$$

Combining the above inequality with the fact that $\tilde{g}_h$ is smooth outside $A_{\delta,h}$ and is with compact support, we get that there exists $K_m > 0$ that depends on $\delta, r_1, r_2$ and $d_{\max}$ such that

$$\int_{\mathbb{R}^2} |\partial_h \tilde{g}_h(y)|^2 dy \leq K_m + C\int_0^{2\delta} \frac{y_1^2}{\psi_{top}(y_1)-\psi_b(y_1)} dy_1.$$

Hence,

$$\int_{\mathbb{R}^2} |\partial_h \tilde{g}_h(y)|^2 dy \leq K_m + \frac{C}{a}\int_0^{2\delta} \frac{ay_1^2}{h+ay_1^2} dy_1,$$

and as $ay_1^2 \leq h + ay_1^2$, we get the estimate (4.14). $\square$

The following proposition shows why the vector field $v$ is a good candidate to our problem.

**Proposition 4.1** *Let $h \in (0, d_{\max})$ and $\overline{u} \in \mathbf{H}^1(\mathbb{R}^2)$ such that for all $i \in \{1, \ldots, k\}$, we have*

$$\overline{u}(y) = V_{\overline{u},i} + \omega_i(y - y_{G_i})^{\perp} \ \text{ on } \ B(G_i, r_i),$$

*where $G_i$ denotes the center of mass of the i-th disk in the local coordinates. Then there exists a positive constant $K_m = K_m(\delta, r_1, r_2, d_{\max})$ such that*

$$\left|2\nu\int_{A_{\delta,h}} D[\overline{v}_h] : D[\overline{u}]dy - \tilde{n}_1(h)\big(V_{\overline{u},1} - V_{\overline{u},2}\big)\cdot e_2\right| \leq K_m\big(\|\overline{u}\|_{\mathbf{L}^2(\Omega_{F,\mathbf{h}})} + \|\overline{u}\|_{\mathbf{L}^\infty(B_1\cup B_2)} + \|\nabla\overline{u}\|_{[L^2(\mathbb{R}^2)]^4}\big), \tag{4.15}$$

*where*

$$\tilde{n}_1(h) = \int_{\partial A_{\delta,h}\cap\partial B_1} \Big(2\nu D[\overline{v}_h] - q_h I\Big) n d\Gamma_1 \cdot e_2.$$

*Moreover, there exists an absolute constant $K = K(\delta, r_1, r_2)$ such that*

$$\tilde{n}_1(h) \geq \frac{K}{h^{\frac{3}{2}}}.$$

Proof. Without loss, we may assume that $\nu = 1$. By noting that

$$\Delta\overline{v}_h \cdot \overline{u} = 2\nabla\cdot(D[\overline{v}_h]u) - 2D[\overline{v}_h] : D[\overline{u}]$$

and performing integration by parts, we get

$$\int_{A_{\delta,h}} \Big(\Delta\overline{v}_h - \nabla q_h\Big)\cdot\overline{u}dy = -\int_{A_{\delta,h}} D[\overline{v}_h] : D[\overline{u}]dy + \int_{\partial A_{\delta,h}} \Big(D[\overline{v}_h]n - q_h n\Big)\cdot\overline{u}d\Gamma, \tag{4.16}$$

for some pressure $q_h$. The idea now is to find a good pressure field $q_h$ on $A_{\delta,h}$ such that (4.16) holds. We start with computing laplacian of $\overline{v}_h$ and we find that

$$\Delta\overline{v}_h = \begin{pmatrix} -\partial_{112}\tilde{g}_h - \partial_{222}\tilde{g}_h \\ \partial_{111}\tilde{g}_h + \partial_{122}\tilde{g}_h \end{pmatrix}.$$

We construct the pressure field $q_h$ such that

$$\Delta\overline{v}_h - \nabla q_h = \begin{pmatrix} -2\partial_{112}\tilde{g}_h - y_1(1-\chi)\partial_{222}\phi \\ \partial_{111}\tilde{g}_h \end{pmatrix}.$$

To match this property, we define

$$q_h(y,t) = \partial_{12}\tilde{g}_h(y) + \int_{-2\delta}^{y_1} \frac{12\, s\, \chi(s)}{(\psi_{top}(s)-\psi_b(s))^3} ds, \ \ \forall y \in A_{\delta,h}. \tag{4.17}$$

On the other hand, we have

$$\int_{A_{\delta,h}} \Big( -\Delta\overline{v}_h + \nabla q_h \Big) \cdot \overline{u} dy = \int_{A_{\delta,h}} \Big( 2\partial_{112}\tilde{g}_h\overline{u}_1 - \partial_{111}\tilde{g}_h\overline{u}_2 \Big) dy + \int_{A_{\delta,h}} y_1(1-\chi(y_1))\partial_{222}\phi(y)\overline{u}_1 dy.$$

By performing integration by parts, we obtain

$$\int_{A_{\delta,h}} \Big( 2\partial_{112}\tilde{g}_h\overline{u}_1 - \partial_{111}\tilde{g}_h\overline{u}_2 \Big) dy \ = \ -\int_{A_{\delta,h}} \partial_{11}\tilde{g}_h \Big( 2\frac{\partial\overline{u}_1}{\partial y_2} - \frac{\partial\overline{u}_2}{\partial y_1} \Big) dy + \int_{\partial A_{\delta,h}} \partial_{11}\tilde{g}_h \Big( 2\overline{u}_1 n_2 - \overline{u}_2 n_1 \Big) d\Gamma. \tag{4.18}$$

For $y \in A_{\delta,h}$, we have

$$\partial_{11}\tilde{g}_h(y) = \partial_{11}\Big( y_1(1-\chi(y_1))\phi(y) \Big) + \partial_{11}\chi(y_1)g_h(y) + 2\partial_1\chi\,(y_1)\partial_1 g_h(y) + \chi(y_1)\partial_{11}g_h(y).$$

Hence there exists $C>0$ and $K = K(\delta, r_1, r_2) > 0$ such that

$$|\partial_{11}\tilde{g}_h(y)| \le K\Big( 1 + \frac{|y_1|}{\psi_{top}(y_1)-\psi_b(y_1)} \Big) + C.$$

