HOROFUNCTION COMPACTIFICATIONS OF SYMMETRIC SPACES

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Abstract. We consider horofunction compactifications of symmetric spaces with respect to invariant Finsler metrics. We show that any (generalized) Satake compactification can be realized as a horofunction compactification with respect to a polyhedral Finsler metric.

Contents

1. Introduction

Symmetric spaces of non-compact type arise in many areas of mathematics. Topologically they are diffeomorphic to a finite-dimensional vector

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space, in particular they are non-compact. The problem of constructing compactifications of symmetric spaces of non-compact type has been a classical problem. For an overview of compactifications of symmetric spaces see [BJ06, GJT98].

Any proper metric space (X, d) can be compactified by embedding X into the space of continuous functions $C_{p_0}(X)$ on X which vanish at a fixed base point p_0:

$$\begin{aligned} X &\longrightarrow C_{p_0}(X) \\ z &\longmapsto d(\cdot, z) - d(p_0, z). \end{aligned} \tag{1}$$

The closure of the image is the horofunction compactification $\overline{X}_d^{hor}$ of (X, d).

In this article we investigate horofunction compactifications of symmetric spaces endowed with invariant Finsler metrics.

It is well known that the visual compactification of a symmetric space X is realized as horofunction compactification with respect to the invariant Riemannian metric. We show that all Satake compactifications of X, and more generally all generalized Satake compactifications as defined in [GKW17] can be realized as horofunction compactifications with respect to polyhedral Finsler metrics on X.

Any G-invariant Finsler metric on X induces a Weyl group invariant norm on a maximal flat $F \cong \mathbb{R}^k$. The Finsler metric is said to be polyhedral, if the unit ball for its Weyl group invariant norm on $F \cong \mathbb{R}^k$ is a finite sided polytope.

Before stating the result more precisely we recall that Satake compactifications $\overline{X}_\tau^S$ are associated to irreducible faithful representations $\tau : G \to \mathrm{PSL}(n, \mathbb{C})$, which give rise to embeddings $X = G/K \to \mathbb{P}(Herm(\mathbb{C}^n))$, $gK \mapsto [\tau(g)^*\tau(g)]$, see [Sat60]. Generalized Satake compactifications are defined the same way, but allowing τ to be reducible. There are finitely many isomorphism classes of Satake compactifications, determined by subsets of the set of simple roots, but infinitely many isomorphism classes of generalized Satake compactifications.

Theorem 1.1 *Let $X = G/K$ be a symmetric space of non-compact type. Any generalized Satake compactification of X can be realized as the horofunction compactification of a polyhedral G-invariant Finsler metric on X. More precisely, if the generalized Satake compactification is given by a representation $\tau : G \to \mathrm{PSL}(n, \mathbb{C})$, then $\overline{X}_\tau^S$ is isomorphic to $\overline{X}_d^{hor}$, where d is the Finsler metric on X whose unit ball in a maximal Cartan subspace $\mathfrak{a}$ is the polytope dual to $-D = -\operatorname{conv}(\mu_1, \cdots, \mu_k)$, where μ_i are the weights of the representation τ.*

Remark 1.2 The idea to realize Satake compactifications as horofunction compactifications with respect to polyhedral Finsler metrics has been sketched in the second authors diploma thesis [S13]. Specific Satake compactifications have been realized as horofunction compactifications of Finsler metrics before. Friedland and Freitas [FF04I, FF04II] describe the horofunction compactification for Finsler p-metrics on $GL(n, \mathbb{C})/U_n$ for $p \in [1, \infty]$, which they show to agree with the visual compactification for $p > 1$, and the horofunction compactification of the Siegel upper half plane of rank n

for the l_1-metric, which they show to agrees with the bounded symmetric domain compactification, a minimal Satake compactification. Kapovich and Leeb realize the maximal Satake compactification of a symmetric space $X = G/K$ of non-compact type as the horofunction boundary with respect to a G-invariant Finsler metric on X [KL16]. Parreau [Pa] shows that the horofunction compactification with respect to the Weyl chamber valued distance function is isomorphic to the maximal Satake compactification. There are related constructions for buildings, too, see for example [Brill]. Another paper by Ciobotaru, Kramer and Schwer [CKS] on horofunction compactifications is in preparation.

In order to describe the horofunction compactification $\overline{X}_d^{hor}$ in more detail and relate it to generalized Satake compactifications, we will make use of the Cartan decomposition $G = KAK$. This allows to reduce the problem to comparing the closure of a maximal flat $F = Ap_0 \subseteq X$ in $\overline{X}_d^{hor}$ with the closure of F in a generalized Satake compactification. The closure of a flat in a generalized Satake compactification is described in [GKW17].

The key result for the proof of Theorem 1.1 is then the following theorem, which states that for polyhedral Finsler metrics, the closure of a maximal flat $F = Ap_0 \subseteq X$ in $\overline{X}_d^{hor}$ is isomorphic to the horofunction compactification of $F \cong \mathbb{R}^k$ with respect to the induced Weyl group invariant polyhedral Finsler norm. The horofunction compactification of $F \cong \mathbb{R}^k$ with respect to a Weyl group invariant polyhedral Finsler norm has been determined in [Wal07, JS16].

Theorem 1.3 *Let $X = G/K$ be a symmetric space of non-compact type. Consider a polyhedral G-invariant Finsler metric on X. Let $\overline{X}^{hor}$ be the horofunction compactification of X with respect to this Finsler metric. Then the closure of a maximal flat F in $\overline{X}^{hor}$ is isomorphic to the horofunction compactification of F with respect to the induced metric.*

Remark 1.4 In fact we prove a more general statement than Theorem 1.3. Instead of requiring that the Finsler metric is polyhedral we only need to require that the Finsler norm on F satisfies a Convexity Lemma, see Section 3.4. We prove the Convexity Lemma for polyhedral Finsler metrics, but expect it to hold in general.

In Section 2, we review the structure theory of symmetric spaces, and recall a characterization of G-invariant Finsler metrics. In Section 3, we review the horofunction compactification of metric spaces, and focus on the case of normed vector spaces. Using a characterization of convergent sequences in the horofunction compactification, we prove a technical statement called the Convexity Lemma for polyhedral norms. In Section 4, we consider a G-invariant Finsler metric on the symmetric space X which satisfies the Convexity Lemma. For each maximal flat F in X, we prove that the closure of F in the horofunction compactification of X is isomorphic to the intrinsic horofunction compactification of F. So we reduce the study of horofunction compactifications of symmetric spaces to the study of horofunction compactifications of maximal flats. In Section 5, we combine the previous results to prove that each (generalized) Satake compactification is a horofunction compactification for a specific polyhedral Finsler norm on the symmetric space.

2. Invariant Finsler metrics on symmetric spaces

In this section we first review the necessary structure theory of semisimple Lie groups, see [Hel78] for details, and recall a characterization of G-invariant Finsler metrics due to Planche [Pla95].

2.1. **Structure Theory.** Throughout the article we denote by G a real semisimple Lie group with finite center, and by $\mathfrak{g}$ its Lie algebra. $K < G$ denotes a maximal compact subgroup, and $\mathfrak{k} \subseteq \mathfrak{g}$ its Lie algebra. The (Riemannian) symmetric space associated to G is $X = G/K$, and $p_0 = eK$ denotes its base point.

2.1.1. *Cartan decomposition.* The Lie algebra of G decomposes as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form κ on $\mathfrak{g}$.

We fix a maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$, and denote by $A = \exp(\mathfrak{a}) < G$ the corresponding connected subgroup of G. Then $F = A \cdot p_0 \subseteq X$ is a maximal flat.

All maximal abelian subalgebras are conjugate to each other, and $\mathfrak{p} = \mathrm{Ad}(K)\mathfrak{a} = \bigcup_{k\in K} \mathrm{Ad}(k)\mathfrak{a}$.

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*$ denote the system of restricted roots, i.e. $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$ if and only if

$$\mathfrak{g}_\alpha := \{V \in \mathfrak{g} \mid \mathrm{ad}(H)V = \alpha(H)V \ \forall H \in \mathfrak{a}\}$$

is non-zero.

For each $\alpha \in \Sigma$ consider the hyperplane $\ker(\alpha) \subseteq \mathfrak{a}$. Each of them divides the vector space $\mathfrak{a}$ into two half-spaces. The connected components of the set $\mathfrak{a} \setminus \bigcup_{\alpha\in\Sigma} \ker(\alpha)$ are called *Weyl chambers*. We fix one of these chambers to be the positive Weyl chamber $\mathfrak{a}^+$, and define *positive roots* by

$$\Sigma^+ := \{\alpha \in \Sigma \mid \alpha(H) > 0 \ \ \forall H \in \mathfrak{a}^+\}.$$

We denote by Δ the set of *simple roots* :

$$\Delta := \{\alpha \in \Sigma^+ \mid \alpha(H) \text{ is not the sum of two positive roots}\}.$$

The simple roots form a basis of Σ in the sense that we can express every root as a linear combination of elements in Δ with integer coefficients which are either all ≥ 0 or all ≤ 0.

Lemma 2.1 (Cartan decomposition; [Hel78], **Thm.V.6.7 and Thm.IX.1.1)** *Let $\mathfrak{a}^+$ be a positive Weyl chamber. Set $A^+ := \exp(\mathfrak{a}^+) \subseteq G$ and denote by $\overline{A^+}$ its closure. Note that $\overline{A^+} = \exp(\overline{\mathfrak{a}^+})$. For every element $g \in G$ there exist $k_1, k_2 \in K$ and some $a \in \overline{A^+}$ such that $g = k_1 a k_2$.*

We shortly write

$$G = K\overline{A^+}K,$$

and call this a Cartan decomposition of G.

2.1.2. *The Weyl group.* Let $\mathcal{C}_K(\mathfrak{a}) = \{k \in K \mid \mathrm{Ad}(k)(H) = H \ \forall H \in \mathfrak{a}\}$ denote the centralizer of $\mathfrak{a}$ in K and $\mathcal{N}_K(\mathfrak{a}) = \{k \in K \mid \mathrm{Ad}(k)\mathfrak{a} \subseteq \mathfrak{a}\}$ its normalizer. Then $\mathcal{C}_K(\mathfrak{a}) \trianglelefteq \mathcal{N}_K(\mathfrak{a})$ is a normal subgroup.

Definition 2.2 The quotient

$$W := {\mathcal{N}_K(\mathfrak{a})}\big/{\mathcal{C}_K(\mathfrak{a})}$$

is the *Weyl group.* It acts simply transitively on the set of Weyl chambers. The Weyl group is generated by the reflections in the hyperplanes $\ker(\alpha)$ for $\alpha \in \Delta$ and can also be expressed as $\mathcal{N}_K(A)/\mathcal{C}_K(A)$.

2.2. **Finsler Geometry.** A Finsler metric on a smooth manifold M generalizes the concept of Riemannian metric. It is a continuous family of (possibly asymmetric) norms on the tangent spaces, which are not necessarily induced by an inner product.

Definition 2.3 Let M be a smooth manifold. A *Finsler metric* on M is a continuous function

$$F : TM \longrightarrow [0, \infty)$$

such that, for each $p \in M$, the restriction $F|_{T_pM} : T_pM \longrightarrow [0, \infty)$ is a (possibly asymmetric) norm.

The length and (forward) distance on a Finsler manifold can be defined in the same way as on a Riemannian manifold:

Definition 2.4 The *length* of a curve $\gamma : [0, 1] \subseteq \mathbb{R} \longrightarrow M$ is defined as

$$L(\gamma) := \int_I F(\gamma(t), \dot{\gamma}(t))dt.$$

The *forward distance* between two points $p, q \in M$ is given by

$$d_F(p, q) := \inf_{\gamma} L(\gamma),$$

where the infimum is taken over all piecewise continuously differentiable curves $\gamma : [0, 1] \longrightarrow M$ with $\gamma(0) = p$ and $\gamma(1) = q$.

