Hochschild cohomology of multi-extension zero algebras

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Abstract

The main objective of this paper is to present a theory for computing the Hochschild cohomology of algebras built on a specific data, namely multi-extension algebras. The computation relies on cohomological functors evaluated on the data, and on the combinatorics of an ad hoc quiver. One-point extensions are occurrences of this theory, and Happel's long exact sequence is a particular case of the long exact sequence of cohomology that we obtain via the study of trajectories of the quiver. We introduce cohomology along paths, and we compute it under suitable Tor vanishing hypotheses. The cup product on Hochschild cohomology enables us to describe the connecting homomorphism of the long exact sequence.

Multi-extension algebras built on the round trip quiver provide square matrix algebras which have two algebras on the diagonal and two bimodules on the corners. If the bimodules are projective, we show that a five-term exact sequences arises. If the bimodules are free of rank one, we provide a complete computation of the Hochschild cohomology. On the other hand, if the corner bimodules are projective without producing new cycles in the data, Hochschild cohomology is that of the product of the algebras on the diagonal for large enough degrees.

The word *algebra* always means an associative k-algebra over a field k; an algebra is not necessarily finite dimensional.

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1 Introduction

Hochschild cohomology of an algebra over a field is an interesting and not fully understood tool, see for instance [2]. It has been defined in 1945 by Hochschild in [18], it provides the theory of infinitesimal deformations and the deformation theory on the variety of algebras of a fixed dimension, see for instance [13]. Moreover it is a Gerstenhaber algebra, and it is related with the representation theory of the given algebra, see for instance [1, 6]. Rephrasing the introduction of [20], observe

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that Hochschild cohomology is not functorial, there is no natural way to relate the Hochschild cohomology of an algebra to that of its quotient algebras or of its subalgebras. In exchange, the idea is to find a way of relating the cohomology of an algebra to that of an easier or smaller algebra. Several articles go in this direction, starting with the work of Happel in [16], see for instance [3, 5, 9, 14, 15, 17, 20, 22].

In this paper we consider associative multi-extension zero algebras over a field k, not necessarily finite dimensional, determined by a set of k-algebras and a set of bimodules over them. They can also be viewed as cleft singular extensions $\Lambda = A \oplus M$ - see [21, p. 284], where in addition the subalgebra A is provided with a finite set E of central orthogonal idempotents; recall that M is an A-bimodule verifying $M^2 = 0$. This setting provides an ad hoc quiver as follows: its vertices are the idempotents of E, and there is an arrow from a vertex x to a different one y in case yMx is not zero. Moreover, this quiver coincides with the Peirce E-quiver, see Definition 2.6.

We develop tools to compute the Hochschild cohomology of Λ, in relation to the value of cohomological functors related to this specific quiver, and to the algebras and the bimodules involved. More precisely, a finite quiver Q is called simply laced (see for instance [19]) if it has neither double arrows nor loops. A Q-data Δ is a collection of algebras $\{A_x\}$ associated to each vertex x, a set of bimodules $\{M_a\}$ associated to each arrow a, and a family of bimodule maps verifying associative constraints - this family will provide the product map amongst the bimodules in the algebra to be built. We set $A = \times_{x \in Q_0} A_x$ and $M = \oplus_{a \in Q_1} M_a$; the multi-extension algebra Λ_Δ is, as a vector space, $A \oplus M$. If all the maps of the family are zero, then Λ_Δ is called a multi-extension zero algebra. In this case, Λ_Δ is a cleft singular extension, provided with the set Q_0 of central orthogonal idempotents of A.

One-point extensions are occurrences of multi-extension zero algebras: indeed they are built on the simply laced quiver which has just an arrow. One of the algebras associated to its vertices is k. In Section 2 we precise the definitions and we provide other examples of multi-extension zero algebras.

In Section 3 we consider a multi-extension algebra which is not necessarily a multi-extension zero algebra, built on a simply laced quiver Q provided with a Q-data. We analyze the complex of cochains relative to the separable subalgebra given by the vertices of Q, which computes the Hochschild cohomology of the multi-extension algebra. The main tool that we introduce are the trajectories over Q. A trajectory is an oriented path of Q provided with non negative integers at each vertex, called waiting times. The duration of the trajectory is the sum of the length of the path, plus the waiting times. The cochains of the complex decompose along the trajectories, and we describe the coboundary relative to this. The results are stated for multi-extension algebras, but they are valid as well for small k-categories, with possibly an infinite number of objects, and for Hochschild-Mitchell cohomology, see [23].

In Section 4, we focalize on multi-extension zero algebras and use the tools we have considered. Each non cycle δ of Q provides a subcomplex, it is given in each degree by the trajectories over δ whose duration equals the degree. This way we obtain a short exact sequence of complexes. One of its interests is that the quotient complex decomposes as a direct sum along cycles of Q, and that this decomposition is again based on trajectories over paths of Q which are now cycles.

The above sketched analysis shows that for any path ω of Q, it is natural to define a cohomology theory $\mathsf{H}^\bullet_\omega(\Delta)$ along ω of the Q-data Δ. We infer a cohomology long exact sequence from the short exact sequence, which makes use of the cohomology theory along paths of Q.