This implies that there exists a positive constant $K_m = K_m(\delta, r_1, r_2, d_{\max})$ such that

$$\|\partial_{11}\tilde{g}_h(s)\|^2_{L^2(A_{\delta,h})} \le K_m\Big( 1 + \int_0^{2\delta} \frac{y_1^2}{h + a y_1^2} dy_1 \Big),$$

and thus

$$\|\partial_{11}\tilde{g}_h(y)\|^2_{L^2(A_{\delta,h})} \le K_m.$$

The above inequality implies that

$$\Big| \int_{A_{\delta,h}} \partial_{11}\tilde{g}_h \Big( 2\frac{\partial\overline{u}_1}{\partial y_2} - \frac{\partial\overline{u}_2}{\partial y_1} \Big) dy \Big| \le K_m \Big( \Big\| \frac{\partial\overline{u}_1}{\partial y_2} \Big\|_{L^2(A_{\delta,h})} + \Big\| \frac{\partial\overline{u}_2}{\partial y_1} \Big\|_{L^2(A_{\delta,h})} \Big).$$

We turn now to estimate the boundary term in (4.18) and in this respect we have

$$\Big| \int_{\partial A_{\delta,h}} \partial_{11}\tilde{g}_h\overline{u}_2 n_1 d\Gamma \Big| \le K_m \Big( ||\overline{u}||_{\mathbf{L}^\infty(B_1)} \int_{-2\delta}^{2\delta} \frac{|y_1||\psi'_{top}(y_1)|^3}{(\psi_{top}(y_1)-\psi_b(y_1))^2} dy_1 + ||\overline{u}||_{\mathbf{L}^\infty(B_2)} \int_{-2\delta}^{2\delta} \frac{|y_1||\psi'_b(y_1)|^3}{(\psi_{top}(y_1)-\psi_b(y_1))^2} dy_1 + \int_{\partial A_{\delta,h}\cap\{|y_1|=2\delta\}} \partial_{11}\tilde{g}_h\overline{u}_2 n_1 d\Gamma \Big). \tag{4.19}$$

Noting that $\partial_{11}\tilde{g}_h$ is regular and odd, we get that

$$\int_{\partial A_{\delta,h}\cap\{|y_1|=2\delta\}} \partial_{11}\tilde{g}_h \overline{u}_2 n_1 d\Gamma = \int_{\psi_b(2\delta)}^{\psi_{top}(2\delta)} \partial_{11}\tilde{g}_h(2\delta, y_2)(\overline{u}_2(2\delta, y_2) - \overline{u}_2(-2\delta, y_2))dy_2$$
$$= \int_{\psi_b(2\delta)}^{\psi_{top}(2\delta)} \partial_{11}\tilde{g}_h(2\delta, y_2) \int_{-2\delta}^{2\delta} \partial_1 \overline{u}_2(s, y_2) ds dy_2.$$

This implies that there exists a positive constant $C$ such that

$$\Big| \int_{\partial A_{\delta,h}\cap\{|y_1|=2\delta\}} \partial_{11}\tilde{g}_h \overline{u}_2 n_1 d\Gamma \Big| \leq C \|\partial_1 \overline{u}_2\|_{L^2([-2\delta,2\delta]\times[\psi_b(2\delta),\psi_{top}(2\delta)])}.$$

Combining (4.19) with the above inequality noting (4.3) and (4.5), we obtain that

$$\Big| \int_{\partial A_{\delta,h}} \partial_{11}\tilde{g}_h \overline{u}_2 n_1 d\Gamma \Big| \leq K_m \Big( ||\overline{u}||_{\mathbf{L}^\infty(B_1\cup B_2)} + \|\partial_1 \overline{u}_2\|_{L^2([-2\delta,2\delta]\times[\psi_b(2\delta),\psi_{top}(2\delta)])} \Big)$$
$$\leq K_m \Big( ||\overline{u}||_{\mathbf{L}^\infty(B_1\cup B_2)} + \|\partial_1 \overline{u}_2\|_{L^2([-2\delta,2\delta]\times[\psi_b(2\delta),\psi_{top}(2\delta)])} \Big).$$

Moreover, we have

$$\Big| \int_{\partial A_{\delta,h}} \partial_{11}\tilde{g}_h \overline{u}_1 n_2 d\Gamma \Big| \leq K_m \Big| \int_{-2\delta}^{2\delta} \partial_{11}\tilde{g}_h(y_1, \psi_{top}(y_1)) \big(V^1_{\overline{u},1} + \omega\, \psi_{top}(y_1)\big) dy_1$$
$$+ \int_{-2\delta}^{2\delta} \partial_{11}\tilde{g}_h(y_1, \psi_b(y_1)) \big(V^1_{\overline{u},2} + \omega\, \psi_b(y_1)\big) dy_1 \Big|.$$

As $\partial_{11}\tilde{g}_h$ is odd with respect to $y_1$ in the time $\psi_{top}$ and $\psi_b$ are even with respect to $y_1$, we get that

$$\int_{\partial A_{\delta,h}} \partial_{11}\tilde{g}_h \overline{u}_1 n_2 d\Gamma = 0.$$

Combining (4.19) with (4.3), (4.5) and the above inequality yields to

$$\Big| \int_{A_{\delta,h}} \big(\Delta \overline{v}_h - \nabla q_h\big) \cdot \overline{u} dy \Big| \leq K_m \Big( \|\overline{u}\|_{\mathbf{L}^2(A_{\delta,h})} + \|\overline{u}\|_{\mathbf{L}^\infty(B_1\cup B_2)} + \Big\| \frac{\partial \overline{u}_1}{\partial y_2} \Big\|_{L^2(A_{\delta,h})}$$
$$+ \Big\| \frac{\partial \overline{u}_2}{\partial y_1} \Big\|_{L^2(A_{\delta,h})} + \Big\| \frac{\partial \overline{u}_2}{\partial y_1} \Big\|_{L^2([-2\delta,2\delta]\times[\psi_b(2\delta),\psi_{top}(2\delta)])} \Big). \quad (4.20)$$