Remark 2.5 As the homogeneity in the definition for a Finsler metric only holds for positive scalars, the norms on the tangent spaces do not have to be symmetric. Therefore in general $d_F(p, q) \neq d_F(q, p)$.

The symmetric space X carries a G-invariant Riemannian metric, which is essentially unique (up to scaling on the irreducible factors). However, X carries many G-invariant Finsler metrics. Such G-invariant Finsler metrics on X and their isometry groups have been investigated by Planche, who proved:

Theorem 2.6 ([Pla95]**, Thm.6.2.1)** *There is a bijection between*

i) the W-invariant convex closed balls B of $\mathfrak{a}$,
ii) the $\mathrm{Ad}(K)$-invariant convex closed balls C of $\mathfrak{p}$,
iii) the G-invariant Finsler metrics on X.

In particular, any G-invariant Finsler metric on X gives rise to a (not necessarily symmetric) norm on the vector space $\mathfrak{a}$, whose unit ball is the

W-invariant convex ball B, and it is in turn completely determined by this norm.

Definition 2.7 A G-invariant Finsler metric on X is said to be polyhedral if its W-invariant convex ball B in $\mathfrak{a}$ is a finite sided polytope.

3. Horofunction compactifications

The horofunction compactifications of normed vector spaces have been described by Walsh in [Wal10]. We will not give a full description of his work, but focus on the case when the convex ball B is a finite sided polytope. In this setting the horofunction compactification has been described in detail by Ji and Schilling in [JS16], see also [KMN06] for a description of horoballs.

The key results for us are a characterization of converging sequences, a convexity lemma, and the identification between the horofunction compactification and the dual convex polytope B°.

3.1. The horofunction compactification of a metric space.

Let (X, d) be a metric space whose metric is possibly non-symmetric and with its topology induced by the symmetrized distance

$$d_{sym}(x, y) := d(x, y) + d(y, x)$$

for all $x, y \in X$. Let $C(X)$ be endowed with the topology of uniform convergence on bounded sets with respect to d_{sym}. Fix a basepoint $p_0 \in X$. Let $C_{p_0}(X)$ be the set of continuous functions on X which vanish at p_0. This space is homeomorphic to the quotient of $C(X)$ by constant functions, $\widetilde{C}(X) := C(X) \,/\, const$. Define the map

$$\begin{aligned} \psi : X &\longrightarrow \widetilde{C}(X) \\ z &\longmapsto \psi_z \end{aligned} \tag{2}$$

where

$$\psi_z(x) = d(x, z) - d(p_0, z)$$

for all $x \in X$. Then ψ is continuous and injective. If X is geodesic, proper with respect to d_{sym} and if d is symmetric with respect to convergence, that is, $d(x_n, x) \to 0$ iff $d(x, x_n) \to 0$ for any sequence $(x_n)_{n\in\mathbb{N}}$ and some $x \in X$, then the closed set $\mathrm{cl}\{\psi_z \mid z \in X\}$ is compact and ψ is an embedding of X into $\widetilde{C}(X)$. For more details see [Wal10, p.4 and Prop. 2.2].

Definition 3.1 The *horofunction boundary* $\partial_{hor}(X)$ of X in $\widetilde{C}(X)$ is defined as

$$\partial_{hor}X := (\mathrm{cl}\{\psi_z \mid z \in X\}) \setminus \{\psi_z \mid z \in X\}.$$

Its elements are called *horofunctions*. If $\mathrm{cl}\{\psi_z \mid z \in X\}$ is compact, then the set $\overline{X}^{\mathrm{hor}} := \mathrm{cl}\{\psi_z \mid z \in X\} = X \cup \partial_{hor}X$ is called the *horofunction compactification* of X.

Remark 3.2 The definition of ψ_z and therefore also those of ψ and $\partial_{hor}X$ depend on the choice of the basepoint p_0. One can show by a short calculation that, if we choose an alternative basepoint, the corresponding boundaries are homeomorphic.

3.2. **Horofunction compactifications of normed vector spaces.** Let in the following $(V, \|\cdot\|)$ always denote a finite-dimensional normed space and let $\langle\cdot|\cdot\rangle$ denote the dual pairing on it.

Notation For a subset $C \subseteq V$ we denote by $V(C) \subseteq V$ the subspace generated by C and by $V(C)^\perp$ its orthogonal complement, for some fixed arbitrary Euclidean structure on V. The orthogonal projection to $V(C)$ will be denoted by Π_C. For an element $v \in V$ we write $v = v_C + v^C$ with $v_C \in V(C)$ and $v^C \in V(C)^\perp$.

As the norm might be asymmetric, note that we use the convention

$$d(x,z) = \|z - x\|.$$

Definition 3.3 Let B be the unit ball with respect to the norm $\|\cdot\|$. The *dual unit ball* B° of B is defined to be the polar of B:

$$B^\circ := \{y \in V^* \mid \langle y|x\rangle \geq -1 \ \ \forall x \in B\},$$

where V^* denotes the dual space of V.

Remark 3.4 Some authors define the polar and therefore the dual unit ball by the condition $\langle y^*|x\rangle \leq 1 \ \ \forall x \in B$. As long as B is symmetric, this makes as a set no difference.

Definition 3.5 The *relative interior* $\operatorname{ri} S$ of a set $S \subseteq V$ is the interior of S when S is seen as a subset of a minimal affine subspace of V containing S. Similary, the *relative boundary* $\partial_{\text{rel}} S$ of S is the boundary of S seen in the minimal affine subspace containing S.

In the following we will consider norms $\|\cdot\|$ which have a polyhedral unit ball B, that is, B is a polytope. We will always assume our polytope to be finite, bounded and convex. Such a polytope can be described in two ways: either as the convex hull of finitely many points or as the intersection of finitely many halfspaces. The interplay of them is strongly related to the relation between the unit ball B and its dual B°:

Remark 3.6 Let the unit ball $B \subseteq V$ be given as the convex hull of a finite set of points, $B = \operatorname{conv}\{a_1, \ldots, a_k\}$. In this description we want all points a_i to be extremal, i.e. $\operatorname{conv}\{a_1, \ldots, a_k\} \neq \operatorname{conv}\{a_1, \ldots, \hat{a}_j, \ldots, a_k\}$ for all $j \in \{1, \ldots, k\}$. Then each vertex $a_i \in V$ determines a halfspace $V_i := \{y \in V^* \mid \langle y|a_i\rangle \geq -1\} \subseteq V^*$ which contains the origin in its interior. The boundary H_i of such a halfspace is a hyperplane for which it holds $\langle H_i|a_i\rangle = -1$, that is, $\langle h_i|a_i\rangle = -1$ for all $h_i \in H_i$. Then the dual unit ball B° is given by:

$$\begin{aligned} B^\circ &= \bigcap_{i=1}^k V_i \\ &= \{y \in V^* \mid \langle y|a_i\rangle \geq -1 \ \ \forall i = 1, \ldots, k\}. \end{aligned}$$

As the unit ball B is closed convex and contains the origin as an interior point, we know by the theory of polars and convex sets that

$$(B^\circ)^\circ = B.$$

Therefore, if B is given as the intersection of halfspaces, with this result we can easily determine B° as the convex hull of a set of points. Starting from this description, we only want to consider relevant halfspaces in the intersection, that is, $B \cap \partial V_i$ is a $m-1$ dimensional face[1] of B, which we will also call a *facet*.

Based on these two descriptions there is a one-to-one correspondence between the faces of B and those of B°.

Lemma 3.7 *Let $B \subseteq V$ be a polyhedral unit ball and $B^\circ \subseteq V^*$ its dual. For a face $F \subseteq B$ define its* dual set *by*

$$F^\circ := \{y \in B^\circ \mid \langle y|x\rangle = -1 \ \forall x \in F\} \subseteq B^\circ.$$

Then F° is a face of B° and it holds

$$\dim F + \dim F^\circ = m - 1.$$

Proof. To show that F° is a face of B° we have to show that it is an extreme set of B°. Recall that $F^\circ \subseteq B^\circ$ is an extreme set, if some interior point of a line in B° lies in F°, then also both endpoints of the line. Therefore let $y_1, y_2 \in B^\circ$ with $y = \frac{1}{2}(y_1 + y_2) \in F^\circ$. For any $x \in F$ we have

$$-1 = \langle y|x\rangle = \frac{1}{2}(\langle y_1|x\rangle + \langle y_2|x\rangle) \geq -1,$$

as both $y_1, y_2 \in B^\circ$. Equality holds if and only if $\langle y_1|x\rangle = \langle y_2|x\rangle = -1$ and therefore $y_1, y_2 \in F^\circ$.

We now show the formula for the dimensions. Let $V(F) \subseteq V$ be the subspace generated by F. We show that the dual $(V(F)^\perp)^* \subseteq V^*$ of its orthogonal complement is parallel to the affine subspace generated by F°. As $F \subseteq B$ is part of a hyperplane defining B, we can find $z \in V^*$ such that $F = \{x \in V | \langle z|x\rangle = -1\}$ and $\langle z|x\rangle >= 1$ for all $x \in B \setminus F$. Then $z \in F^\circ$ and we claim that

$$(V(F)^\perp)^* = V(F^\circ - z).$$

To see this, let $B = \operatorname{conv}\{a_1, \ldots, a_k\}$, and let $S_F \subseteq \{1, \ldots, k\}$ be those indices such that $F = \operatorname{conv}\{a_i | i \in S_F\}$. Let $y \in (V(F)^\perp)^*$ and $\varepsilon > 0$. Then $\langle z + \epsilon y|a_i\rangle = -1$ for all $i \in S_F$ and $\langle z + \varepsilon y|a_j\rangle > -1 + \varepsilon\langle y|a_j\rangle$ for all $a_j \notin S_F$. This implies $z + \varepsilon y \in F^\circ$ for ε small enough and $y \in \frac{1}{\varepsilon}(F^\circ - z) \subseteq V(F^\circ - z)$. The other inclusion follows immediately.
With $\dim(F) = \dim(V(F)) - 1$ (because $0 \notin F$) and $\dim(V(F^\circ - z)) = \dim(F^\circ)$ (because $0 \in (F^\circ - z)$) we obtain

$$\dim(V) = \dim(V(F)) + \dim(V(F)^\perp) = \dim(F) + 1 + \dim(F^\circ),$$

which finishes the proof. □

In the case of a finite-dimensional normed space with polyhedral norm, Walsh gives a criterion ([Wal07, Thm. 1.1 and Thm 1.2]) to calculate the horofunctions explicitely by using the Legendre-Fenchel transform of some special map. We rewrite these functions using some kind of pseudo-norm, see [Wal07, p.5] or [S13] for more details:

[1]A *k-face* of a polytope $P = \operatorname{conv}\{p_1, \ldots, p_r\}$ is the k-dimensional intersection of P with one or more hyperplanes H_i $(i \in \{1, \ldots, r\})$ defining P.

Definition 3.8 Let $C \subseteq V^*$ be a convex set. For $p \in V$ we set

$$|p|_C := - \inf_{q \in C} \langle q|p \rangle.$$

Remark 3.9 $|\cdot|_C$ is in general not a norm but

$$|\cdot|_{B^\circ} = \|\cdot\|.$$

Now we define the functions that will turn out to be the horofunctions in the horofunction compactification of V with respect to the norm with unit ball B. Let $E \subseteq B^\circ$ be a face of B° and $p \in V(E^\circ)^\perp$ be a point. Then we define

$$\begin{aligned} h_{E,p} : V &\longrightarrow \mathbb{R} \\ y &\longmapsto |p - y|_E - |p|_E. \end{aligned}$$

A short calculation shows that only the orthogonal part of p makes a contribution:

$$h_{E,p} = h_{E,p^F},$$

with $F = E^\circ$ and $p^F \in V(F)^\perp$. If we choose not a proper face E but the entire dual unit ball, we bet by Remark 3.9 that

$$\psi_z = h_{B^\circ,z}$$

for all $z \in V$.