Section 5 mostly concerns the computation of the cohomology along paths of a Q-data. If the path is a vertex x, trajectories over it are just waiting times on x, and the cohomology theory along x is the Hochschild cohomology of the algebra A_x. A main result is that the cohomology along an arrow a is Ext of M_a with itself, with a shift of one in the degrees. This already shows that the long exact sequence that we obtain coincides with the long exact sequence of Happel for one-point extensions [16], as well as with its generalization for corner algebras obtained independently in [8], [22] and [15]. In order to go further in the computation of the cohomology theory along paths, Tor vanishing hypotheses are needed. More precisely, let ω be a path of length two. If Tor between the bimodules of the Q-data is zero in positive degrees, then $\mathsf{H}^\bullet_\omega(\Delta)$ is an Ext functor, shifted by two in the degree. We provide a generalization of this result for paths of higher length. Observe that the Tor vanishing conditions that we require in order to compute the cohomology along paths resemble to the ones required for an ideal to be stratified (see [11, 20]) as well as to the hypotheses used recently in [17].

The connecting homomorphism ∇ of the long exact sequence that we have obtained is important for computations. For describing ∇, in Section 6 we consider the multiplicative structure involved. We show that cohomology along paths has a cup product verifying the graded Leibniz rule, and which is compatible with composition of paths. In our context, there is a canonical element in the cohomology along paths, that is the sum of the identity maps as endomorphisms of each bimodule, which provides a 1-cocycle in the sum of the cohomologies along arrows. This enables to describe ∇, for a multi-extension zero algebra, as the graded commutator with this canonical element - this way we recover a result in [9] as a particular case. In this process we reobtain part of the results of [14], for instance that for one-point extensions the connecting morphism is a graded algebra map.

In Section 7 we specialize our results to square algebras, namely multi-extension algebras built on the so-called round trip quiver $Q = \cdot \rightleftarrows \cdot$ provided with a Q-data. If the two bimodules associated to the arrows - that is the corner bimodules of the 2×2 matrix algebra - are projective, and if the bimodule maps of the data are zero, the algebra is called a null-square projective algebra. For these algebras we show that the cohomology long exact sequence that we have obtained splits into five-term exact sequences. Using this fact, we provide explicit formulas for the Hochschild cohomology of a null-square projective algebra Λ, in terms of the Hochschild cohomology of the algebras at the vertices of Q - that is the algebras on the diagonal - and the kernel and cokernel in even degrees of the non trivial part ∇' of ∇. As a consequence, the Hochschild cohomology of Λ contains that of the algebras of the Q-data: the inclusion is canonical in even degrees while it is obtained by choosing a splitting of a canonical short exact sequence in the odd ones. Next we prove in Section 7 that a square algebra which corner bimodules are free of rank one is necessarily a null-square algebra. In other words, the family of bimodule maps is necessarily zero for a Q-data based on the round trip quiver Q, and if the bimodules associated to the arrows are free of rank one. We also show that for such an algebra, ∇' is injective in even positive degrees. In the finite dimensional case this leads to explicit formulas for the dimension of the Hochschild cohomology. Moreover, we describe the presentation of an algebra of this sort by means of its Gabriel quiver and admissible relations, relying on such presentations for the algebras on the diagonal.

In Section 8 we consider once more null-square algebras, but focussing on an opposite family to the one considered in Section 7. Here the corner bimodules are projective, provided by some couples of idempotents belonging to complete systems of orthogonal idempotents of the algebras which are on the diagonal. This leads to a combinatorial data encoded in the Peirce quiver which conveys enough information for Hochschild cohomology purposes. We prove that if the corner projective bimodules do not produce new oriented cycles in the Peirce quiver, then the Hochschild cohomology of the null-square algebra coincides with that of the diagonal algebra in large enough degrees. This fact differs from the results of Section 7, indeed free rank one corner bimodules produce plenty of new cycles in the Peirce quiver.

To end this Introduction, we point out a possible follow-up to our work. In [24] the authors introduced a theory of support variety for modules over any Artin algebra A. This is defined in terms of a quotient of the Hochschild cohomology ring. They consider the Hochschild cohomology ring and divide by the ideal generated by the homogeneous nilpotent elements. Let us call this associated ring $\mathcal{R}(A)$. Observe that the odd degree elements have square zero, so they are nilpotent. The ring $\mathcal{R}(A)$ is always a commutative ring, if it is finitely generated then its spectrum defines a variety, and it is over this variety that they define the support variety of a module. We believe that our results can be used to compare the associated ring of the algebras involved in our work. For instance Theorem 6.5 shows that for a multi-extension algebra, the image of the considered map is in the ideal generated by the nilpotent elements of the Hochschild cohomology. Also, from Corollary 7.4 we infer that for $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ a null-square projective algebra, with $\mathbb{A} = A \times B$ and $\mathbb{M} = M \oplus N$, if there is a positive integer h such that $\mathbb{M}^{\otimes_{\mathbb{A}} h} = 0$, then the associated ring $\mathcal{R}(\Lambda)$ is isomorphic to $\mathcal{R}(\mathbb{A})$ in large enough degrees.

2 Multi-extension algebras

A *simply laced quiver* is a finite quiver Q without multiple parallel arrows nor loops (see for instance [19, p. 112]). More precisely, a quiver Q is given by a finite set Q_0 of vertices, a finite set Q_1 of arrows and two maps s and t from Q_1 to Q_0 called source and target. The quiver is simply laced if there is at most one arrow from one vertex to another one and if there is no arrow where the source and target vertices are the same.

Let k be a field and let Q be a simply laced quiver.