We turn now to compute the line integral on (4.16). It is not difficult to check that

$$q_h, \ \frac{\partial \overline{v}_{h,1}}{\partial y_1} \ \text{ and } \ \frac{\partial \overline{v}_{h,2}}{\partial y_2}$$

are even with respect to $y_1$ whereas

$$\frac{\partial \overline{v}_{h,1}}{\partial y_2} \ \text{and} \ \frac{\partial \overline{v}_{h,2}}{\partial y_1}$$

are odd with respect to $y_1$. This implies that

$$\int_{\partial A_{\delta,h}} \Big(2D[\overline{v}_h]n - q_h n\Big) \cdot \overline{u}(t) d\Gamma = \tilde{n}_1(h)\, V_{\overline{u},1} \cdot e_2 + \tilde{n}_2(h)\, V_{\overline{u},2} \cdot e_2 + \int_{\partial A_{\delta,h}\cap|y_1|=2\delta} \Big(2D[\overline{v}_h]n - q_h n\Big) \cdot \overline{u} d\Gamma$$

with

$$\tilde{n}_i(h) = \int_{\partial A_{\delta,h}\cap\partial B_i} \Big(2D[\overline{v}_h]n - q_h n\Big) d\Gamma_i \cdot e_2, \ \ i = 1, 2.$$

As $\nabla\overline{v}_h$ and $q_h$ are regular on $\partial A_{\delta,h}\cap\{|y_1|=2\delta\}$, then there exists a constant $K$ independent of $h$ such that

$$\int_{\partial A_{\delta,h}\cap|y_1|=2\delta}\Big(D[\overline{v}_h]n-q_hn\Big)\cdot\overline{u}d\Gamma\leq K\|u\|_{\mathbf{H}^1(A_{\delta,h})}.$$

This implies that

$$\Big|2\int_{A_{\delta,h}}D[\overline{v}_h]:D[\overline{u}]dy-\tilde{n}_1(h)V_{\overline{u},1}\cdot e_2-\tilde{n}_2(h)V_{\overline{u},2}\cdot e_2\Big|\leq K_m\Big(\|\overline{u}\|_{\mathbf{L}^2(\Omega_{F,h})}+\|u\|_{\mathbf{L}^\infty(B_1\cup B_2)}+\|\nabla\overline{u}\|_{[L^2(\mathbb{R}^2)]^4}\Big),\quad(4.21)$$

By integration by parts, we have

$$\int_{A_{\delta,h}}(\Delta\overline{v}_h-\nabla q_h)dy\cdot e_2=-\int_{\partial A_{\delta,h}}\Big(2D[\overline{v}_h]n-q_hn\Big)d\Gamma\cdot e_2.$$

Since

$$\int_{\partial A_{\delta,h}\cap\{|y_1|=2\delta\}}\Big(2D[\overline{v}_h]n-q_hn\Big)d\Gamma\cdot e_2=2\int_{\psi_b(2\delta)}^{\psi_{top}(2\delta)}\Big(\frac{\partial\overline{v}_{h,1}}{\partial y_2}(-2\delta,y_2)+\frac{\partial\overline{v}_{h,1}}{\partial y_2}(2\delta,y_2)\Big)dy_2$$
$$+2\int_{\psi_b(2\delta)}^{\psi_{top}(2\delta)}\Big(\frac{\partial\overline{v}_{h,2}}{\partial y_1}(-2\delta,y_2)+\frac{\partial\overline{v}_{h,2}}{\partial y_1}(2\delta,y_2)\Big)dy_2,$$

and as $\dfrac{\partial\overline{v}_{h,2}}{\partial y_1}$ and $\dfrac{\partial\overline{v}_{h,2}}{\partial y_1}$ are odd with respect to $y_1$, then the above integral vanishes and hence we get

$$\int_{A_{\delta,h}}(-\Delta\overline{v}_h+\nabla q_h)dy\cdot e_2=\tilde{n}_1(h)+\tilde{n}_2(h).$$

Setting $\overline{u}=e_2$ in (4.20), we get that

$$\Big|\int_{A_{\delta,h}}(\Delta\overline{v}_h-\nabla q_h)dy\cdot e_2\Big|\leq K_m.$$

This implies that

$$\tilde{n}_2(h)=-\tilde{n}_1(h)+O(K_m).$$

Combining the above result with (4.21), we obtain that

$$\Big|2\int_{A_{\delta,h}}D[\overline{v}_h]:D[\overline{u}]dy-\tilde{n}_1(h)\Big(V_{\overline{u},1}-V_{\overline{u},2}\Big)\cdot e_2\Big|\leq K_m\Big(\|\overline{u}\|_{\mathbf{L}^2(\Omega_{F,\mathbf{h}})}+\|u\|_{\mathbf{L}^\infty(B_1\cup B_2)}+\|\nabla\overline{u}\|_{[L^2(\mathbb{R}^2)]^4}\Big),\quad(4.22)$$

Thus, (4.15) holds.

By similar way, one has:

$$\Big|2\int_{A_{\delta,h}}|D[\overline{v}_h]|^2dy-\tilde{n}_1(h)V_{\overline{v}_h,1}\cdot e_2-\tilde{n}_2(h)V_{\overline{v}_h,2}\cdot e_2\Big|$$
$$\leq K_m\Big(\|\overline{v}_h\|_{\mathbf{L}^2(A_{\delta,h}))}+\|\overline{v}_h\|_{\mathbf{L}^\infty(B_1\cup B_2)}+\Big\|\frac{\partial\overline{v}_{h,1}}{\partial y_2}\Big\|_{L^2(A_{\delta,h})}+\Big\|\frac{\partial\overline{v}_{h,2}}{\partial y_1}\Big\|_{L^2(A_{\delta,h})}\Big).$$

By Lemma 4.1, we have $\overline{v}_h=e_2$ on $B_1$ and vanishes on $B_2$. This implies that

$$\tilde{n}_1(h)\geq2\int_{A_{\delta,h}}|D[\overline{v}_h^1]|^2dy-K_m\Big(\|\overline{v}_h\|_{\mathbf{L}^2(A_{\delta,h}))}+\|\overline{v}_h\|_{\mathbf{L}^\infty(B_1\cup B_2)}+\Big\|\frac{\partial\overline{v}_{h,1}}{\partial y_2}\Big\|_{L^2(A_{\delta,h})}+\Big\|\frac{\partial\overline{v}_{h,2}}{\partial y_1}\Big\|_{L^2(A_{\delta,h})}\Big).\quad(4.23)$$

Standard calculations show that

$$\frac{\partial\overline{v}_{h,1}}{\partial y_2}(y)=-y_1(1-\chi(y_1)\partial_{22}\phi-6y_1\chi(y_1)\Big(\frac{1}{(\psi_{top}(y_1)-\psi_b(y_1))^2}-2\frac{y_2-\psi_b(y_1)}{(\psi_{top}(y_1)-\psi_b(y_1))^3}\Big).$$