Combining the results of Walsh with some calculations that can be found in [JS16] we obtain

Theorem 3.10 ([Wal10], **Thm. 1.1 ,** [JS16], **p.10)** *Let $(V, \|\cdot\|)$ be a finite-dimensional normed space where the unit ball B is a polytope. Then the set of horofunctions is given as*

$$\partial_{hor}(V) = \{h_{E,p} | E \subseteq B \text{ is a proper face and } p \in V(E^\circ)^\perp\}.$$

Example 3.11 As an example let us consider $\mathbb{R}^2$ equipped with the L^1-norm. For notations of the faces see Figure 1.

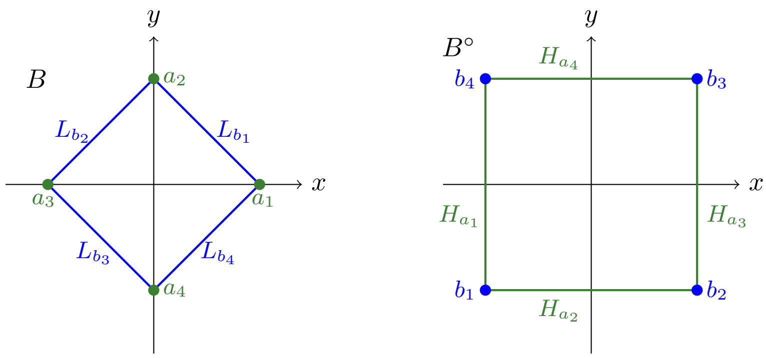

FIGURE 1. The unit ball B and its dual B°

Then its unit ball is given as the convex set

$$\begin{aligned} B &= \operatorname{conv}\{(1,0),(0,1),(-1,0),(0,-1)\} = \operatorname{conv}\{a_i | i=1,\dots,4\} \\ &= \bigcap_{i=1}^{4} \{x \in \mathbb{R}^2 \mid \langle b_i | x \rangle \geq -1\} \end{aligned}$$

with $b_1 = (-1,-1), b_2 = (1,-1), b_3 = (1,1)$ and $b_4 = (-1,1)$. By Remark 3.6 the dual unit ball is given by

$$\begin{aligned} B^\circ &= \bigcap_{i=1}^{4} \{y \in \mathbb{R}^2 \mid \langle y | a_i \rangle \geq -1\} \\ &= \operatorname{conv}\{b_1, b_2, b_3, b_4\}. \end{aligned}$$

The faces of B are $\mathcal{F} = \{a_i, L_{b_i} | i = 1,\dots,4\}$ where the facets are given by $L_{b_i} := \{x \in \mathbb{R}^2 | \langle b_i | x \rangle = -1\} \cap B$. Similary, the faces of B° are $\mathcal{E} = \{b_j, H_{a_j} | j = 1,\dots,4\}$ with $H_{a_j} := \{y \in \mathbb{R}^2 | \langle y | a_j \rangle = -1\} \cap B^\circ$.
As indicated by the notation, the dual faces are:

$$\begin{aligned} \{a_j\}^\circ &= H_{a_j} \\ (L_{b_i})^\circ &= \{b_i\}. \end{aligned}$$

The horofunctions then are for example given by

$$\begin{aligned} h_{\{b_j\},p}(y) &= |p-y|_{\{b_j\}} - |p|_{\{b_j\}} = \langle b_j | y \rangle \quad \text{for all } j = 1,\dots,4 \\ h_{H_{a_1},p}(y) &= -y_1 + |p_2 - y_2| - |p_2|. \end{aligned}$$

Remark 3.12 It is a general result that if $E \subseteq B^\circ$ is a vertex, then $h_{E,p}$ is independent of the point p.

3.3. **Characterization of sequences.** To describe the topology of $\overline{X}^{\text{hor}}$ we use the description given in the following theorem from [JS16]. For a convex set $C \subseteq V$ we denote by K_C the cone generated by C.

Theorem 3.13 ([JS16] **Theorem 3.10**) *Let $B \subseteq V$ be a convex polyhedral unit ball and B° its dual. Then $\psi_{z_n}(\cdot) = \|z_n - \cdot\| - \|z_n\|$ converges to a horofunction $h_{E,p}$ where $E \subseteq B^\circ$ is a proper face and $p \in V(E^\circ)^\perp$ if and only if the following conditions are satisfied with respect to the proper face $F = E^\circ \subseteq B$:*

- *i)* $\exists N \in \mathbb{N}$ *such that for all* $n \geq N : z_{n,F} \in K_F$.
- *ii)* $d(z_{n,F}, \partial_{\text{rel}} K_F) \longrightarrow \infty$ *as* $n \longrightarrow \infty$.
- *iii)* $\|z_n^F - p\| \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Example 3.14 As an example let us consider the same situation as in Example 3.11, that is, $\mathbb{R}^2$ equipped with the L^1-norm. Let $z_{n,1}$ be any unbounded sequence lying in the open first quadrant of $\mathbb{R}^2$ such that the distance to the coordinate axes goes to infinity. Choose $F_1 = L_{\{b_1\}}$ as a proper facet of B. Then K_{F_1} is the closed first quadrant and $V(F_1) = \mathbb{R}^2$. Therefore all conditions of the theorem are satisfied and as $F_1 = \{b_1\}^\circ$, it follows $\psi_{z_{n,1}} \longrightarrow h_{\{b_1\},p}$. On the other hand, if we take a sequence $z_{n,2}$ following a straight line l_2 parallel to the positive x-axis, we have to choose $F_2 = \{a_1\}$ as extreme set of B and some $p \in \mathbb{R}^2$ with $|p_2|$ the distance of the line l_2 to the x-axis. Then $\psi_{z_{n,2}} \longrightarrow h_{H_{\{a_1\}},p}$. Note that if $p, p' \in \mathbb{R}^2$ with $p_2 = p'_2$ then $h_{H_{\{a_1\}},p} = h_{H_{\{a_1\}},p'}$.

3.4. **Convexity Lemma.** Let $(V, \|\cdot\|)$ be a normed vector space. Let $(x_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ be two sequences converging to the same boundary point $\xi \in \overline{V}^{hor}$ in the horofunction compactification of $(V, \|\cdot\|)$, that is, $\psi_{x_n}, \psi_{z_n} \longrightarrow \xi$. Let y_n be a sequence lying between (x_n) and (z_n) in the sense that for each $n \in \mathbb{N}$ there exists a $t_n \in [0, 1]$ such that

$$y_n = (1 - t_n)x_n + t_n z_n.$$

We say that $(V, \|\cdot\|)$ satisfies the *Convexity Lemma* if then y_n converges to the same boundary point: $\psi_{y_n} \longrightarrow \xi$.

Lemma 3.15 (Convexity lemma for polyhedral norms) *Let $\|\cdot\|$ be a polyhedral norm on V and $\|\cdot\|^\circ$ its dual. Their unit balls are $B \subseteq V$ and $B^\circ \subseteq V^*$, respectively, both convex polytopes. Then $(V, \|\cdot\|)$ satisfies the Convexity Lemma.*

Proof. Let $\xi \in \overline{V}^{hor}$. Then, by Walsh, there is an extreme set $E \subseteq B^\circ$ with dual extreme set $F \subseteq B$ and a point $p \in V(F)^\perp$ such that we can write $\xi = h_{E,p}$. Let x_n, z_n be sequences converging to ξ, i.e. $\psi_{x_n}, \psi_{z_n} \longrightarrow \xi$. Let y_n be a sequence lying between (x_n) and (z_n), that is, for each $n \in \mathbb{N}$ $\exists t_n \in [0, 1]$ such that

$$y_n = (1 - t_n)x_n + t_n z_n.$$

We need to show that $\psi_{y_n} \longrightarrow \xi = h_{E,p}$.

We show this convergence by checking the criteria for converging sequences given by Theorem 3.13. As $\xi = h_{E,p}$, take $F = E^\circ$ and p as given. To facilitate readability, denote the projection of a point $q \in V$ to $V(F)$ by q_F. By the linearity of the orthogonal projection we have

$$y_{n,F} = \Pi_F\big((1 - t_n)x_n + t_n z_n\big) = (1 - t_n)x_{n,F} + t_n z_{n,F}.$$

As the sequences x_n and z_n satisfy the conditions of the theorem for this special F and as K_F is convex, it is $y_{n,F} \in K_F$ for n large enough.

Again by linearity of Π_F, an easy calculation shows:

$$\begin{aligned}\|y_n - y_{n,F} - p\| &= \|(1 - t_n)x_n + t_n z_n - (1 - t_n)x_{n,F} - t_n z_{n,F} - (1 - t_n)p - t_n p\| \\ &= \|(1 - t_n)(x_n - x_{n.F} - p) + t_n(z_n - z_{n,F} - p)\| \longrightarrow 0.\end{aligned}$$

Next we prove unboundedness of $(y_n)_n$. We already know that $x_{n,F}$, $y_{n,F}$, $z_{n,F} \in K_F$ for n large enough. Let denote $m_n := \|y_{n,F}\|$, then for each n, the point $y_{n,F}$ lies on the boundary of the scaled unit ball $m_n B$. As the boundary of $m_n B$ is the union of the scaled extreme sets of B and as $y_{n,F} \in K_F \cap \partial(m_n B)$, it follows $y_{n,F} \in m_n F$. The point $y_{n,F}$ lies on a straight line between $x_{n,F}$ and $z_{n,F}$, so we have to distinguish two cases for each n now. Either both $x_{n,F}$ and $z_{n,F}$ are also lying on the boundary $m_n F$, or one of them lies within $K_F \cap \operatorname{int}(m_n B)$. In any case, at least one of $x_{n,F}$ or $z_{n,F}$ has norm $\leq \|y_{n,F}\|$ for each $n \in \mathbb{N}$. By Theorem 3.13 we know that $\|x_n\|, \|z_n\| \longrightarrow \infty$, and as by the last condition of the theorem the orthogonal parts $x_n - x_{n,F}$ and $z_n - z_{n,F}$ are bounded, it follows $\|y_{n,F}\| \longrightarrow \infty$. Similary we can split up our sequence such that $y_n = y_{n,F} + p_n$ with $p_n \in V(F)^\perp$ and $p_n \longrightarrow p$. Then by the triangle inequality

$$\|y_{n,F}\| = \|y_{n,F} + p_n - p_n\| \leq \|y_{n,F} + p_n\| + \|p_n\| = \|y_n\| + \|p_n\|.$$

As $\|y_{n,F}\|$ goes to infinity and $\|p_n\|$ is bounded, we conclude that $\|y_n\| \to \infty$.

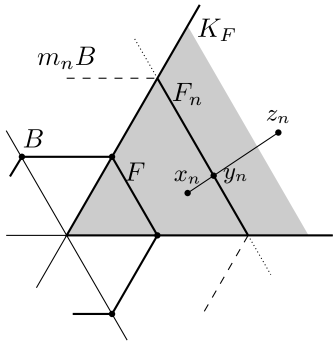

FIGURE 2. Sketch to the proof ($\mathbb{R}^2$ with hexagonal norm)

To show that the distance of $y_{n,F}$ to the relative boundary of K_F goes to infinity, remark that the relative boundary of K_F is a union of cones with vertex $\{0\}$ which are subsets of linear hyperplanes around K_F. Assume $y_{n,F}$ remains within bounded distance to one such hyperplane H. For each n let H_n be an affine hyperplane parallel to H such that $y_{n,F} \in H_n \cap K_F$ and with $d_n := d(H_n, H) < \infty$ for all n. Then with similar arguments as before, we have for each n the two cases that either both $x_{n,F}, z_{n,F} \in H_n$ or one of them lies between H_n and H and has therefore bounded distance $< d_n$ to $\partial_{rel} K_F$. But this is a contradiction to the condition $d(x_{n,F}, \partial_{\mathrm{rel}} K_F), d(z_{n,F}, \partial_{\mathrm{rel}} K_F) \longrightarrow \infty$ of Theorem 3.13. □

4. The compactification of flats in symmetric spaces

In this section we give, for any G-invariant Finsler metric on the symmetric space which satisfies the Convexity Lemma, an explicit homeomorphism between the intrinsic compactification of a flat and the closure of a flat in the horofunction compactification of the symmetric space.