Definition 2.1 *A Q-*data *Δ consists in a set of algebras $\{A_x\}_{x\in Q_0} A_x$, a set $\{M_a\}_{a\in Q_1}$ of bimodules, where M_a is an $A_{t(a)} - A_{s(a)}$-bimodule, and a family of bimodule maps*

$$\alpha = \{\alpha_{z,y,x}\}_{z,y,x\in Q_0}$$

satisfying obvious associativity constraints in order to make $A \oplus M$ an associative algebra, where $A = \times_{x\in Q_0} A_x$ and $M = \oplus_{a\in Q_1} M_a$.

Let Δ be a Q-data. Each M_a is an A-bimodule by extending the actions by zero, hence M is an A-bimodule as well.

Definition 2.2 *The* multi-extension algebra *Λ_Δ is, as vector space $A \oplus M$, where A is a subalgebra. If $a \in A$ and $m \in M$, then the products am and ma are given by the actions. The product on M is determined by the family α.*

The algebra Λ_Δ is a multi-extension zero algebra *if all the maps of the family α are zero.*

Example 2.3 *Recall that if A_y is an algebra and M is a left A_y-module, the one-point extension $A_y[M]$ is the algebra $\begin{pmatrix} k & 0 \\ M & A_y \end{pmatrix} = \begin{pmatrix} A_y & M \\ 0 & k \end{pmatrix}$. This is also an instance of a multi-extension zero algebra, where $Q = x\cdot \xrightarrow{a} \cdot y$.*

Example 2.4 *More generally, let A_x and A_y be algebras, let M_a be an $A_y - A_x$-bimodule and let M_b be an $A_x - A_y$-bimodule. The null-square algebra (see for instance [10] or [5]) is $\begin{pmatrix} A_x & M_b \\ M_a & A_y \end{pmatrix}$ with matrix multiplication given by the bimodule structures of M_a and M_b, and setting $m_a m_b = 0 = m_b m_a$ for all $m_a \in M_a$ and $m_b \in M_b$. This is again an instance of a multi-extension zero algebra, where Q is the round trip quiver ${}_x\cdot \rightleftarrows \cdot_y$.*

Definition 2.5 *A* system E *of an algebra Λ is a finite set of orthogonal idempotents of Λ which is complete, that is $\sum_{x\in E} x = 1$.*

Definition 2.6 *Let Λ be an algebra provided with a system E. The* Peirce E-quiver *has set of vertices E. For $x, y \in E$, there is an arrow from x to y if and only if $x \neq y$ and $y\Lambda x \neq 0$.*

Remark 2.7

- *A Peirce E-quiver is simply laced.*
- *Let Λ_Δ be a multi-extension algebra given by a quiver Q and a Q-data Δ. The Peirce Q_0-quiver of Λ_Δ is Q.*

Let Q be a finite quiver and kQ be its path algebra. Let $\langle Q_1\rangle$ be the two-sided ideal of kQ generated by the arrows. A two-sided ideal I of kQ is admissible if $I \subset \langle Q_1\rangle^2$ and if there exist a positive integer n such that $I \subset \langle Q_1\rangle^n$. The Gabriel quiver - also called Ext-quiver - of $\Lambda = kQ/I$ is Q, its vertices are the simple modules and the number of arrows is the k-dimension of Ext^1 between them.

Proposition 2.8 *Let $\Lambda = kQ/I$ be an algebra presented as above. The Peirce Q_0-quiver of Λ is obtained as follows.*

- *First delete loops,*
- *for $x \neq y$, add an arrow from x to y if there exists a path from x to y in Q which is not in I - that is the path is not zero in Λ,*
- *finally replace multiple parallel arrows by only one arrow.*

We assert that an algebra Λ provided with a system E has an evident multi-extension algebra structure as follows. Let Q be its Peirce E-quiver, and let Δ be the Q-data given by the algebras $x\Lambda x$ for each $x \in E$, the bimodules $y\Lambda x$ for $x, y \in E$, and the bimodule maps

$$\alpha_{z,y,x} : z\Lambda y \otimes_{y\Lambda y} y\Lambda x \to z\Lambda x$$

determined by the product of Λ. It is evident that $\Lambda = \Lambda_\Delta$.

Proposition 2.9 *An algebra Λ is a multi-extension zero with respect to a system E if and only if $(z\Lambda y)(y\Lambda x) = 0$ for all $z \neq y$ and $y \neq x$ in E.*

We will say that Λ is a multi-extension zero algebra without referring to a system E.

Recall that a *cleft singular extension algebra* (see [21, p. 284]) is an algebra Λ with a decomposition $\Lambda = A \oplus M$, where A is a subalgebra and M is a two-sided ideal of Λ verifying $M^2 = 0$. These algebras are also called trivial extensions, see for instance [1] and [4, 3] where the natural generalization for abelian categories is considered.

A multi-extension zero algebra is thus a cleft singular extension $A \oplus M$, with in addition a complete set of central orthogonal idempotents of A.