This implies that

$$\Big|\frac{\partial \overline{v}_{h,1}}{\partial y_2}(y)\Big| \leq C\Big(1+\frac{|y_1|}{(\psi_{top}(y_1)-\psi_b(y_1))^2}\Big).$$

Combining the above estimates with the fact that $\dfrac{\partial \overline{v}_{h,2}}{\partial y_1} = -\partial_{11}\tilde{g}_h$, we get that

$$\Big\|\frac{\partial \overline{v}_{h,1}}{\partial y_2}\Big\|_{L^2(A_{\delta,h})} \leq \frac{K_m}{h^{\frac{3}{4}}}, \tag{4.24}$$

$$\Big\|\frac{\partial \overline{v}_{h,2}}{\partial y_1}\Big\|_{L^2(A_{\delta,h})} \leq K_m. \tag{4.25}$$

To bound from below $D[\overline{v}_h]$ in $[L^2(A_{\delta,h})]^4$, it suffices to bound from below $\dfrac{\partial \overline{v}_{h,1}}{\partial y_2}$ in $L^2(A_{\delta,h})$.In this respect, there exists $K = K(\delta, r_1, r_2)$ such that

$$\|\frac{\partial \overline{v}_{h,1}}{\partial y_2}\|_{L^2(A_{\delta,h})} \geq \frac{K}{h^{\frac{3}{4}}}.$$

Combining (4.23) with (4.24), (4.25), and the above result, we obtain

$$\tilde{n}_1(h) \geq \frac{K}{h^{\frac{3}{2}}}.$$

□

Now, we give the proof of Theorem 4.1.

**Proof of Theorem 4.1** We plug the test function $v$ defined in (4.7) into

$$\int_{\mathbb{R}^2}\big(\rho u\cdot\partial_t v + \rho u\otimes u : D[v] - 2\nu D[u] : D[v] + \rho f\cdot v\big)dx = \frac{d}{dt}\int_{\mathbb{R}^2} u\cdot v dx, \tag{4.26}$$

and we start to estimate each term separately. Lemma 4.2 and Lemma 4.3 imply that there exists a positive constant $K_m = K_m(\delta, d_{\max})$ such that

$$\Big|\int_{\mathbb{R}^2}\rho(s)f(s)\cdot v(s)dx\Big| \leq K_m\|\rho\|_{L^\infty([0,T_0)\times\mathbb{R}^2)}\|f\|_{\mathbf{L}^2(\mathbb{R}^2)}. \tag{4.27}$$

We turn now to bound the non-linear term and we have

$$\Big|\int_{\mathbb{R}^2}\rho(s)u(s)\otimes u(s) : D[v(s)]dx\Big| \leq K_m\|\rho\|_{L^\infty([0,T_0)\times\mathbb{R}^2)}\Big(\|u\|^2_{\mathbf{H}^1(\mathbb{R}^2)} + \Big|\int_{A_{\delta,d_{1,2}(s)}} u(s)\otimes u(s) : D[v(s)]dx\Big|\Big).$$

By performing integration by parts, applying Holder inequality and noting that the vector field $v$ is uniformly bounded outside the $A_{\delta,h}$, we get

$$\begin{aligned}\Big|\int_{A_{\delta,d_{1,2}(s)}} u(s)\otimes u(s) : D[v(s)]dx\Big| &\leq \Big|\int_{A_{\delta,d_{1,2}(s)}}(u(s)\cdot\nabla)u(s)\cdot v(s)dx\Big| + \Big|\int_{\partial A_{\delta,d_{1,2}(s)}}(u(s)\cdot v(s))(u(s)\cdot n)d\Gamma\Big| \\ &\leq C\|u(s)\|^2_{\mathbf{H}^1(\mathbb{R}^2)}\|\overline{v}_{d_{1,2}(s)}\|_{\mathbf{L}^{5/2}(A_{\delta,d_{1,2}(s)})} + K_m(\|u(s)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + \|\nabla u(s)\|^2_{[L^2(\mathbb{R}^2)]^4}).\end{aligned}$$

Combining the above result with Lemma 4.3, we obtain

$$\Big|\int_{\mathbb{R}^2}\rho(s)u(s)\otimes u(s) : D[v(s)]dx\Big| \leq K_m\|\rho\|_{L^\infty([0,T_0)\times\mathbb{R}^2)}\Big(\|u\|^2_{L^\infty([0,T_0),\mathbf{L}^2(\mathbb{R}^2))} + \|\nabla u(s)\|^2_{[L^2(\mathbb{R}^2)]^4}\Big). \tag{4.28}$$

For simplicity, we denote $d_{1,2}(t)$ by $h(t)$. With this notation and from the definition of the vector field $v$ in (4.7), we have

$$\begin{aligned}\int_{\mathbb{R}^2}\rho(s)u(s)\cdot\partial_t v(s)dx &= \int_{\mathbb{R}^2}\rho(s)u(x,s)\cdot\partial_t\Big(J_X(Y(x,s),s)\overline{v}(Y(x,s),s)\Big)dx \\ &= \int_{\mathbb{R}^2}\rho(s)u(x,s)\cdot\partial_t\Big(J_X(Y(x,s),s)\big(\nabla_y^\perp\tilde{g}_{h(s)}\big)(Y(x,s))\Big)dx.\end{aligned}$$

By noting that

$$\frac{\partial \tilde{g}_{h(s)}}{\partial y_i}((Y(x,s),s) = \sum_{j=1}^{2} \frac{\partial X_j}{\partial y_i}(Y(x,s),s)\partial_{x_j}\Big(\tilde{g}_{h(s)}(Y(x,s))\Big),$$

we obtain

$$J_X(Y(x,s),s)\big(\nabla_y^{\perp}\tilde{g}_{h(s)}\big)(Y(x,s),s) = \nabla_x^{\perp}\Big(\tilde{g}_{h(s)}(Y(x,s))\Big),$$

and thus

$$\int_{\mathbb{R}^2} \rho(s)u(s)\cdot\partial_t v(s)dx = \int_{\mathbb{R}^2} \rho(s)u(x,s)\cdot\partial_t\nabla_x^{\perp}\Big(\tilde{g}_{h(s)}(Y(x,s))\Big)dx.$$