Let d be the distance function associated to a G-invariant Finsler metric on the symmetric space $X = G/K$, and $\psi : X \to \widetilde{C}(X)$ the embedding defined in Subsection 3.1 on page 6. Let us state some basic observations.

Lemma 4.1 *The function ψ_{p_0} is K-invariant. Moreover for every $g \in G$, the function $\psi_{g \cdot p_0}$ is gKg^{-1}-invariant.*

Proof. Fix $g \in G$ and $k \in K$. Then, for any $x \in X$, we have

$$\begin{aligned} \psi_{g\cdot p_0}((gkg^{-1}) \cdot x) &= d((gkg^{-1}) \cdot x, g \cdot p_0) - d(p_0, g \cdot p_0) \\ &= d(x, g \cdot p_0) - d(p_0, g \cdot p_0) = \psi_{g \cdot p_0}(x). \end{aligned}$$

So $\psi_{g \cdot p_0}$ is gKg^{-1}-invariant. □

Lemma 4.2 *The map $\psi : X \to \widetilde{C}(X)$ is K-equivariant, that is, $\psi_{k \cdot z}(x) = k \cdot \psi_z(x)$, where the action on $\widetilde{C}(X)$ is given by $k \cdot f(x) := f(k^{-1}x)$.*

Proof. Fix $x, z \in X$ and $k \in K$. Then

$$\begin{aligned}\psi_{k\cdot z}(x) &= d(x, k\cdot z) - d(p_0, k\cdot z) = d(k^{-1}\cdot x, z) - d(p_0, z)\\ &= \psi_z(k^{-1}\cdot x) = k\cdot\psi_z(x).\end{aligned}$$

□

Lemma 4.3 *Let $G = K\overline{A^+}K$ be a Cartan decompostion, and $X = K\overline{A^+}\cdot p_0$. Then*

$$\overline{\psi(X)}^{\widetilde{C}(X)} = \overline{\psi(K\overline{A^+}\cdot p_0)}^{\widetilde{C}(X)} = K\overline{\psi(\overline{A^+}\cdot p_0)}^{\widetilde{C}(X)}.$$

In particular, the horofunction compactification $\overline{\psi(X)}^{\widetilde{C}(X)}$ is determined by the horofunction compactification of the flat $F = A\cdot p_0$, or more precisely of a closed Weyl chamber $F^+ = \overline{A^+}\cdot p_0$.

Proof. Since $\overline{\psi(\overline{A^+}\cdot p_0)}^{\widetilde{C}(X)}$ is a compact subspace of $\widetilde{C}(X)$ and K is a compact subgroup of G, which acts continuously on $\widetilde{C}(X)$, we deduce that the space $K\overline{\psi(\overline{A^+}\cdot p_0)}^{\widetilde{C}(X)}$ is a compact subspace of $\widetilde{C}(X)$. Since it contains $\psi(K\overline{A^+}\cdot p_0)$, we conclude that $\overline{\psi(K\overline{A^+}\cdot p_0)}^{\widetilde{C}(X)} \subseteq K\overline{\psi(\overline{A^+}\cdot p_0)}^{\widetilde{C}(X)}$. As the converse inclusion is clear, we conclude that $\overline{\psi(K\overline{A^+}\cdot p_0)}^{\widetilde{C}(X)} = K\overline{\psi(\overline{A^+}\cdot p_0)}^{\widetilde{C}(X)}$. □

In order to understand the horofunction compactification $\overline{\psi(\overline{A^+}\cdot p_0)}^{\widetilde{C}(X)}$ of a closed Weyl chamber, we will first compare it to its closure in the intrinsic horofunction compactification in $\widetilde{C}(A\cdot p_0)$. We will show that these two compactifications are in fact isomorphic.

4.1. **The intrinsic compactification of a flat.** The intrinsic compactification of the flat $A\cdot p_0$ is the horofunction compactification of $A\cdot p_0$ within the space of continuous functions on $A\cdot p_0$. Since the exponential map is a diffeomorphism $\exp : \mathfrak{a} \to A\cdot p_0$, the intrinsic compactification is homeomorphic to the horofunction compactification of the normed vector space $\mathfrak{a}$ with respect to the norm defined by the W-invariant convex ball B. This has been determined explicitely in [JS16]:

Theorem 4.4 ([JS16] **Theorem 1.2.**) *Let $(V, \|\cdot\|)$ be a normed vector space with polyhedral unit ball B. Then the horofunction compactification $\overline{V}^{hor}$ is homeomorphic to the dual convex polyhedron B°.*

4.2. **The closure of a flat.** The aim of this section is to compare the intrinsic compactification of $A\cdot p_0$ with the closure of the flat $A\cdot p_0$ in the horofunction compactification of X. To minimize confusion, we introduce the following notation:

Let d be the distance function of a G-invariant Finsler metric on $X = G/K$ and

$$\begin{aligned}\psi^X : X &\longrightarrow \widetilde{C}(X)\\ z &\longmapsto \psi^X_z := d(\cdot, z) - d(p_0, z)\end{aligned} \tag{3}$$

the embedding of X into the space of continuous functions on X vanishing at p_0.

We denote by d also the restriction of the distance to the flat $F = A \cdot p_0 \subseteq X$ and let

$$\begin{aligned} \psi^F : F &\longrightarrow \widetilde{C}(F) \\ z &\longmapsto \psi^F_z := d(\cdot, z) - d(p_0, z). \end{aligned} \tag{4}$$

denote the embedding of F into the space of continuous functions on F vanishing at p_0. The closure of $\psi^F(F) \subseteq \widetilde{C}(F)$ is the intrinsic compactification of F. We set $F^+ := \overline{A^+} \cdot p_0$.

We assume that d satisfies the Convexity Lemma, see Section 3.4.

4.3. **Groups associated with subsets of simple roots.** We want to associate to each horofunction on the flat a horofunction on the whole symmetric space X. We will start by setting up notation.

Let Δ be the set of positive roots. Given a subset $I \subseteq \Delta$ we denote by

- $W_I < W$ the subgroup generated by the reflections in the hyperplanes $\ker(\alpha)$ for $\alpha \in I$,
- $\mathfrak{a}_I = \bigcap_{\alpha \in I} \ker \alpha$, and $\mathfrak{a}^I$ its orthogonal complement in $\mathfrak{a}$,
- A_I, A^I the connected subgroups of A with Lie algebras $\mathfrak{a}_I$ and $\mathfrak{a}^I$ respectively,
- $M = C_K(A)$ the centralizer of A in K,
- G^I the derived subgroup of the centralizer of A_I in G,
- $K^I = G^I \cap K$, so that $K^I M$ is the centralizer of A_I in K,
- $W^I = N_{K^I}(A^I)/Z_{K^I}(A^I)$ the Weyl group of G^I,
- N the connected subgroup with Lie algebra $\bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$.
- N_I the connected subgroup of N with Lie algebra $\bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma^I} \mathfrak{g}_\alpha$, where Σ^I is the root subsystem spanned by I.

Definition 4.5 Two subsets I, J of Δ are said to be orthogonal if, for every $\alpha \in I$ and $\beta \in J$, the roots α and β are orthogonal. A subset $I \subseteq \Delta$ is called irreducible if it is not a disjoint union of two proper orthogonal subsets.

Lemma 4.6 *Fix a subset I of Δ, and consider a linear subspace V of $\mathfrak{a}^I$ which is invariant under the action of W^I. Then there exists $J \subseteq I$ such that $V = \mathfrak{a}^J$, and J and $I \backslash J$ are orthogonal.*

Proof. Let $I = J_1 \sqcup J_2 \sqcup \dots J_r$ be the decomposition of I into irreducible subsets. The linear representation of W^I on $\mathfrak{a}^I$ decomposes as the direct sum of the irreducible representations $\mathfrak{a}^I = \bigoplus_{j=1}^{r} \mathfrak{a}^{J_j}$. Since V is a W^I-invariant subspace, there exists $R \subseteq \{1, 2, \dots, r\}$ such that $V = \bigoplus_{j \in R} \mathfrak{a}^{J_j}$. As a consequence, we have $V = \mathfrak{a}^J$, where $J = \bigsqcup_{j \in R} J_j$. □

Lemma 4.7 *Let C be a non-discrete subset of A^I. Let $J' \subseteq I$ denote the smallest subset such that*

i) $\forall c \in C, C \subseteq cA^{J'}$ *and*
ii) *the roots in* J' *and in* $I \backslash J'$ *are orthogonal.*

Then the smallest closed subgroup of $W^I A$ *containing all conjugates* $\{cW^I c^{-1}, c \in C\}$ *is equal to* $W^I A^{J'}$.

Proof. In this proof, we will identify A with its Lie algebra, and thus consider A as a vector space. Up to conjugating, we can assume that the affine subspace of A spanned by C contains 0. Let $\Gamma \subseteq W^I A$ denote the smallest closed subgroup containing all conjugates $\{cW^I c^{-1}, c \in C\}$. Since C is non-discrete, Γ is not discrete and the linear part of Γ is equal to W^I. So the identity component Γ_0 of Γ is a vector subspace of A^I containing C. Since Γ_0 is invariant under W^I, so according to Lemma 4.6, we deduce that $\Gamma_0 = A^{J'}$, for some $J' \subseteq I$ such that J' and $I \backslash J'$ are orthogonal. □

4.3.1. *Generalized horocyclic decompositions.* We will make use of the generalized Iwasawa decompositions of G, respectively the generalized horocyclic decompositions of X.

Lemma 4.8 *For every* $I \subseteq \Delta$ *and* $a^I \in A^I$, *we have the following decomposition:*

$$X = a^I K^I {a^I}^{-1} N_I A \cdot p_0,$$

where the A *component is unique up to the following condition: for every* $a, a' \in A$, *we have* $a^I K^I {a^I}^{-1} N_I a \cdot p_0 = a^I K^I {a^I}^{-1} N_I a' \cdot p_0$ *if and only if* $(a^I)^{-1} a$ *and* $(a^I)^{-1} a'$ *are conjugated by some element in* W^I. *The classical Iwasawa and horocyclic decompositions* $G = NAK$ *resp.* $X = NA \cdot p_0$ *correspond to* $I = \emptyset$.

Proof. Up to translating by ${a^I}^{-1}$, we can assume for simplicity that $a^I = e$. According to [GJT98, Corollary 2.16], we have the following generalized horocyclic decomposition: $X = A_I N_I X^I$, where X^I is the relative symmetric space $X^I = G^I / K^I$ identified as the orbit $X^I = G^I \cdot p_0$ of p_0 in X. Furthermore, in this decomposition $X = A_I N_I X^I = A_I N_I G^I \cdot p_0$, the components in A_I, N_I and $X^I \simeq G^I \cdot p_0$ are unique.

The group K^I is a maximal compact subgroup of the semisimple group G^I, and A^I is a Cartan subgroup of G^I, so we can consider the Cartan decomposition of G^I as $G^I = K^I A^I K^I$, where the component in A^I is unique up to conjugation by some element in W^I.

Fix some point $p \in X$. According to the two previous decompositions, we have $p = b_I u_I k^I b^I \cdot p_0$, where $b_I \in A_I$, $u_I \in N_I$, $k^I \in K^I$ and $b^I \in A^I$, and furthermore b_I and u_I are unique and b^I is unique up to conjugation by some element in W^I. Since A_I commutes with K^I, we also have $p = (b_I u_I b_I^{-1}) k^I b_I b^I \cdot p_0$. Furthermore, since A^I and $K^I M$ normalize N_I, we have $(b_I u_I b_I^{-1}) k^I \in K^I N_I$.