Example 2.10 *Let R be the quiver given by two quivers Q_1 (which we view horizontally upstairs) and Q_2 (horizontally downstairs), together with a finite set of* vertical down *arrows joining some of the vertices of Q_1 to vertices of Q_2, as well as an analogous set of* vertical up *arrows. Let I be the two-sided ideal of kR generated by the paths which contain two vertical arrows.*

The algebra $\Lambda = kR/I$ can be viewed in a natural way as a multi-extension zero algebra. Indeed, for $i = 1, 2$, let x_i be the sum of the vertices of Q_i. The set $E = \{x_1, x_2\}$ is a system of Λ. The Peirce E-quiver is the round trip quiver $Q = {}_{x_1}\cdot \rightleftarrows \cdot_{x_2}$, where the arrow from x_1 to x_2 is called a and the reverse one is called b. The algebras of the Q-data are $A_{x_1} = kQ_1$ and $A_{x_2} = kQ_2$. Notice that the bimodules M_a and M_b are projective bimodules.

Finally we record that a multi-extension algebra Λ_Δ encodes a k-category: its objects are the vertices of the quiver, the endomorphisms of the objects are the algebras, and the morphisms between distinct objects are the bimodules. The composition of morphisms between different objects is given by the family α.

3 Hochschild cohomology of multi-extension algebras

In this section we provide tools for computing the Hochschild cohomology of a multi-extension algebra, which will be mainly used for multi-extension zero algebras.

Let Λ be an algebra and let Z be a Λ-bimodule. By definition, the Hochschild cohomology of Λ with coefficients in Z is

$$\mathsf{H}^n(\Lambda, Z) = \mathsf{Ext}^n_{\Lambda \otimes \Lambda^{\mathsf{op}}}(\Lambda, Z).$$

For $Z = \Lambda$ it is usual to write $\mathsf{HH}^n(\Lambda)$ instead of $\mathsf{H}^n(\Lambda, \Lambda)$.

The following result is well-known, the proof is analogous to the one sketched in [10] for Hochschild homology:

Lemma 3.1 *Let Λ be an algebra, let D be a separable subalgebra of Λ and let Z be a Λ-bimodule. The cohomology of the complex $\mathsf{J}^\bullet(Z)$*

$$0 \to Z^D \xrightarrow{d} \mathsf{Hom}_{D-D}(\Lambda, Z) \xrightarrow{d} \mathsf{Hom}_{D-D}(\Lambda \otimes_D \Lambda, Z) \cdots \xrightarrow{d} \mathsf{Hom}_{D-D}\left(\Lambda^{\otimes_D n}, Z\right) \xrightarrow{d} \cdots$$

is $\mathsf{H}^*(\Lambda, Z)$, where $Z^D = \{z \in Z \mid dz = zd \text{ for all } d \in D\}$ and Hom_{D-D} stands for $\mathsf{Hom}_{D \otimes D^{\mathsf{op}}}$. The definition of the maps d is provided by the same formulas that those for computing Hochschild cohomology:

- *for $n > 0$*

$$
\begin{aligned}
(df)(x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1}) = {} & x_1 f(x_2 \otimes x_3 \otimes \cdots \otimes x_{n+1}) \\
& + \sum_1^n (-1)^i f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) \\
& + (-1)^{n+1} f(x_1 \otimes x_2 \otimes \cdots \otimes x_n) x_{n+1},
\end{aligned}
$$

- *for $n = 0$ and $z \in \Lambda^D$, we have $(dz)(x) = xz - zx$.*

Let Λ_Δ be a multi-extension algebra and let $D = \times_{x \in Q_0} k$ be the separable subalgebra of $A = \times_{x \in Q_0} A_x$ given by the inclusions $k \subset A_x$. Next we will provide a canonical decomposition of the space of n-cochains $\mathsf{J}^n(\Lambda_\Delta)$.

For any quiver Q and $m > 0$, a *path of length* m is a sequence of arrows $\omega = a_m \ldots a_1$ which are *concatenated*, that is $t(a_i) = s(a_{i+1})$ for all i. The set of vertices Q_0 is the set of paths of length 0; if $x \in Q_0$, then $s(x) = t(x) = x$. The set of paths of length m is denoted Q_m. The set of paths of length less or equal to m is denoted $Q_{\leq m}$. The maps s and t are extended to the set of paths by $s(\omega) = s(a_1)$ and $t(\omega) = t(a_m)$.

A path ω is a *cycle* if $s(\omega) = t(\omega)$. We denote CQ_m the set of *cycles* of length m, note that $CQ_0 = Q_0$. Its complement in Q_m is the set of *non cycles* of length m that we denote DQ_m.

Definition 3.2 *Let Q be a simply laced quiver, let $m > 0$ and let $\omega = a_m \cdots a_1 \in Q_m$. The set $T_n(\omega)$ of trajectories of duration n over ω is the set of sequences*

$$\tau = t(a_m)^{p_{m+1}}, a_m, s(a_m)^{p_m}, \cdots, s(a_2)^{p_2}, a_1, s(a_1)^{p_1}$$

where each p_i is a non negative integer and $n = m + \sum_1^{m+1} p_i$. The integer $n - m$ is the total waiting time *of the trajectory. For $p > 0$ and $x \in Q_0$, the symbol x^p denotes the sequence $(x, x, \ldots, x)$ where x is repeated p times. The* waiting time *of the trajectory x^p over x is p. If $p = 0$, the symbol x^0 is the empty sequence, it corresponds to a 0 waiting time at x.*

We record the following facts:

- If x is a vertex, then $T_n(x) = \{x^n\}$ for $n \geq 0$.
- If $\omega \in Q_m$ and $n < m$, then $T_n(\omega) = \emptyset$.
- If $\omega \in Q_m$, then $T_m(\omega)$ has a unique element

$$t(a_m)^0, a_m, s(a_m)^0, \ldots, s(a_2)^0, a_1, s(a_1)^0$$

  of total waiting time zero and duration m.