By performing integration by parts on the space variable, we get that

$$\int_{\mathbb{R}^2} u(x,s)\cdot\partial_t\nabla_x^{\perp}\Big(\tilde{g}_{h(s)}(Y(x,s))\Big)dx = \int_{\mathbb{R}^2}\Big(\frac{\partial u_1}{\partial x_2}(x,s) - \frac{\partial u_2}{\partial x_1}(x,s)\Big)\partial_t\Big(\tilde{g}_{h(s)}(Y(x,s))\Big)dx.$$

By noting that

$$\partial_t\Big(\tilde{g}_{h(t)}(Y(x,s)\Big) = h'(t)\partial_h\tilde{g}_{h(t)}((Y(x,t)) + \sum_{i=1}^{2} Y_i'(x,t)\partial_{y_i}\tilde{g}_{h(t)}(Y(x,t)),$$

and

$$\Big\|\frac{\partial X_i}{\partial y_j}\Big\|_{L^\infty(\mathbb{R}^2)} \leq 1,\ \ \Big\|\frac{\partial Y_i}{\partial x_j}\Big\|_{L^\infty(\mathbb{R}^2)} \leq 1,\ \ \|Y_i'\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}^2)} \leq c\sum_{i=1}^{2}|h_i'(t)|,$$

we get that there exists a positive constant $C$ such that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} u(x,s)\cdot\partial_t\nabla_x^{\perp}\Big(\tilde{g}_{h(s)}(Y(x,s),s)\Big)dx\Big| \leq C\|\nabla u(s)\|_{[L^2(\mathbb{R}^2)]^4}\Big\{|h'(s)|\Big(\|\partial_h\tilde{g}_{h(s)}\|_{\mathbf{L}^2(\mathbb{R}^2\setminus A^i_{\delta,h(s)})} \\ + \Big[\int_{A^i_{\delta,h(s)}}|\partial_h\tilde{g}_{h(t)}(y)|^2dy\Big]^{\frac{1}{2}}\Big) + K_m\sum_{i=1}^{2}|h_i^{'}(t)|\Big\}.\end{aligned}$$

By Lemma 4.4, we get that

$$\Big|\int_{\mathbb{R}^2}\rho(s)u(s)\cdot\partial_t v(s)dx\Big| \leq K_m\|\rho\|_{L^\infty(\mathbb{R}^2\times[0,T_0))}\big(\sup_{s\in[0,T_0)}|h'(s)| + \sum_{i=1}^{2}|h_i'(t)|\big)\|\nabla u(s)\|_{[L^2(\mathbb{R}^2)]^4}. \tag{4.29}$$

Adding the term $\tilde{n}_1(d_{1,2}(s))(V_{\overline{u},1} - V_{\overline{u},2})\cdot e_2(h)$ to both sides of the weak formulation (4.26) and combining the resulting equation with Proposition 4.1, Lemma 4.2 and the estimates in (4.27), (4.28) and (4.29), we get that

$$\Big|\frac{d}{dt}\int_{\mathbb{R}^2}\rho(s)u(s)\cdot v(s)dx + \tilde{n}_1(d_{1,2}(s))(V_{\overline{u},1} - V_{\overline{u},2})\cdot e_2(h)\Big| \leq K_m'\big(1 + \|u\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + \|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4}\big),$$

where $K_m' = (\delta, d_M, \|\rho\|_{L^\infty([0,T_0)\times\mathbb{R}^2)}, \|f\|_{\mathbf{L}^2(\mathbb{R}^2)})$ is a positive real constant.
By noting that

$$\overline{u}(y,s) = J_Y h_1'(s) + \omega_1(s)y^{\perp},\ \ y\in\partial B(G_1,r_1),$$
$$\overline{u}(y,s) = J_Y h_2'(s) + \omega_2(s)(y - y_{G_2})^{\perp},\ \ y\in\partial B(G_2,r_2),$$

we obtain

$$\big(V_{\overline{u},1} - V_{\overline{u},2}\big)\cdot e_2 = d_{1,2}^{'}(s).$$

This implies that

$$\Big|\frac{d}{dt}\int_{\mathbb{R}^2}\rho(s)u(s)\cdot v(s)dx + d_{1,2}^{'}(s)\tilde{n}_1(d_{1,2}(s))\Big| \leq K_m'\big(1 + \|u\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + \|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4}\big).$$

Integrating the above inequality from $t_0$ to $t < T_0$, we get that

$$\Big| \int_{\mathbb{R}^2} \rho(t)u(t) \cdot v(t)dx - \int_{\mathbb{R}^2} \rho(t)u(t_0) \cdot v(t_0)dx + \int_{t_0}^{t} d_{1,2}^{'}(s)\tilde{n}_1(d_{1,2}(s))ds \Big|$$
$$\leq K_m' \big(T_0 + \sup_{t\in[0,T_0)} \|u\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + \int_{t_0}^{t} \|\nabla u\|^2_{[L^2(\mathbb{R}^2)]^4}\big),$$

Combining together Lemma 3.1 with Lemma 4.3, we get that there exist $M > 0$ that depends on $T_0$ and the initial data such that

$$\Big| \int_{t_0}^{t} d_{1,2}^{'}(s)\tilde{n}_1(d_{1,2}(s))ds \Big| \leq K_m' M.$$

With the change of variable $h(s) = d_{1,2}(s)$, we get that

$$\Big| \int_{d_{1,2}(t_0)}^{d_{1,2}(t)} \tilde{n}_1(h)dh \Big| \leq K_m' M.$$

Again by Proposition 4.1, we get that

$$\Big| \int_{d_{1,2}(t_0)}^{d_{1,2}(t)} \frac{dh}{h^{\frac{3}{2}}} \Big| \leq K_m' M,$$

and thus

$$\frac{1}{[d_{1,2}(t)]^{\frac{1}{2}}} \leq \frac{1}{[d_{1,2}(t_0)]^{\frac{1}{2}}} + K_m' M.$$