As a consequence, $p \in K^I N_I b_I b^I \cdot p_0$, where $b_I b^I \in A$ is unique up to conjugation by some element in W^I (notice that W^I commutes with $b_I \in A_I$). □

4.3.2. *Types of sequences and horofunctions.* We have seen in Lemma 4.1 that each function $\psi_{g\cdot p_0}$ is invariant under the conjuagte gKg^{-1} of the maximal compact subgroup K. In order to study the invariance properties of horofunctions, we will use the study of limits of conjugates of K (see [GJT98, Chapter IX]). In order to describe such limits, we need to introduce the notion of type of a diverging sequence of elements in A. Roughly speaking, the type of a sequence encodes the roots "along which" the sequence goes to infinity.

Definition 4.9 A sequence $(a_n)_{n\in\mathbb{N}}$ in $\overline{A^+}$ is said to be of type (I, a^I), where I is a proper subset of Δ and $a^I \in A^I$, if

i) for $\alpha \in I$, $\lim_{n\to\infty} \alpha(\log a_n)$ exists and is equal to $\alpha(\log a^I)$,
ii) for $\alpha \in \Delta\backslash I$ there holds $\alpha(\log a_n) \to +\infty$.

The main result on limits of conjugates of K is the following.

Proposition 4.10 [GJT98, Proposition 9.14] *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{A^+}$ of type (I, a^I). In the space of closed subgroups of G, endowed with the Chabauty topology, the sequence $(a_nKa_n{}^{-1})_{n\in\mathbb{N}}$ converges to $a^IK^IM(a^I)^{-1}N_I$.*

Remark 4.11 Since the groups $a^IK^IMa^{I^{-1}}N_I$ arise as limits of the maximal compact subgroups under conjugations by sequences of type I in A, the (generalized) Iwasawa decompositions can thus be seen as limits of the Cartan decomposition.

We will now use this result to deduce some invariance for horofunctions.

Lemma 4.12 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{A^+}$ of type (I, a^I) such that $(\psi^X_{a_n\cdot p_0})_{n\in\mathbb{N}}$ converges to ξ. Then ξ is $a^IK^IM(a^I)^{-1}N_I$-invariant.*

Proof. For each $n \in \mathbb{N}$, the function $\psi^X_{a_n\cdot p_0}$ is invariant under $a_nKa_n^{-1}$. Since the sequence $(a_nKa_n^{-1})_{n\in\mathbb{N}}$ converges to $a^IK^IM(a^I)^{-1}N_I$ in the Chabauty topology (see Proposition 4.10), for every $g \in a^IK^IM(a^I)^{-1}N_I$ there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in K such that the sequence $(a_nk_na_n^{-1})_{n\in\mathbb{N}}$ converges to g. Therefore, for every $p \in X$ we have

$$\begin{aligned}\xi(g\cdot p) - \xi(p) &= \lim_{n\to+\infty} d(a_n\cdot p_0, g\cdot p) - d(a_n\cdot p_0, p)\\ &= \lim_{n\to+\infty} d(a_n\cdot p_0, a_nk_na_n^{-1}\cdot p) - d(a_n\cdot p_0, p) = 0.\end{aligned}$$

As a consequence, ξ is invariant under $a^IK^IM(a^I)^{-1}N_I$. □

Definition 4.13 A horofunction $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ is said to be of type (I, a^I), where I is a proper subset of Δ and $a^I \in A^I$, if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A of type (I, a^I) such that the sequence $(\psi^F_{a_n\cdot p_0})_{n\in\mathbb{N}}$ converges to η in $\widetilde{C}(F)$. Note that a horofunction may have several types, but has at least one type.

Lemma 4.14 *Let $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ be a horofunction which has two types (I, a^I) and (J, b^J), with $I, J \subseteq \Delta$ and $a^I \in A^I, b^J, \in A^J$. Then η also has type $(I \cap J, c^{I\cap J})$ for some $c^{I\cap J} \in A^{I\cap J}$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences in A of different types (I, a^I) and (J, b^J) respectively, such that the sequences $(\psi^F_{a_n\cdot p_0})_{n\in\mathbb{N}}$ and $(\psi^F_{b_n\cdot p_0})_{n\in\mathbb{N}}$ both converge to η. For every $n \in \mathbb{N}$, we define $c_n = \exp(\frac{1}{2}\log(a_n) + \frac{1}{2}\log(b_n))$. The sequence $(c_n)_{n\in\mathbb{N}}$ has type $(I\cap J, c^{I\cap J})$, where $c^{I\cap J} \in \Big(\exp(\frac{1}{2}\log(a^I) + \frac{1}{2}\log(b^J))A_{I\cap J}\Big)\cap A^{I\cap J}$. According to Lemma 3.15, the sequence $(\psi^A_{c_n\cdot p_0})_{n\in\mathbb{N}}$ also converges to η. As a consequence, η has type $(I\cap J, c^{I\cap J})$. ☐

Lemma 4.15 *Let* $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ *be a horofunction of type* (I, a^I)*, where* $I \subsetneq \Delta$ *and* $a^I \in A^I$. *If* η *is invariant under* $A^{J'}$ *with* $J' \subseteq I$*, then* η *has type* $(I\backslash J', c^{I\backslash J'})$ *for some* $c^{I\backslash J'} \in A^{I\backslash J'}$.

Proof. Fix $c \in A^{J'+}$. For each $k \in \mathbb{N}$, the sequence $(\psi^F_{c^k a_n\cdot p_0})_{n\in\mathbb{N}}$ converges to $c^k\cdot\eta = \eta$, since η is invariant under $A^{J'}$. As a consequence, there exists $n_k \in \mathbb{N}$ such that, for every $n \geq n_k$, and for every $a \in A$ such that $d(p_0, a\cdot p_0) \leq k$, we have

$$|d(a\cdot p_0, c^k a_n\cdot p_0) - d(a\cdot p_0, a_n\cdot p_0)| \leq \frac{1}{k+1}.$$

We can furthermore assume that the sequence $(n_k)_{k\in\mathbb{N}}$ is increasing. Fix $a \in A$. For every $k \geq d(p_0, a\cdot p_0)$, we have $|d(a\cdot p_0, c^k a_{n_k}\cdot p_0) - d(a\cdot p_0, a_{n_k}\cdot p_0)| \leq \frac{1}{k+1}$ and $|d(p_0, c^k a_{n_k}\cdot p_0) - d(p_0, a_{n_k}\cdot p_0)| \leq \frac{1}{k+1}$. Therefore, we have

$$\begin{aligned}
&\lim_{k\to+\infty} d(a\cdot p_0, c^k a_{n_k}\cdot p_0) - d(p_0, c^k a_{n_k}\cdot p_0)\\
&= \lim_{k\to+\infty} d(a\cdot p_0, a_{n_k}\cdot p_0) - d(p_0, a_{n_k}\cdot p_0)\\
&= \eta(a\cdot p_0) - \eta(p_0).
\end{aligned}$$

As a consequence, the sequence $(\psi^F_{c^k a_{n_k}\cdot p_0})_{k\in\mathbb{N}}$ converges to η.

To conclude, observe that the sequence $(c^k a_{n_k})_{k\in\mathbb{N}}$ has type $(I\backslash J', c^{I\backslash J'})$, for some $c^{I\backslash J'} \in A^{I\backslash J'}$. ☐

4.4. **The intrinisc compactification versus the closure of a flat.** In this section we define an explicit map from the intrinsic compactification of the flat F into the horofunction compactification of X. For this we use the invariance shown in Lemma 4.12 and the generalized horocyclic decomposition $X = a^I K^I a^{I^{-1}} N_I A\cdot p_0$ in Lemma 4.8.

Theorem 4.16 *The following map*

$$\begin{aligned}
\phi : \overline{\psi^F(F^+)}^{\widetilde{C}(F)} &\longrightarrow \overline{\psi^X(X)}^{\widetilde{C}(X)}\\
\psi^F_z,\ \text{where } z \in F^+ &\longmapsto \psi^X_z\\
\eta \text{ of type } (I, a^I) &\longmapsto \Big(a^I k^I a^{I^{-1}} u_I a\cdot p_0 \in X \mapsto \eta(a\cdot p_0)\Big)
\end{aligned}$$

is a well-defined, continuous embedding.

Proof. The fact that ϕ is well-defined will be proved in Section 4.4.1, and the fact that ϕ is continuous will be proved in Section 4.4.2. Since the restiction to F^+ is a left inverse to ϕ, we deduce that ϕ is injective. Since $\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ is compact, ϕ is then an embedding. □

4.4.1. *Well-definedness.* We want to prove that the map ϕ in Theorem 4.16 is well-defined.

Consider first a horofunction $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ which has some type (I, a^I), and consider two decompositions $a^I k^I a^{I^{-1}} u_I a \cdot p_0 = a^I k'^I a^{I^{-1}} u'_I a' \cdot p_0$ of the same point in X. According to Lemma 4.8, there exists $w \in W^I$ such that $(a^I)^{-1} a' = w(a^I)^{-1} a w^{-1}$. According to Lemma 4.12, η is invariant under $a^I W^I (a^I)^{-1}$, so

$$\eta(a' \cdot p_0) = \eta(a^I w(a^I)^{-1} a w^{-1} \cdot p_0) = \eta((a^I w(a^I)^{-1}) a \cdot p_0) = \eta(a \cdot p_0).$$

This means that the formula defining ϕ does not depend on the choice of the A component in the decomposition $X = a^I K^I a^{I^{-1}} N_I A \cdot p_0$.

Consider now a horofunction $\eta \in \partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$, which has two types (I, a^I) and (J, b^J). We will prove that the two formulas defining $\phi(\eta)$, for each type, agree. Let the notations be as in Lemma 4.14. Up to passing to a subsequence, we may assume that the sequences $(\psi^X_{a_n \cdot p_0})_{n\in\mathbb{N}}$ and $(\psi^X_{b_n \cdot p_0})_{n\in\mathbb{N}}$ converge to ξ and ξ' respectively. We precisely need to prove that $\xi = \xi'$, which will be done by induction on $|I| + |J|$. By Lemma 4.14 assume from now on that $J \subseteq I$.

Assume first that $|I| + |J| = 0$, so $I = J = \emptyset$. According to Lemma 4.12, ξ and ξ' are both N-invariant, so for every $p = ua \cdot p_0 \in X = NA \cdot p_0$, we have $\xi(p) = \eta(a \cdot p_0) = \xi'(p)$. So $\xi = \xi'$.

By induction, fix $m \in \mathbb{N}$ and assume that if $|I| + |J| \leq m$, then $\xi = \xi'$. Consider now I, J such that $|I| + |J| = m + 1$. We will distinguish the two cases $J = I$ and $J \subsetneq I$.

<u>The case $J = I$</u>. Assume that $J = I$. We will first show that η has extra invariance and then define a subset $J' \subseteq I$ to show that η has also a type smaller than I. The result will then follow by two inductions.

Lemma 4.17 *Assume that $J = I$. Then there exists $J' \subseteq I$ such that :*

i) $a^I \in b^I A^{J'}$,
ii) the roots in J' and $I \backslash J'$ are orthogonal, and
iii) η is $W^I A^{J'}$-invariant.

Proof. For simplicity, up to translating by $(a^I)^{-1}$, we may assume that $a^I = 1$.