Remark 3.3 *By definition, a trajectory τ over a path ω is a sequence of vertices and arrows which are concatenated, that is the product of two successive entries of τ is not zero in the path algebra kQ. Moreover, the product of all successive entries of τ is equal to ω in kQ.*

Definition 3.4 *Let Q be a simply laced quiver with a Q-data Δ and let $\omega = a_m \dots a_1 \in Q_m$ for $m > 0$. For $n \geq m$, let*

$$\tau = t(a_m)^{p_{m+1}}, a_m, s(a_m)^{p_m}, \cdots, s(a_2)^{p_2}, a_1, s(a_1)^{p_1} \in T_n(\omega).$$

The evaluation of τ at Δ *is the vector space*

$$\tau_\Delta = A_{t(a_m)}^{\otimes p_{m+1}} \otimes M_{a_m} \otimes A_{s(a_m)}^{\otimes p_m} \otimes \cdots \otimes A_{s(a_2)}^{\otimes p_2} \otimes M_{a_1} \otimes A_{s(a_1)}^{\otimes p_1}$$

where all the tensor products are over k. If $x \in Q_0$ and $\tau \in T_n(x)$, then $\tau_\Delta = A_x^{\otimes n}$. By definition, if A is a an algebra, then $A^{\otimes 0} = k$.

Proposition 3.5 *Let Q be a simply laced quiver with a Q-data Δ, let Λ_Δ be the corresponding multi-extension algebra, and let Z be a Λ_Δ-bimodule. For $n > 0$ the following decompositions hold:*

$$(\Lambda_\Delta)^{\otimes_D n} = \bigoplus_{\omega \in Q_{\leq n}} \left[\bigoplus_{\tau \in T_n(\omega)} \tau_\Delta \right] \tag{3.1}$$

$$\mathsf{J}^n(Z) = \bigoplus_{\omega \in Q_{\leq n}} \left[\bigoplus_{\tau \in T_n(\omega)} \mathsf{Hom}_k \left(\tau_\Delta,\ t(\omega) Z s(\omega)\right) \right]. \tag{3.2}$$

Moreover

$$\mathsf{J}^0(\Lambda) = \Lambda^D = \times_{x \in Q_0} A_x = \bigoplus_{x \in Q_0} \bigoplus_{\tau \in T_0(x)} \tau_\Delta.$$

Proof. The proof of (3.1) is by induction on n, we only describe in detail the low degree cases. Recall that $\Lambda_\Delta = A \oplus M$, hence

$$(\Lambda_\Delta) \otimes_D (\Lambda_\Delta) = (A \otimes_D A) \oplus (A \otimes_D M) \oplus (M \otimes_D A) \oplus (M \otimes_D M).$$

For $x \in Q_0$, let e_x be the idempotent of D with value 1 at x and 0 at other vertices. Note that $\{e_x\}_{x \in Q_0}$ is a complete set of central orthogonal idempotents of A. Actually

$$Ae_x = A_x = e_x A.$$

Observe that if $x \neq y$, then

$$ae_x \otimes e_y a' = ae_x^2 \otimes e_y a' = ae_x \otimes e_x e_y a' = 0,$$

so $A_x \otimes_D A_y = 0$. Moreover $A_x \otimes_D A_x = A_x \otimes A_x$. Hence

$$A \otimes_D A = \bigoplus_{x \in Q_0} A_x \otimes A_x.$$

The direct summand $A_x \otimes A_x$ corresponds to the trajectory x^2 of total waiting time 2 and duration 2 at the vertex x.

Observe that if $a \in Q_1$, then $M_a = e_{t(a)} M e_{s(a)} = e_{t(a)} (\Lambda_\Delta) e_{s(a)}$, while $e_y M e_x = 0$ if there is no arrow from x to y in Q.

$$M = \bigoplus_{a \in Q_1} M_a = \bigoplus_{x,y \in Q_0} e_y M e_x.$$

If $z \neq t(a)$, then $A_z \otimes_D M_a = 0$, while $A_{t(a)} \otimes_D M_a = A_{t(a)} \otimes M_a$. Hence

$$A \otimes_D M = \bigoplus_{a \in Q_1} A_{t(a)} \otimes M_a.$$

Similarly

$$M \otimes_D A = \bigoplus_{a \in Q_1} M_a \otimes A_{s(a)}.$$

Each direct summand above corresponds to a path of length 1 - that is an arrow a - and the trajectories $(t(a)^1, a, s(a)^0)$ or $(t(a)^0, a, s(a)^1)$ over a, which are of total waiting time 1 and duration 2. Analogously, we obtain the decomposition

$$M \otimes_D M = \bigoplus_{\{\omega = a_2 a_1 \in Q_2\}} M_{a_2} \otimes M_{a_1}.$$

Each direct summand corresponds to the unique trajectory over $a_2 a_1$ of total waiting time 0 and duration 2.