The last inequality implies that

$$\sup_{t\leq T_0} \frac{1}{[d_{1,2}(t)]^{\frac{1}{2}}} \leq K_m' M.$$

□

**Proof of Theorem 1.2** By applying Theorem 1.1 and Proposition 3.1, our proof reduces to obtaining that no

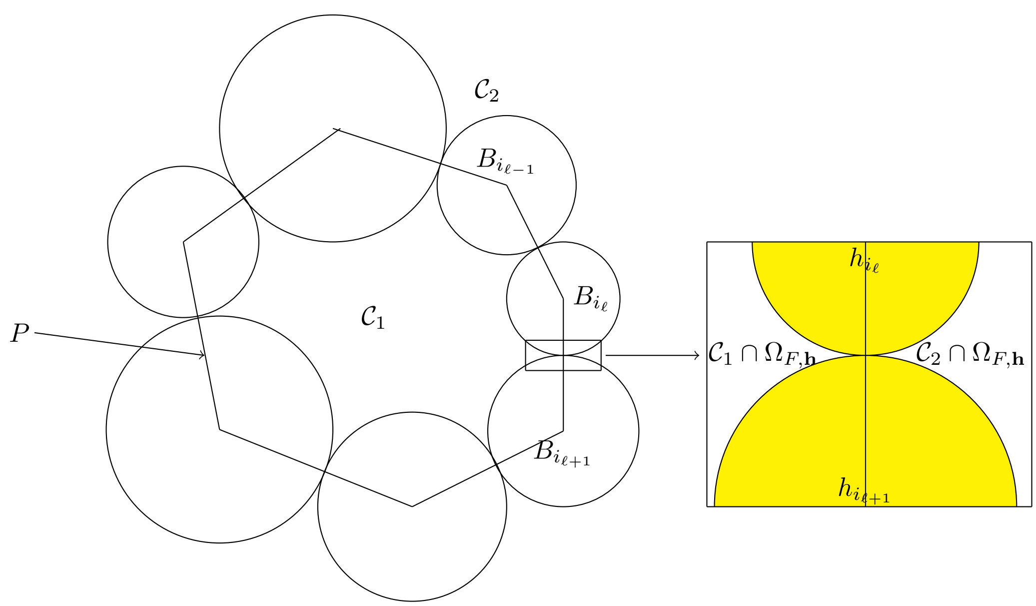


Figure 3: Collision between particles dividing the fluid domain into two connected components.

collision occurs in finite time under the hypothesis (H1). We act by contradiction and we assume that collision could take place in finite time. We define the the non-empty set $J$ of cardinal $2 \leq m \leq k$ as follows

$$J = \Big\{ j \in \{1, \dots, k\} : \exists i \neq j, 1 \leq i \leq k, \ \ d(B_i, B_j)(T_0) = 0 \Big\}.$$

For $i \in J$, we define the non-empty set of indices $J_i$ by

$$J_i = \Big\{ j \in J : \ j \neq i, \ \ d(B_i, B_j)(T_0) = 0 \Big\}.$$

We claim that there exists $i \in J$ such that $\operatorname{card}(J_i) = 1$. Otherwise, we have $\operatorname{card}(J_i) \geq 2$ for all $i \in J$. Hence for a fixed $i_0 \in J$, there exists $i_1 \in J_{i_0}$ and as $\operatorname{card}(J_{i_1}) \geq 2$, then there exists $i_2 \in J_{i_1} \backslash \{i_0\}$. By recurrence, we construct a sequence $\{i_\ell\}_{\ell \in \mathbb{N}}$ such that for all $\ell \in \mathbb{N}$, we have $i_{\ell+1} \in J_{i_\ell} \backslash \{i_{\ell-1}\}$. Since card $J$ is finite, then there exists two positive integers $\ell$ and $p$ such that $i_{\ell+p} = i_\ell$ and a simple draw shows that the center of masses $h_{i_\ell}, \dots, h_{i_{\ell+p}}$ of the disks $B_{i_\ell}, \dots, B_{i_{\ell+p}}$ form a set of vertices of a simple polygon $P$, whose complement is the union of two connected components $\mathcal{C}_1$ and $\mathcal{C}_2$. Furthermore, the fluid domain $\Omega_{F,\mathbf{h}} \subset P^c$ and we have $\Omega_{F,\mathbf{h}} \cap \mathcal{C}_i \neq \emptyset$, for $i = 1, 2$ (see Figure 3). This contradicts the assumption (H1) in Theorem 1.2.
Let $j$ denote the index of the disk that the disk $B_i$ only collide with at time $T_0$. Up to a renumbering, we assume that $i = 1$ and $j = 2$, so that (H2) holds true. We apply then Theorem 4.1 and we obtain a contradiction.

□

# Appendix

# A Proof of Lemma 3.3

Let $t < T_0$, $0 < \varepsilon < \gamma$ and $u \in \mathbf{H}^2(\Omega_F(t))$. We consider a family of smooth functions $\{\chi_i\}_{i=1,\dots,k}$ each of compact support included in $[-r_i - \frac{\varepsilon}{2}, r_i + \frac{\varepsilon}{2}]$ and equals to one on $[-r_i, r_i]$. For each $i \in \{0, \dots, k\}$, we define the function $u^{(i)} : \Omega_F(t) \to \mathbb{R}^2$, such that

$$u^{(i)}(x) = \chi_i(|x - h_i(t)|)u(x), \ \ 1 \leq i \leq k$$

and

$$u^{(0)} = u - \sum_{i=1}^{k} u^{(i)}.$$

Moreover, for $i \in \{1, \dots, k\}$, we define the function $v^{(i)} : B(h_i(0), r_i + \frac{\varepsilon}{2}) \backslash B_i(0) \to \mathbb{R}^2$ by

$$v^{(i)} = u^{(i)}(x + h_i(t) - h_i(0)).$$

We note that $v^{(i)} \in \mathbf{H}^2\big(B(h_i(0), r_i + \frac{\varepsilon}{2}) \backslash B_i(0)\big)$ for all $i \in \{1, \dots, k\}$. We set $\overline{v}^{(i)} = Ev^{(i)}$, where $E$ is a strong 2-extension operator for $\Omega_F$. By Theorem 5.22 in [1], there exists a constant $k = k(\varepsilon)$ such that

$$\|\overline{v}^{(i)}\|_{\mathbf{L}^2(\mathbb{R}^2)} \quad \leq k \|u^{(i)}\|_{\mathbf{L}^2(\Omega_F(t))}, \tag{A.1}$$

$$\|\overline{v}^{(i)}\|_{\mathbf{H}^1(\mathbb{R}^2)} \quad \leq k \|u^{(i)}\|_{\mathbf{H}^1(\Omega_F(t))}, \tag{A.2}$$

$$\|\overline{v}^{(i)}\|_{\mathbf{H}^2(\mathbb{R}^2)} \quad \leq k \|u^{(i)}\|_{\mathbf{H}^2(\Omega_F(t))}. \tag{A.3}$$

We note that $\overline{v}^{(i)}$ vanishes outside $B(h_j(t), r_j + \frac{\varepsilon}{2})$ for all $j \neq i$. Finally, we set

$$Eu = \tilde{u}^{(0)} + \sum_{i=1}^{k} \overline{u}^{(i)},$$

where $\overline{u}^{(i)}(x) = \overline{v}^{(i)}(x - h_i(t) + h_i(0)), \forall i \in \{1, \ldots, k\}$ and $\tilde{u}^{(0)}$ is the extension of $u^{(0)}$ by zero over the disks. We remark here that $\tilde{u}^{(0)} \in \mathbf{H}^2(\mathbb{R}^2)$ and for simplicity we remove the tilde.