Fix $\lambda \in [0, 1]$. For each $n \in \mathbb{N}$, let $c_n = \exp((1-\lambda)\log a_n + \lambda \log b_n) \in A$. According to Lemma 3.15, the sequence $(\psi^F_{c_n \cdot p_0})_{n\in\mathbb{N}}$ converges to η. The sequence $(c_n)_{n\in\mathbb{N}}$ is of type $(I, (b^I)^\lambda)$, where $(b^I)^\lambda$ denotes $\exp(\lambda \log b^I)$. Since the sequence $(\psi^F_{c_n \cdot p_0})_{n\in\mathbb{N}}$ converges to η, by Lemma 4.12 we deduce that η is $(b^I)^\lambda W^I ((b^I)^\lambda)^{-1}$-invariant, for every $\lambda \in [0, 1]$.

According to Lemma 4.7, we deduce that η is invariant under $W^I A^{J'}$, where $J' \subseteq I$ is the smallest subset such that $b^I \in A^{J'}$ and such that the roots in J' and in $I \backslash J'$ are orthogonal. $\square$

According to Lemma 4.15, we deduce that η has also type $(I\backslash J', c^{I\backslash J'})$, for some $c^{I\backslash J'} \in A^{I\backslash J'}$. Let $(c_n)_{n\in\mathbb{N}}$ denote a sequence of type $(I\backslash J', c^{I\backslash J'})$ such that the sequence $(\psi^F_{c_n \cdot p_0})_{n\in\mathbb{N}}$ converges to η. Up to passing to a subsequence, assume that the sequence $(\psi^X_{c_n \cdot p_0})_{n\in\mathbb{N}}$ converges to some ξ''.

Since $a^I \in b^I A^{J'}$ and $a^I \neq b^I$, we know that $J' \neq \emptyset$. Therefore we have $|I| + |I\backslash J'| < |I| + |I|$ so $|I| + |I\backslash J'| \leq m$. By induction applied to the sequences $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$, we deduce that $\xi = \xi''$. By induction applied to the sequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$, we deduce that $\xi' = \xi''$. In conclusion, we have $\xi = \xi'$. This concludes the induction, and finishes the proof that $\xi = \xi'$ in the case where $J = I$.

The case $J \subsetneq I$. Assume that $J \subsetneq I$. Similarly to the case before, we will first show an extra invariance of η and that it has a smaller type with respect to a new subset $J' \subseteq I$. To conclude the result by induction, we have to distinguish again two cases depending on whether $I \setminus J' = J$ or not.

Lemma 4.18 *There exists $J' \subseteq I$ such that :*

i) $J \cup J' = I$,
ii) the roots in J' and $I\backslash J'$ are orthogonal, and
iii) η is $W^I A^{J'}$-invariant.

Proof. Let a_n, b_n be the sequences of type I and J converging to η. For simplicity, up to translating by $(a^I)^{-1}$, we may assume that $a^I = 1$. Up to passing to a subsequence, let us partition $I\backslash J$ into $I\backslash J = I_1 \sqcup I_2 \sqcup \dots I_p$ such that:

- $\forall 1 \leq i \leq p, \forall \alpha, \beta \in I_i, \lim_{n\to+\infty} \frac{\alpha(\log b_n)}{\beta(\log b_n)} \in (0, +\infty)$,
- $\forall 1 \leq i < j \leq p, \forall \alpha \in I_i, \forall \beta \in I_j, \lim_{n\to+\infty} \frac{\alpha(\log b_n)}{\beta(\log b_n)} = 0$.

Fix $1 \leq i \leq p$, and for some $\alpha \in I_i$ define $t_n := \frac{1}{\alpha(\log b_n)}$ such that $t_n \longrightarrow 0$ as $n \to +\infty$. Fix $\lambda > 0$. For each $n \in \mathbb{N}$, let $c_n = \exp((1 - \lambda t_n) \log a_n + \lambda t_n \log b_n) \in A$. According to Lemma 3.15, the sequence $(\psi^F_{c_n \cdot p_0})_{n\in\mathbb{N}}$ converges to η. Let us define

$$c^{I_i} := \lim_{n\to+\infty} \left(\pi^{I_i}(b_n)\right)^{t_n} \in A^{I_i},$$

where $\pi^{I_i}(b_n)$ denotes the orthogonal projection of b_n onto A^{I_i}. Note that this sequence converges: for any $\beta \in I_i$, we have

$$\beta\left(\log\left(\pi^{I_i}(b_n)\right)^{t_n}\right) = t_n \beta(\log b_n) = \frac{\beta(\log b_n)}{\alpha(\log b_n)},$$

so $\lim_{n\to+\infty} \beta\left(\log\left(\pi^{I_i}(b_n)\right)^{t_n}\right) \in (0, +\infty)$. On the other hand, for any $\beta \in \Delta\backslash I_i$, we have $\beta\left(\log\left(\pi^{I_i}(b_n)\right)^{t_n}\right) = 0$, so the limit $c^{I_i} \in A^{I_i}$ exists.

Furthermore, we have $c^{I_i} \in (A^{I_i})^+$. Let

$$J_i := J \sqcup I_1 \sqcup \cdots \sqcup I_i.$$

For every $\alpha \in \Delta \backslash J_i$, we have

$$\alpha(\log c_n) = (1 - \lambda t_n)\alpha(\log a_n) + \lambda t_n \alpha(\log b_n) \longrightarrow +\infty.$$

For every $\alpha \in J \cup I_1 \cup \dots I_{i-1}$, we have

$$\alpha(\log c_n) = (1 - \lambda t_n)\alpha(\log a_n) + \lambda t_n \alpha(\log b_n) \longrightarrow \alpha(\log a^I) = 0.$$

For every $\alpha \in I_i$, we have

$$\alpha(\log c_n) = (1 - \lambda t_n)\alpha(\log a_n) + \lambda t_n \alpha(\log b_n)$$
$$\longrightarrow \alpha(\log a^I) + \lambda\alpha(\log c^{I_i}) = \lambda\alpha(\log c^{I_i}).$$

As a consequence, the sequence $(c_n)_{n\in\mathbb{N}}$ is of type $(J_i, (c^{I_i})^\lambda)$, where $(c^{I_i})^\lambda$ denotes $\exp(\lambda \log c^{I_i})$. Since the sequence $(\psi^F_{c_n \cdot p_0})_{n\in\mathbb{N}}$ converges to η, by Lemma 4.12 we deduce that η is $(c^{I_i})^\lambda W^{I_i}((c^{I_i})^\lambda)^{-1}$-invariant.

As $c^{I_i} \in (A^{I_i})^+$, we deduce by Lemma 4.7 that η is invariant under A^{I_i}. Because this is true for every $1 \leq i \leq p$, we conclude that η is invariant under $A^{I \backslash J}$.

Since the sequence $(a_n)_{n\in\mathbb{N}}$ is of type $(I, 1)$, and the sequence $(\psi^F_{a_n \cdot p_0})_{n\in\mathbb{N}}$ converges to η, we know by Lemma 4.12 that η is W^I-invariant. In conclusion, η is invariant under W^I and $A^{I\backslash J}$. The smallest closed subgroup of $W^I A^I$ containing both W^I and $A^{I\backslash J}$ is $W^I A^{J'}$, where $J' \subseteq I$ is the smallest subset containing $I\backslash J$ such that the roots in J' and in $I\backslash J'$ are orthogonal. Therefore η is invariant under $W^I A^{J'}$. □

Since η is invariant under $A^{J'}$, we deduce according to Lemma 4.15 that η has type $(I\backslash J', c^{I\backslash J'})$, for some $c^{I\backslash J'} \in A^{I\backslash J'}$.

If $I\backslash J' \subsetneq J$, then $|I| + |I\backslash J'| < |I| + |J|$ and $|J| + |I\backslash J'| < |I| + |J|$, so by applying the induction twice, we know that $\xi = \xi'$.

So we are left with the case $I\backslash J' = J$. In this case J and $I\backslash J$ are orthogonal.

Lemma 4.19 *If η is $A^{I\backslash J}$-invariant and J and $I\backslash J$ are orthogonal, then $\xi = \xi'$.*

Proof. As J and $J' = I\backslash J$ are orthogonal, we have the orthogonal decomposition $A^I = A^J A^{J'}$. Let us decompose $a^I = a^J a^{J'} \in A^J A^{J'}$. Up to translating by $(b^J a^{J'})^{-1}$, we can assume that $b^J = 1$ and $a^I = a^{J'} \in A^{J'}$.

As Σ^J and $\Sigma^{J'}$ are orthogonal, we have the decomposition $K^I = K^J K^{J'}$, with K^J and $K^{J'}$ commuting. Furthermore K^J and A_J are commuting. Since $A^{J'} \subseteq A_J$, we deduce that $a^{J'}$ commutes with K^J. In particular,

$$a^{J'} K^I (a^{J'})^{-1} = K^J a^{J'} K^{J'} (a^{J'})^{-1}.$$

Fix any point $p \in X$, we will show that $\xi'(p) = \xi(p)$. Consider the decomposition $X = a^{J'} K^I (a^{J'})^{-1} N_I A \cdot p_0 = K^J N_I a^{J'} K^{J'} (a^{J'})^{-1} A \cdot p_0$, and write

$$p = k^J u_I a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0 \in X,$$

where $k^J \in K^J$, $u_I \in N_I$, $k^{J'} \in K^{J'}$ and $c \in A$. According to Lemma 4.12, ξ' is invariant under $K^J M N_J$. Since $N_I \subseteq N_J$, we deduce that

$$\xi'(p) = \xi'(a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0).$$

In the decomposition $A = A_{J'} A^{J'}$, let us write $c = c_{J'} c^{J'}$. So

$$a^{J'} k^{J'} (a^{J'})^{-1} c = c_{J'} a^{J'} k^{J'} (a^{J'})^{-1} c^{J'} \in c_{J'} G^{J'}.$$

According to the Iwasawa decomposition $G^{J'} = N^{J'} A^{J'} K^{J'}$, there exists $u^{J'} \in N^{J'}$ and $d^{J'} \in A^{J'}$ such that $u^{J'} a^{J'} k^{J'} (a^{J'})^{-1} c^{J'} \in d^{J'} K^{J'}$. As a consequence,

$$u^{J'} \cdot (a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) = c_{J'} d^{J'} \cdot p_0.$$

We claim that

$$\xi'(p) = \xi'(c_{J'} d^{J'} \cdot p_0).$$

Showing this is equivalent to showing that $\xi'(a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) = \xi'(u^{J'} a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0)$. Since the sequence $(b_n K b_n^{-1})_{n \in \mathbb{N}}$ converges to $K^J M N_J$ in the Chabauty topology (see Proposition 4.10), and as $u^{J'} \in N^{J'} \subseteq N_J$, there exists a sequence $(k_n)_{n \in \mathbb{N}}$ such that the sequence $(b_n k_n {b_n}^{-1})_{n \in \mathbb{N}}$ converge to $u^{J'}$. As a consequence,

$$\begin{aligned}
\xi'(u^{J'} a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) - \xi'(a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) &= \\
\lim_{n \to +\infty} d(b_n \cdot p_0, u^{J'} a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) - d(b_n \cdot p_0, a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) &= \\
\lim_{n \to +\infty} d(b_n \cdot p_0, b_n k_n b_n^{-1} a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) - d(b_n \cdot p_0, a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) &= 0.
\end{aligned}$$

Hence $\xi'(u^{J'} a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) = \xi'(a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0)$, so

$$\xi'(p) = \xi'(c_{J'} d^{J'} \cdot p_0).$$

By assumption, η is invariant under $A^{I \setminus J} = A^{J'}$. As a consequence, we have

$$\xi'(p) = \xi'(c_{J'} d^{J'} \cdot p_0) = \eta(c_{J'} d^{J'} \cdot p_0) = \eta(c_{J'} \cdot p_0).$$

On the other hand, according to Lemma 4.12, we have

$$\xi(p) = \xi(k^J u_I a^{J'} k^{J'} (a^{J'})^{-1} c \cdot p_0) = \xi(c \cdot p_0) = \eta(c \cdot p_0).$$

Since $c = c_{J'} c^{J'}$ and η is invariant under $A^{J'}$, we conclude that $\xi(p) = \eta(c_{J'} \cdot p_0)$. Therefore, $\xi'(p) = \xi(p)$. So $\xi = \xi'$. □

This conludes the proof by induction that $\xi = \xi'$. So we have proven that the map ϕ in Theorem 4.16 is well-defined.