The next observations will prove (3.2). Recall that $D = \times_{x \in Q_0} k e_x$ is a semisimple algebra. Hence a D-bimodule U has a canonical decomposition into its isotypic components $U = \oplus_{x,y \in Q_0} e_y U e_x$. Observe that each direct summand of (3.1) is a D-bimodule. More precisely for $\tau \in T_n(\omega)$, we have that τ_Δ is a direct summand of the isotypic component

$$e_{t(\omega)} \left[(\Lambda_\Delta)^{\otimes_D n} \right] e_{s(\omega)}.$$

Let U and V be D-bimodules. The following is immediate - and it is an instance of Schur's Lemma: if $y \neq t$ or $x \neq u$, then

$$\mathsf{Hom}_D(e_y U e_x, e_t V e_u) = 0,$$

while

$$\mathsf{Hom}_D(e_y U e_x, e_y V e_x) = \mathsf{Hom}_k(e_y U e_x, e_y V e_x).$$

$\diamond$

Our next aim is to use the decomposition of cochains that we have obtained in Proposition 3.5 in order to describe the coboundary d of Lemma 3.1.

Definition 3.6 *Let*

$$\tau = t(a_m)^{p_{m+1}}, a_m, s(a_m)^{p_m}, \cdots, s(a_2)^{p_2}, a_1, s(a_1)^{p_1}$$

be a n-trajectory over a path $\omega = a_m \dots a_1$. The set τ^+ is the union of:

- τ_0^+, *the set of $n+1$-trajectories obtained by increasing a waiting time of τ by one.*
- τ_1^+, *the set of $n+1$-trajectories*

  $$t(c)^0, c, s(c)^{p_{m+1}}, a_m, s(a_m)^{p_m}, \cdots, s(a_2)^{p_2}, a_1, s(a_1)^{p_1}$$

  where $c \in Q_1$ is any arrow after ω, that is verifying $s(c) = t(a_m)$.

  Similarly, τ_1^+ contains also the trajectories obtained by adding any arrow c before ω, that is verifying $t(c) = s(a_1)$.

- τ_2^+, *the set of* $n+1$*-trajectories obtained by replacing any arrow* a_i *by some path* $a_i''a_i'$ *of length two which is* parallel *to* a_i*, that is* $s(a_i) = s(a_i')$ *and* $t(a_i) = t(a_i'')$*.*

Definition 3.7 *Let* ω *be a path of* Q*. We denote by* Δ_ω *the* $A_{t(\omega)} - A_{s(\omega)}$*-bimodule*

$$t(\omega)(\Lambda_\Delta)s(\omega).$$

Observe that if there is no parallel arrow to ω - that is there is no arrow a such that $s(a) = s(\omega)$ and $t(a) = t(\omega)$ - and ω is a non cycle, then $\Delta_\omega = 0$. On the other hand if such an arrow a exists, then $\Delta_\omega = M_a \neq 0$. If ω is a cycle, then $\Delta_\omega = A_{s(\omega)} = A_{t(\omega)}$.

Definition 3.8 *The space of* n-cochains *over a trajectory* $\tau \in T_n(\omega)$ *is*

$$\mathsf{J}_\tau = \mathsf{Hom}_k\left(\tau_\Delta,\ \Delta_\omega\right).$$

Proposition 3.9 *Let* d *be one of the coboundaries of Lemma 3.1. The following holds*

$$d\mathsf{J}_\tau \subset \bigoplus_{\sigma\in\tau^+} \mathsf{J}_\sigma.$$

Proof. Let T_n be the set of trajectories of duration n and let $\tau \in T_n$. Let $f_\tau \in \mathsf{J}_\tau$, write $df_\tau = \sum_{\sigma\in T_{n+1}} (df_\tau)_\sigma$ the decomposition of df_τ according to (3.2). We assert that if $\sigma \notin \tau^+$, then $(df_\tau)_\sigma = 0$. Recall that

$$\begin{aligned}(df_\tau)(x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1}) = {}& x_1 f_\tau(x_2 \otimes x_3 \otimes \cdots \otimes x_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f_\tau(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) \\ &+ (-1)^{n+1} f_\tau(x_1 \otimes x_2 \otimes \cdots \otimes x_n)x_{n+1}.\end{aligned}$$

Let $x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1} \in \sigma_\Delta$ where $\sigma \notin \tau^+$. We will prove that each summand above is zero.

- If $x_1 f_\tau(x_2 \otimes x_3 \otimes \cdots \otimes x_{n+1}) \neq 0$, then $f_\tau(x_2 \otimes x_3 \otimes \cdots \otimes x_{n+1}) \neq 0$, hence $0 \neq x_2 \otimes x_3 \otimes \cdots \otimes x_{n+1} \in \tau_\Delta$. Moreover, x_1 belongs either to $A_{t(\omega)}$ or to a bimodule M_c for $c \in Q_1$ with $s(c) = t(\omega)$. This is equivalent respectively to $\sigma \in \tau_0^+$ or $\sigma \in \tau_1^+$.
- If $f_\tau(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) \neq 0$, then $0 \neq x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} \in \tau_\Delta$.
  - If $x_i x_{i+1}$ belongs to an algebra A_z, then $x_i \in A_z$ and $x_{i+1} \in A_z$. Hence σ is obtained from τ by increasing by one the waiting time at the vertex z, that is $\sigma \in \tau_0^+$.
  - If $x_i x_{i+1}$ belongs to a bimodule M_a for some arrow a of ω, then
    - either $x_i \in A_{t(a)}$ and $x_{i+1} \in M_a$, or $x_i \in M_a$ and $x_{i+1} \in A_{s(a)}$, that is $\sigma \in \tau_0^+$.
    - or $x_i \in M_{a''}$ and $x_{i+1} \in M_{a'}$ for a path of length two $a''a'$ which is parallel to a, that is $\sigma \in \tau_2^+$.