Hence, for $x \in \Omega_F(t)$ we have

$$Eu(x) = u^{(0)}(x) + \sum_{i=1}^{k} \overline{v}^{(i)}(x - h_i(t) + h_i(0)).$$

If $x \in B(h_j(t), r_j + \frac{\varepsilon}{2}) \backslash B_j(t)$, then $x - h_j(t) + h_j(0) \in B(h_j(0), r_j + \frac{\varepsilon}{2}) \backslash B_j(0)$ and $x \notin B(h_i(t), r_i + \frac{\varepsilon}{2}) \backslash B_i(t)$ for all $i \neq j$. Hence, for all $i \neq j$, we have $x - h_i(t) + h_i(0) \notin B(h_i(0), r_i + \frac{\varepsilon}{2}) \backslash B_i(0)$ and thus

$$\begin{aligned} Eu(x) &= u^{(0)}(x) + \overline{v}^{(j)}(x - h_j(t) + h_j(0)) \\ &= u^{(0)}(x) + v^{(j)}(x - h_j(t) + h_j(0)) \\ &= u^{(0)}(x) + u^{(j)}(x) \\ &= u(x). \end{aligned}$$

Now, if $x \in \Omega_F(t) \backslash \bigcup_{i=1}^{k} B(h_i(0), r_i + \dfrac{\varepsilon}{2})$, then $x - h_i(t) + h_i(0) \notin B(h_i(0), r_i + \frac{\varepsilon}{2})$ and thus

$$Eu(x) = u^{(0)}(x) = u(x).$$

Moreover, there exists a positive real constant $k = k(\varepsilon)$ such that

$$\begin{aligned} \|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)} &\leq \|u^{(0)}\|_{\mathbf{L}^2(\mathbb{R}^2)} + \sum_{i=1}^{k} \|\overline{u}^{(i)}\|_{\mathbf{L}^2(\mathbb{R}^2)} \\ &\leq k\|u\|_{\mathbf{L}^2(\Omega_F(t))} + \sum_{i=1}^{k} \|\overline{v}^{(i)}\|_{\mathbf{L}^2(\mathbb{R}^2)} \\ &\leq k\|u\|_{\mathbf{L}^2(\Omega_F(t))} + k\sum_{i=1}^{k} \|u^{(i)}\|_{\mathbf{L}^2(\Omega_F(t))} \end{aligned}$$

Hence, we get

$$\|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{L}^2(\Omega_F(t))}. \tag{A.4}$$

In a similar way, we can prove

$$\|Eu\|_{\mathbf{H}^1(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^1(\Omega_F(t))}, \tag{A.5}$$

$$\|Eu\|_{\mathbf{H}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^2(\Omega_F(t))}. \tag{A.6}$$

# B Proof of Lemma 3.4

We can consider that the solution $(u,p)$ is a solution of the following problem at a fixed time $t>0$:

$$\begin{cases} u-\nu\Delta u+\nabla p=\tilde{f}, & \text{in } \Omega_F(t),\\ \nabla\cdot u=0, & \text{in } \Omega_F(t),\\ u(x,t)=h_i'(t)+\omega_i(t)(x-h_i(t))^{\perp}, & x\in B_i(t),\ \forall i\in\{1,\dots,k\}, \end{cases} \tag{B.1}$$

where

$$\tilde{f}=-\frac{\partial u}{\partial t}-(u\cdot\nabla)u+f+u. \tag{B.2}$$

By using the change of variables $X$ defined in (2.3), we see that $(U,P)$ as defined in (2.4) satisfies the following problem:

$$\begin{cases} U-\nu\Delta U+\nabla P=\tilde{g}, & \text{in } \Omega_F,\\ \nabla\cdot U=0, & \text{in } \Omega_F,\\ U|_{\partial B_i}=\overline{\Lambda}|_{\partial B_i}, & \forall i\in\{1,\dots,k\}, \end{cases} \tag{B.3}$$

with

$$\tilde{g}=\nu[(L-\Delta)U]-[(G-\nabla)P]-[MU]-[NU]-\frac{\partial U}{\partial t}+F+U, \tag{B.4}$$

where $[LU],[MU],[NU]$, and $[GP]$ are defined as in (2.12)-(2.15).
By Theorem 2.1 in [6], there exists a unique $(U,P)\in\mathbf{H}^2(\Omega_F)\times\dot{H}^1(\Omega_F)$ solution of problem (B.3). Moreover, there exists a constant $C_3=C_3(\nu,\Omega_F)>0$ such that

$$\|U\|_{[H^2(\Omega_F)]^2}+\|\nabla P\|_{[L^2(\Omega_F)]^2}\le C_3\big(\|\tilde{g}\|_{[L^2(\Omega_F)]^2}+\|\overline{\Lambda}\|_{[H^2(\mathbb{R}^2)]^2}\big). \tag{B.5}$$