4.4.2. *Continuity.* We want to prove that the map ϕ in Theorem 4.16 is continuous. It is clear that ϕ is continuous on the interior $\psi^F(F^+)$. Fix $\eta \in \partial \overline{\psi^F(F^+)}^{\widetilde{C}(F)}$, we will show that ϕ is continuous at η.

Lemma 4.20 *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\overline{A^+}$ such that the sequence $(\psi^F_{a_n \cdot p_0})_{n \in \mathbb{N}}$ converges to η. Then $(\psi^X_{a_n \cdot p_0})_{n \in \mathbb{N}}$ converges to $\phi(\eta)$.*

Proof. Up to passing to a subsequence, we may assume that the sequence $(a_n)_{n\in\mathbb{N}}$ has some type (I, a^I) and that the sequence $(\psi^X_{a_n\cdot p_0})_{n\in\mathbb{N}}$ converges to some ξ. According to Lemma 4.12, ξ is invariant under $a^I K^I M (a^I)^{-1} N_I$, so for every $p = a^I k^I (a^I)^{-1} u_I a \cdot p_0 \in X = a^I K^I (a^I)^{-1} N_I A \cdot p_0$, we have $\xi(p) = \xi(a \cdot p_0) = \eta(a \cdot p_0)$.

Furthermore, since ϕ is well-defined and η has type (I, a^I), we can use this type in the definition of $\phi(\eta)$, and thus $\phi(\eta)(p) = \eta(a \cdot p_0) = \xi(p)$. In conclusion, $\xi = \phi(\eta)$, so $(\psi^X_{a_n\cdot p_0})_{n\in\mathbb{N}}$ converges to $\phi(\eta)$ in $C(X)$. □

Lemma 4.21 *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\partial\overline{\psi^F(F^+)}^{\widetilde{C}(F)}$ converging to η in $\widetilde{C}(F)$. Then $(\phi(\eta_n))_{n\in\mathbb{N}}$ converges to $\phi(\eta)$ in $\widetilde{C}(X)$.*

Proof. Up to passing to a subsequence, we may assume that the sequence $(\phi(\eta_n))_{n\in\mathbb{N}}$ converges to some horofunction ξ in $\widetilde{C}(X)$. Up to passing again to a subsequence, we may assume that there exists $I \subsetneq \Delta$ such that for each $n \in \mathbb{N}$, η_n is of type (I, a^I_n) for some $a^I_n \in A^I$. For each $n \in \mathbb{N}$, consider a sequence $(a_{n,m})_{m\in\mathbb{N}}$ of type (I, a^I_n) converging to η_n. Up to passing to a subsequence, we may assume that the sequence $(a^I_n)_{n\in\mathbb{N}}$ is of type (J, a^J) for some $J \subseteq I$ and some $a^J \in A^J$. For each $n \in \mathbb{N}$, one can find some $m_n \in \mathbb{N}$ such that the sequence $(a_{n,m_n})_{n\in\mathbb{N}}$ is of type (J, a^J) and converges to η.

Fix
$$p = a^J k^J {a^J}^{-1} u_J c \cdot p_0 \in X = a^J K^J {a^J}^{-1} N_J A \cdot p_0.$$
Since the sequence $(a^I_n K^I M {a^I_n}^{-1} N_I)_{n\in\mathbb{N}}$ converges to $a^J K^J M {a^J}^{-1} N_J$ in the Chabauty topology (see Proposition 4.10), there exist sequences $(k^I_n)_{n\in\mathbb{N}}$ in $K^I M$ and $(u_{n,I})_{n\in\mathbb{N}}$ in N_I such that the sequence $(a^I_n k^I_n {a^I_n}^{-1} u_{n,I})_{n\in\mathbb{N}}$ converges to $a^J k^J {a^J}^{-1} u_J$. Hence
$$\begin{aligned}\xi(p) &= \lim_{n\to+\infty} \phi(\eta_n)(a^I_n k^I_n {a^I_n}^{-1} u_{n,I} c \cdot p_0)\\ &= \lim_{n\to+\infty} \eta_n(c \cdot p_0)\\ &= \eta(c \cdot p_0)\\ &= \phi(\eta)(a^J k^J {a^J}^{-1} u_J c \cdot p_0) = \phi(\eta)(p).\end{aligned}$$

As a consequence, we have $\xi = \phi(\eta)$, so the sequence $(\phi(\eta_n))_{n\in\mathbb{N}}$ converges to $\phi(\eta)$. □

So we have proved that the map ϕ in Theorem 4.16 is continuous. This concludes the proof of Theorem 4.16.

5. Realizing classical compactifications of symmetric spaces

In this section we prove that all Satake and generalized Satake compactifications can be realized as horofunction compactifications of polyhedral G-invariant Finsler metrics on X.

5.1. Generalized Satake compactifications.

Generalized Satake compactifications. We first recall the construction of generalized Satake compactifications. Let $X = G/K$ be a symmetric space of non-compact type.

Consider the space

$$\mathcal{P}_n := {}^{\mathrm{PSL}(n,\mathbb{C})}\Big/_{\mathrm{PSU}(n)}$$

and identify it via the map $m\,\mathrm{PSU}(n) \longmapsto mm^*$ with the space of positive definite Hermitian matrices. Here m^* denotes the conjugate transpose of $m \in \mathrm{PSL}(n,\mathbb{C})$. Let $\mathcal{H}_n$ be the real vector space of Hermitian matrices and $\mathrm{P}(\mathcal{H}_n)$ the corresponding compact projective space. For $A \in \mathcal{H}_n$ we denote the corresponding equivalence class in $\mathrm{P}(\mathcal{H}_n)$ by $[A]$. As $\mathcal{P}_n \subseteq \mathcal{H}_n$ the map $A \longmapsto [A]$ is a $\mathrm{PSL}(n,\mathbb{C})$-equivariant embedding and we define

$$\overline{\mathcal{P}_n}^S := \overline{i(\mathcal{P}_n)} \subseteq \mathrm{P}(\mathcal{H}_n)$$

to be the *Standard-Satake compactification.*

Now let $\tau : G \longrightarrow \mathrm{PSL}(n,\mathbb{C})$ be a faithful projective representation of G. Via the map

$$\begin{aligned} i_\tau : X = G/K &\longrightarrow \mathcal{P}_n \\ gK &\longmapsto \tau(g)\tau(g)^* \end{aligned} \tag{5}$$

we can embed X into $\mathcal{P}_n$ as totally geodesic submanifold. There is a 1-to-1-correspondence between such embeddings and faithful projective representations of G into $\mathrm{PSL}(n,\mathbb{C})$ with the additional condition $\tau(\vartheta(g)) = (\tau(g)^*)^{-1}$ for all $g \in G$, where ϑ denotes the Cartan involution on G. With this we define

$$\overline{X}^S_\tau := \overline{i_\tau(X)} \subseteq \overline{\mathcal{P}_n}^S$$

as the *generalized Satake compactification* of X with respect to the representation τ. By the action of G on $\mathcal{P}_n$, $g \cdot A = \tau(g)A\tau(g)^*$ for $g \in G$ and $A \in \mathcal{P}_n$, the first embedding i_τ is G-equivariant and therefore $\overline{X}^S_\tau$ is a G-compactification, that is, the G-action on X extends to a continuous action on $\overline{X}^S_\tau$. When τ is an irreducible representation, the compactification $\overline{X}^S_\tau$ is a classical Satake compactification, which has been introduced and described by Satake in [Sat60]. In general, when τ is reducible, the compactification $\overline{X}^S_\tau$ is a generalized Satake compactification as introduced and described in [GKW17].

Note that there are finitely many isomorphism classes of Satake compactifications, one associated to any proper subset $I \subseteq \Delta$, but infinitely many isomorphism classes of generalized Satake compactifications.

5.2. **The compactification of a flat in a generalized Satake compactification.** We now compare the generalized Satake compactification with the horofunction compactification of X with respect to an appropriate polyhedral G-invariant Finsler metric.

With the Cartan decomposition (see Lemma 2.1 on page 4) we can write $X = Ke^{\overline{\mathfrak{a}^+}}.p_0$, and since K is compact $\overline{X} = \overline{Ke^{\overline{\mathfrak{a}^+}}.p_0} = K.\overline{e^{\overline{\mathfrak{a}^+}}p_0}$. Thus it is sufficient to show that the we have an W-equivariant homeomorphism between the closures of $e^{\mathfrak{a}}.p_0$ in the horofunction compactification and the generalized Satake compactification respectively.

For the closure of the flat $e^{\mathfrak{a}}.p_0$ in the generalized Satake compactification we have the following:

Theorem 5.1 ([Ji97] **Prop.4.1,** [GKW17]) *Let $\tau : G \to \mathrm{PSL}(n,\mathbb{C})$ be a faithful projective representation. Let $\mu_1, \dots, \mu_k$ be the weights of τ. Then the closure of the flat $e^{\mathfrak{a}}.p_0$ in the generalized Satake compactification $\overline{X}^S_\tau$ is W-equivariantly isomorphic to* $\mathrm{conv}(2\mu_1, \dots, 2\mu_k) \subseteq \mathfrak{a}^*$.

Remark 5.2 Because of the symmetry of the weights with respect to the Weyl chambers, the convex hull of all weights is the same as the convex hull of the Weyl-group orbit of $\chi_1, \cdots, \chi_l$, where χ_i are the highest weights of the irreducible components τ_i of τ:

$$\mathrm{conv}(2\mu_1, \dots, 2\mu_k) = \mathrm{conv}\left(W(2\chi_1), \cdots, W(2\chi_l)\right).$$

Example 5.3 Let us look at an example. Take $X = \mathrm{SL}(3,\mathbb{C})/\,\mathrm{SU}(3)$ with the adjoint representation ad of $\mathfrak{g}$ which induces a representation on G. The weights of the adjoint representation are exactly the roots $\alpha_{ij} \in \mathfrak{a}^*$ with $1 \le i \neq j \le 3$, where

$$\alpha_{ij}(H) = h_i - h_j$$

for any diagonal matrix $H = \mathrm{diag}(h_1, h_2, h_3) \in \mathfrak{a}$. The highest weight with respect to the positive Weyl chamber

$$\mathfrak{a}^+ = \{\mathrm{diag}(h_1, \dots, h_3) \in \mathfrak{sl}(3,\mathbb{C}) \mid \sum_{i=1}^{3} h_i = 0\}$$

is α_{13}.

We identify $\mathfrak{a}$ with $\mathfrak{a}^*$ using the Killing form κ. Then the convex hull of the weights is shown in Figure 3.

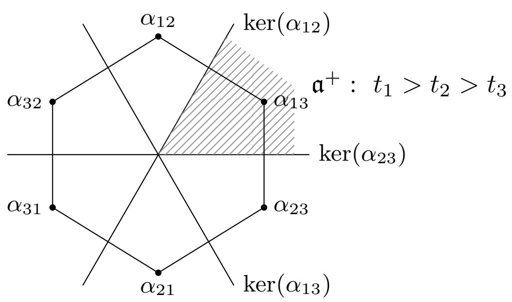

FIGURE 3. $\mathrm{conv}(\mu_1, \dots, \mu_6)$ for the representation $\tau = \mathrm{ad}$ in $\mathfrak{a}^*$.

By 5.1 we know that the closure of the flat in $\overline{X}^S_\tau$ is homeomorphic to $\mathrm{conv}(2\mu_1, \dots, 2\mu_k) \subseteq \mathfrak{a}^*$. If the highest weight is regular, like for $\tau = \mathrm{ad}$ in the example above, we obtain the maximal Satake compactification.
To get the two minimal Satake compactifications, the highest weight has to lie on a singular direction, see Figure 4 for a picture. The representations here are the standard and the dual standard representation.