- If the last summand is non zero, the proof is analogous to the first case.

$\diamond$

The results of this section are given for multi-extension algebras, that is for algebras provided with a system E. Recall that E is a finite set, hence the requirement that E is complete makes sense. However the results of this section are actually true as well for Hochschild-Mitchell cohomology (see [23]) of a small k-category, with possibly an infinite number of objects.

4 A long exact sequence for multi-extension zero algebras

The result of Proposition 3.9 can be made more precise for a multi-extension zero algebra. Let Q be a simply laced quiver provided with a Q-data $\Delta = (A, M)$, where $A = \times_{x\in Q_0} A_x$ and $M = \oplus_{a\in Q_1} M_a$. Let Λ_Δ be the corresponding multi-extension zero algebra.

Definition 4.1 *Let $\omega \in Q_m$. The vector space of n-cochains along ω is*

$$\mathsf{J}^n_\omega = \bigoplus_{\tau\in T_n(\omega)} \mathsf{J}_\tau = \bigoplus_{\tau\in T_n(\omega)} \mathsf{Hom}_k\left(\tau_\Delta, \Delta_\omega\right).$$

Remark 4.2 *Let $\omega \in Q_m$. If $n < m$, then $\mathsf{J}^n_\omega = 0$. If $n = m$ and $\omega = a_m \dots a_1$, then*

$$\mathsf{J}^m_\omega = \mathsf{Hom}_k(M_{a_m} \otimes \cdots \otimes M_{a_1}, \Delta_\omega).$$

Proposition 4.3 *If $\delta \in DQ_m$, then $\mathsf{J}^\bullet_\delta$ is a subcomplex of $\mathsf{J}^\bullet(\Lambda_\Delta)$.*

Proof. We suppose $\Delta_\delta \neq 0$ since otherwise $\mathsf{J}^\bullet_\delta = 0$. The path δ is not a cycle, so there exists $a \in Q_1$ parallel to δ, and $\Delta_\delta = M_a$.

Let $\delta = a_m \dots a_1$, let $\tau = t(a_m)^{p_{m+1}}, a_m, s(a_m)^{p_m}, \cdots, s(a_2)^{p_2}, a_1, s(a_1)^{p_1}$ be a trajectory of duration n over δ, and let $f_\tau \in \mathsf{J}_\tau$. If σ is a trajectory of duration $n+1$ which is not over δ, we will prove that $(df_\tau)_\sigma = 0$. We already know from Proposition 3.9 that if $\sigma \notin \tau^+$, then $(df_\tau)_\sigma = 0$.

- If $\sigma \in \tau^+_0$, there is nothing to prove since σ is over δ, as σ is obtained by increasing some waiting time of τ by one.
- If $\sigma \in \tau^+_1$, then σ is not over δ and we will show that $(df_\tau)_\sigma = 0$. Suppose firstly that

$$\sigma = t(c)^0, c, s(c)^{p_{m+1}}, a_m, s(a_m)^{p_m}, \cdots, s(a_2)^{p_2}, a_1, s(a_1)^{p_1}$$

for some arrow c such that $s(c) = t(a_m)$. Let $x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1} \in \sigma_\Delta$, we recall that

$$\begin{aligned}(df_\tau)(x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1}) &= x_1 f_\tau(x_2 \otimes x_3 \otimes \cdots \otimes x_{n+1}) \\ &+ \sum_1^n (-1)^i f_\tau(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) \\ &+ (-1)^{n+1} f_\tau(x_1 \otimes x_2 \otimes \cdots \otimes x_n)x_{n+1}.\end{aligned}$$

We will show that each of the previous summands is zero. For the first one, $x_1 \in M_c$ and $f_\tau(x_2 \otimes x_3 \otimes \cdots \otimes x_{n+1}) \in M_a$. Hence this summand belongs to the product $M_c M_a$, which is zero because Λ_Δ is a multi-extension zero algebra.

We consider now $f_\tau(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1})$ for $i = 1, \ldots, n$. Notice that $x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} \in \tau'_\Delta$ where τ' is a trajectory over $c\delta$, hence $\tau' \neq \tau$. Then this summand is 0.

For the last summand, note that $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in \tau'_\Delta$, where τ' is a trajectory over a path which last arrow is c. Note that $c \neq a_m$, since there are no loops in Q. Hence $\tau' \neq \tau$ and the summand is 0.

The case where $\sigma \in \tau_1^+$ is obtained from τ by adding at the beginning an arrow c such that $t(c) = s(a_1)$ is analogous.

- If $\sigma \in \tau_2^+$, suppose

$$\sigma = t(a_m)^{p_{m+1}}, a_m, s(a_m)^{p_m}, \cdots, a''_j, a'_j, \cdots, s(a_2)^{p_2}, a_1, s(a_1)^{p_1}$$

and let $x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1} \in \sigma_\Delta$.

The first summand of $(df_\tau)(x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1})$ is zero since $x_2 \otimes \cdots \otimes x_{n+1}$ belongs to τ'_Δ for $\tau' \neq \tau$ beacuse τ' is a trajectory over a path different from δ. Then

$$f_\tau(x_2 \otimes x_3 \otimes \cdots \otimes x_{n+1}) = 0.$$

The proof that the last summand is zero is analogous.