We start with estimating the first term in the expression of $\tilde{g}$. We have:

$$\begin{aligned} \big\|[(L-\Delta)U]_i\big\|_{L^2(\Omega_F)} \le{}& \sum_{j,k=1}^{2}\|g^{jk}-\delta_k^j\|_{L^\infty(\Omega_F)}\Big\|\frac{\partial^2 U_i}{\partial y_j\partial y_k}\Big\|_{L^2(\Omega_F)}\\ &+\sum_{j,k=1}^{2}\Big\|\frac{\partial g^{jk}}{\partial y_j}\Big\|_{L^\infty(\Omega_F)}\Big\|\frac{\partial U_i}{\partial y_k}\Big\|_{L^2(\Omega_F)}+2\sum_{j,k,\ell=1}^{2}\|g^{k\ell}\|_{L^\infty(\Omega_F)}\|\Gamma^i_{j,k}\|_{L^\infty(\Omega_F)}\Big\|\frac{\partial U_j}{\partial y_\ell}\Big\|_{L^2(\Omega_F)}\\ &+\sum_{j,k,\ell=1}^{2}\Big\{\Big\|\frac{\partial g^{k\ell}}{\partial y_k}\Big\|_{L^\infty(\Omega_F)}\|\Gamma^i_{j,\ell}\|_{L^\infty(\Omega_F}+\|g^{k\ell}\|_{L^\infty(\Omega_F)}\Big\|\frac{\partial\Gamma^i_{j,\ell}}{\partial y_\ell}\Big\|_{L^\infty(\Omega_F)}\\ &+\sum_{m=1}^{2}\|g^{k\ell}\|_{L^\infty(\Omega_F)}\|\Gamma^m_{j,\ell}\|_{L^\infty(\Omega_F)}\|\Gamma^i_{k,m}\|_{L^\infty(\Omega_F)}\Big\}\|U_j\|_{L^2(\Omega_F)}, \end{aligned} \tag{B.6}$$

In what follows, we denote by $K$ a positive constant that depends on $\Omega_F, B_i, \overline{\rho}_i, \nu, T_0, \|u_0\|_{\mathbf{L}^2(\mathbb{R}^2)}$ and $\|f\|_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))}$ that may changes between lines.
From the definition of $g^{ij}$, $g_{i,j}$ and $\Gamma^k_{i,j}$ respectively in (2.18), (2.16) and (2.17), and by applying the same technique of proof of Lemma 6.4 and Corollary 6.5 in [24], we get for all $1\le i,j,k\le 2$:

$$
\begin{aligned}
\|g^{ij}-\delta^i_j\|_{L^\infty(\mathbb{R}^2)} &\le KT_1, \quad \|g_{ij}-\delta^i_j\|_{L^\infty(\mathbb{R}^2)} \le KT_1,\\
\Big\|\frac{\partial g^{ij}}{\partial y_k}\Big\|_{L^\infty(\mathbb{R}^2)} &\le KT_1, \quad \Big\|\frac{\partial g_{ij}}{\partial y_k}\Big\|_{L^\infty(\mathbb{R}^2)} \le KT_1,\\
\|\Gamma^k_{i,j}\|_{L^\infty(\mathbb{R}^2)} &\le KT_1, \quad \Big\|\frac{\partial \Gamma^k_{i,j}}{\partial y_\ell}\Big\|_{L^\infty(\mathbb{R}^2)} \le KT_1.
\end{aligned}
$$

Combining the above estimates with (B.6), we obtain that

$$
\big\|[(L-\Delta)U]_i\big\|_{L^2(\Omega_F)} \le KT_1\|U\|_{\mathbf{H}^2(\Omega_F)}. \tag{B.7}
$$

By the same way, we get that there exists some positive constant $C$ such that

$$
\begin{aligned}
\big\|[(\nabla-G)P]_i\big\|_{L^2(\Omega_F)} &\le KT_1\|\nabla P\|_{\mathbf{L}^2(\Omega_F)},\\
\big\|[MU]_i\big\|_{L^2(\Omega_F)} &\le C\|U\|_{\mathbf{H}^1(\Omega_F)},\\
\big\|[NU]_i\big\|_{L^2(\Omega_F)} &\le \|(U\cdot\nabla)U\|_{\mathbf{L}^2(\Omega_F)} + KT_1\|U\|^2_{\mathbf{L}^2(\Omega_F)},
\end{aligned}
$$

and thus

$$
\begin{aligned}
\|\tilde{g}\|_{[L^2(\Omega_F)]^2} \le \Big\|\frac{\partial U}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F)} &+ \|(U\cdot\nabla)U\|_{\mathbf{L}^2(\Omega_F)} + KT_1(\|U\|_{\mathbf{H}^2(\Omega_F)} + \|\nabla P\|_{\mathbf{L}^2(\Omega_F)})\\
&+ C(\|F\|_{\mathbf{L}^2(\Omega_F)} + \|U\|^2_{\mathbf{L}^2(\Omega_F)} + \|\nabla U\|^2_{\mathbf{L}^2(\Omega_F)} + 1).
\end{aligned} \tag{B.8}
$$

Combining the above inequality with the estimate in (B.5), we obtain for $T_1$ is small enough:

$$
\begin{aligned}
\|U\|_{\mathbf{H}^2(\Omega_F)} + \|\nabla P\|_{\mathbf{L}^2(\Omega_F)} \le \frac{C_3}{1-KT_1}\Big\{\Big\|\frac{\partial U}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F)} &+ \|(U\cdot\nabla)U\|_{\mathbf{L}^2(\Omega_F)} + \|U\|^2_{\mathbf{L}^2(\Omega_F)} + \|\nabla U\|^2_{\mathbf{L}^2(\Omega_F)}\\
&+ \|F\|_{\mathbf{L}^2(\Omega_F)} + \|\overline{\Lambda}\|_{\mathbf{H}^2(\mathbb{R}^2)} + 1\Big\}.
\end{aligned} \tag{B.9}
$$

Bounding the transform $X$ and its derivatives up to order 3 from above as in Lemma 6.4 in [24], we get that

$$
\|u\|_{\mathbf{L}^2(\Omega_F(t))} \le K\|U\|_{\mathbf{L}^2(\Omega_F)}, \tag{B.10}
$$

$$
\|U\|_{\mathbf{L}^2(\Omega_F)} \le K\|u\|_{\mathbf{L}^2(\Omega_F(t))}, \tag{B.11}
$$

$$
\Big\|\frac{\partial U}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F)} \le K\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F(t))} + \|u\|_{\mathbf{H}^1(\Omega_F(t))}\Big), \tag{B.12}
$$

$$
\|(U\cdot\nabla)U\|_{\mathbf{L}^2(\Omega_F)} \le K\|u\|^2_{\mathbf{H}^1(\Omega_F(t))}, \tag{B.13}
$$

$$
\|\nabla u\|_{[L^2(\Omega_F)]^4} \le K\|U\|_{\mathbf{H}^1(\Omega_F)}, \tag{B.14}
$$

$$
\|\nabla^2 u\|_{[L^2(\Omega_F)]^8} \le K\|U\|_{\mathbf{H}^2(\Omega_F)}. \tag{B.15}
$$

Combining these estimates with that in (B.9), we obtain the required inequality in (3.21).


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