If we now take the convex hull of these two triangles, we again obtain a hexagon but now with its vertices on the singular directions, see Figure 5. This compactification of the flat corresponds to a generalized Satake compactification associated to the direct sum of the standard and the dual standard representation. It is the same as the polyhedral compactification

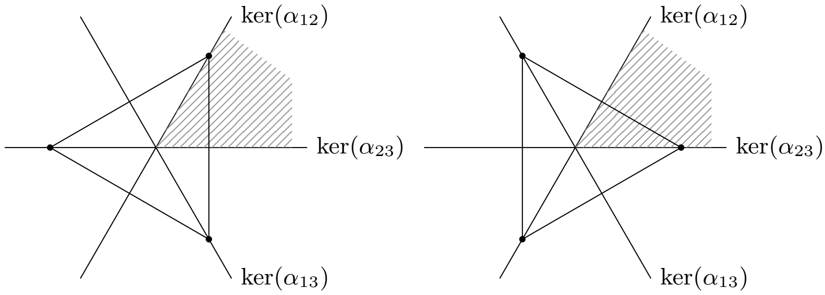

FIGURE 4. These two convex hulls correspond to the standard and the dual standard representations.

of the flat with respect to the polyhedral decomposition of $\mathfrak{a}$ with respect to the Weyl chambers.

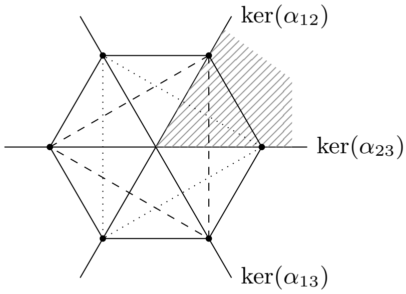

FIGURE 5. The convex hull of the two balls above give a hexagon with vertices on the singular directions.

Proposition 5.4 *Let $X = G/K$ be a symmetric space of non-compact type. Let τ be a faithful projective representation of G, and $\mu_1, \dots, \mu_n$ its weights. Let $D := \mathrm{conv}(2\mu_1, \dots, 2\mu_n) \subseteq \mathfrak{a}^*$. Let $B = -D^\circ$ the dual closed convex set in the maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$. Then the closure of the flat $e^{\mathfrak{a}}.p_0$ in the generalized Satake compactification is W-equivariantly isomorphic to the closure of the flat $e^{\mathfrak{a}}.p_0$ in the horofunction compactification of X with respect to the Finsler metric defined by B.*

Proof. By Theorem 4.16, it suffices to compare the closure of $e^{\mathfrak{a}^+}.p_0$ in the generalized Satake compactifications with the closure of $e^{\mathfrak{a}}.p_0$ in the flat compactification of $e^{\mathfrak{a}}.p_0$ with respect to the norm defined by B. By Theorem 5.1 and Theorem 4.4, both are W-equivariantly homeomorphic to the closed convex $\mathrm{conv}(2\mu_1, \dots, 2\mu_n) = D = -B^\circ$.

Note that in Theorem 5.1 and Theorem 4.4, the identification of the closure of $e^{\mathfrak{a}}.p_0$ with D relies on a moment map, and a direct comparison shows that a sequence $H_n \in \mathfrak{a}$ converges in the Satake compactification $\overline{X}^S_\tau$ if and only if it converges in the horofunction compactification $\overline{X}^{\mathrm{hor}}$ with respect to the G-invariant Finsler metric defined by B. □

Theorem 5.5 *Let $X = G/K$ be a symmetric space of non-compact type. Let τ be a faithful projective representation of G and $\mu_1, \dots, \mu_n$ its weights.*

Let $D := \mathrm{conv}(\mu_1, \dots, \mu_n)$. Let $B = -D^\circ$ define a unit ball in the maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$. Then the generalized Satake compactification $\overline{X}^S_\tau$ is G-equivariantly isomorphic to the horofunction compactification of X with respect to the Finsler metric defined by B.

Proof. We show that a sequence converges in the generalized Satake compactification $\overline{X}^S_\tau$ if and only if it converges in the horofunction compactification $\overline{X}^{\mathrm{hor}}$ with respect to the G-invariant Finsler metric defined by B. Let $x_n \in X$ be a sequence. Then we can write $x_n = k_n \cdot a_n p_0$, where $k_n \in K$ and $a_n \in \overline{A^+}$ is uniquely determined. Up to passing to a subsequence we can assume that x_n converges in $\overline{X}^S_\tau$ and that k_n converges to an element $k \in K$. Therefore Theorem 5.5 is a consequence of the Proposition 5.4. □

Remark 5.6 Note that Theorem 5.5 describes explicitly the convex unit ball of the Finsler metric which induces the horofunction compactification realizing the (generalized) Satake compactifications. For classical Satake compactifications the convex D (and hence also the unit ball B) has a particularly simple description as it is just the convex hull of the Weyl group orbit of the highest weight vector of τ. In order to obtain the Satake compactification determined by a subset $I \subseteq \Delta$ one has to choose a representation τ, whose highest weight vector has support equal to I.

Example 5.7 We consider $X = \mathrm{SL}(4, \mathbb{C}) / \mathrm{SU}(4)$ with the same notations as in Example 5.3 above. Let again $\tau = \mathrm{ad}$ be the considered representation. Then the highest weight with respect to the positive Weyl chamber $\mathfrak{a}^+ = \{\mathrm{diag}(t_1, \dots, t_4) \in \mathfrak{sl}(4, \mathbb{C}) \mid \sum_{i=1}^4 t_i = 0\}$ is $\mu_\tau = \alpha_{14}$. Let $M_\tau \in \mathfrak{a}$ be the element corresponding to μ_τ by identifying $\mathfrak{a}$ and $\mathfrak{a}^*$ with the Killing form. Note that as $\alpha_{23}(M_\tau) = 0$, M_τ lies on a Weyl chamber wall. The Weyl chamber system is shown in Figure 6. The picture on the left illustrates the structure of the Weyl chamber walls while the one on the right shows the positive Weyl chamber we chose.

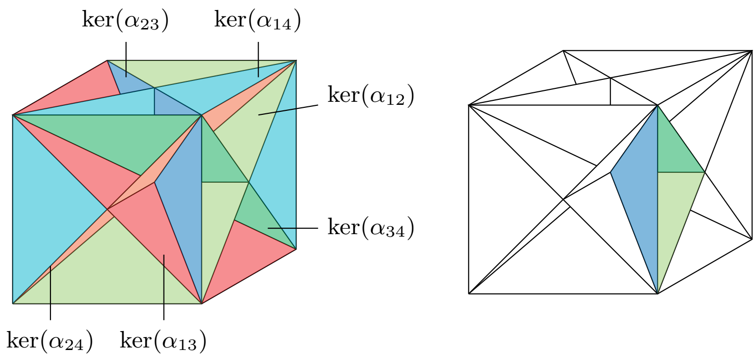

FIGURE 6. The Weyl chamber system of $\mathrm{SL}(4, \mathbb{R})$ with positive Weyl chamber and the Weyl chamber walls

The convex hull D of the weights is a regular polyhedral ball with 12 vertices and 14 maximal dimensional faces. Accordingly, the unit ball $B = -D^\circ$ has 14 vertices and 12 maximal dimensional faces, a picture

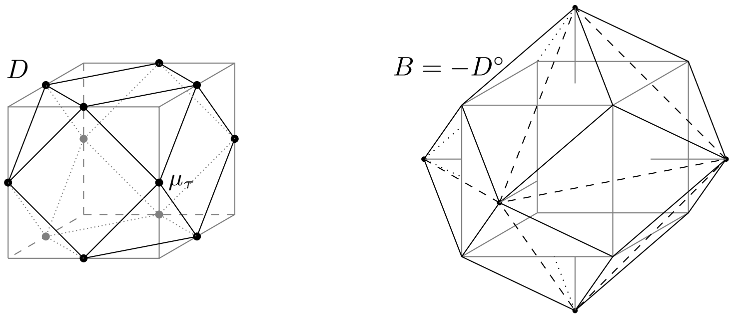

FIGURE 7. $D = \mathrm{conv}(W(\mu_\tau))$ and $B = -D^\circ$ for $\tau = \mathrm{ad}$.

of both is given in Figure 7. The dashed lines in the right picture are to indicate that always two triangles together form a rhomb. If we chose the Finsler metric corresponding to B as unit ball, we obtain a horofunction compactification of the flat which is isomorphic to the Satake compactification with respect to $\tau = \mathrm{ad}$.

Let us now consider other representations. If the representative M_τ of the highest weight lies completely inside of $\mathfrak{a}^+$ we get D and B as shown in Figure 8. The polyhedron $D = \mathrm{conv}(W(\mu_\tau))$ is then called the permutohedron of dimension 3. More generally, if τ is a faithful representation of $\mathrm{SL}(n-1, \mathbb{R})$ with regular M_τ, then the polyhedron $D = \mathrm{conv}(W(\mu_\tau))$ is the $(n-1)$-dimensional permutohedron.

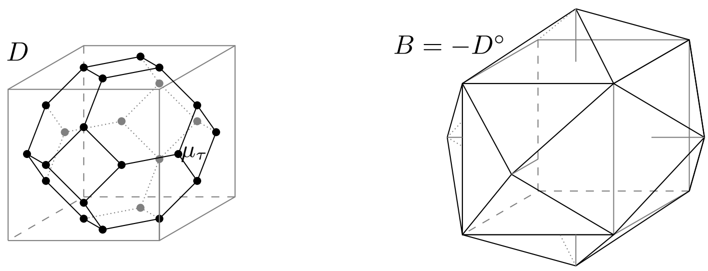

FIGURE 8. D and $B = -D^\circ$ for a representation with highest weight in a regular direction.

On the other hand if M_τ lies in more than one Weyl chamber wall, the convex hull D of the weights and the unit ball B of the Finsler norm are like shown in Figure 9 or a rotated version of it, depending on which pair of Weyl chamber walls M_τ lies.

5.3. **Dual generalized Satake compactifications.** The realization of generalized Satake compactifications as horofunction compactifications for polyhedral G-invariant Finsler metrics allows us to define the dual generalized Satake compactification $\overline{X}_\tau^{S^*}$.

Definition 5.8 Let $\tau : G \to \mathrm{PSL}(n, \mathbb{C})$ be a faithful projective representations and $\overline{X}_\tau^S$ the associated generalized Satake compactification. The dual generalized Satake compactification $\overline{X}_\tau^{S^*}$ is defined to be the horofunction

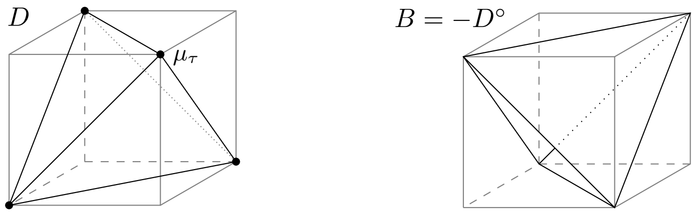

FIGURE 9. D and $B = -D^\circ$ for a representation with highest weight in a singular direction.

compactification of X with respect to the polyhedral G-invariant Finsler metric defined by the unit ball $B = D = \mathrm{conv}(\mu_1, \ldots, \mu_k)$.

Question 5.9 Is there a geometric way to interpret the duality between $\overline{X}_\tau^S$ and $\overline{X}_\tau^{S^*}$?

There are many polyhedral G-invariant Finsler metrics which are not related to generalized Satake compactifications, and even more G-invariant Finsler metrics which are not polyhedral. Since any Weyl group invariant convex set containing 0 defines a G-invariant Finsler metric, it is very natural to ask whether natural operations on convex sets extend to natural operations on the corresponding horofunction compactifications.

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