The other summands are also zero: if $x_{i+1} \in M_{a''_j}$ and $x_i \in M_{a'_j}$, then $x_{i+1}x_i = 0$ because the products of elements of the bimodules are zero in Λ_Δ. Otherwise, observe that $x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} \in \tau'_\Delta$ where τ' is a trajectory over a path different from δ, then

$$f_\tau(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) = 0.$$

$\diamond$

Definition 4.4 *Let Λ_Δ be a multi-extension zero algebra with respect to a Q-data $\Delta = (A, M)$ and let $\delta \in DQ_m$.*

For $n \geq m$, the cohomology of Δ along δ *is denoted $\mathsf{H}^n_\delta(\Delta)$, and it is the cohomology of the complex of cochains $\mathsf{J}^\bullet_\delta$.*

From the above proof, we record the next result for the coboundary of $\mathsf{J}^\bullet_\delta$.

Proposition 4.5 *Let $\delta \in DQ_m$ and let $\tau \in T_n(\delta)$. Let $f_\tau \in \mathsf{J}_\tau$. The coboundary $d : \mathsf{J}^n_\delta \to \mathsf{J}^{n+1}_\delta$ is given by*

$$df_\tau = \sum_{\sigma \in \tau_0^+} (df_\tau)_\sigma.$$

Proof. We know from Proposition 3.9 that

$$df_\tau = \sum_{\sigma \in \tau^+} (df_\tau)_\sigma = \sum_{\sigma \in \tau_0^+} (df_\tau)_\sigma \; + \sum_{\sigma \in \tau_1^+} (df_\tau)_\sigma + \sum_{\sigma \in \tau_2^+} (df_\tau)_\sigma. \tag{4.1}$$

We have just shown that if δ is not a cycle, and if $\sigma \in \tau_1^+ \cup \tau_2^+$, then $(df_\tau)_\sigma = 0$.

$\diamond$

Definition 4.6 *Let $\omega = a_m \dots a_1 \in Q_m$. The* bimodule along ω *is*

$$M_\omega = M_{a_m} \otimes_{A_{s(a_m)}} \cdots \otimes_{A_{s(a_2)}} M_{a_1}.$$

Note that M_ω is an $A_{t(\omega)} - A_{s(\omega)}$-bimodule.

Lemma 4.7 *Let Δ be a Q-data, and let $\delta \in DQ_m$. The following holds:*

$$\mathsf{H}^m_\delta(\Delta) = \mathsf{Hom}_{A_{t(\delta)}-A_{s(\delta)}}(M_\delta, \Delta_\delta).$$

In particular if $a \in Q_1$,

$$\mathsf{H}^1_a(\Delta) = \mathsf{End}_{A_{t(a)}-A_{s(a)}} M_a.$$

Proof. The complex of cochains $\mathsf{J}^\bullet_\delta(\Lambda)$ begins as follows:

$$0 \to \mathsf{Hom}_k(M_{a_m} \otimes \cdots \otimes M_{a_1},\ \Delta_\delta) \xrightarrow{d} \bigoplus_{\tau \in T_{m+1}(\delta)} \mathsf{Hom}_k(\tau_\Delta,\ \Delta_\delta) \xrightarrow{d} \cdots$$

Observe that $T_{m+1}(\delta) = \{\tau_i\}_{i=0,\dots,m+1}$, where τ_i is obtained from the trajectory with 0 total waiting time by increasing the waiting time at $s(a_i)$ by one, for $i = 0, \dots, m$, and at $t(a_m)$ also by one for $i = m+1$. Hence, for $i = 0 \dots, m$:

$$(\tau_i)_\Delta = M_{a_m} \otimes \cdots \otimes M_{a_i} \otimes A_{s(a_i)} \otimes M_{a_{i-1}} \cdots \otimes M_{a_1}$$

and

$$(\tau_{m+1})_\Delta = A_{t(a_m)} \otimes M_{a_m} \otimes \cdots \otimes M_{a_1}.$$

Let $f \in \mathsf{Hom}_k(M_{a_m} \otimes \cdots \otimes M_{a_1},\ \Delta_\delta)$ be such that $df = 0$, that is $(df)_{\tau_i} = 0$ for all i. For $0 < i < m+1$, this shows that f is $A_{s(i)}$-balanced, while the cases $i = 0$ and $i = m+1$ prove that f is a morphism of bimodules. $\diamond$

Remark 4.8 *For $r \geq 0$ and $\delta \in DQ_m$, we will prove in the next section that assuming adequate hypotheses,*

$$\mathsf{H}^{m+r}_\delta(\Delta) = \mathsf{Ext}^r_{A_{t(\delta)}-A_{s(\delta)}}(M_\delta, \Delta_\delta).$$

We have proven in Proposition 4.3 that each non cycle δ provides a subcomplex $\mathsf{J}^\bullet_\delta$ of the complex $\mathsf{J}^\bullet(\Lambda_\Delta)$. This enables to consider their direct sum as follows.

Definition 4.9 *Let Λ be a multi-extension zero algebra given by a simply laced quiver Q and a Q-data $\Delta = (A, M)$. The* non cycle subcomplex *of cochains $\mathsf{D}^\bullet(\Lambda_\Delta)$ of $\mathsf{J}^\bullet(\Lambda_\Delta)$ is*

$$\mathsf{D}^n(\Lambda_\Delta) = \bigoplus_{\delta \in DQ} \mathsf{J}^n_\delta.$$

Remark 4.10 *We already know that if $\omega \in Q_m$ and $n < m$, then $T_n(\omega) = \emptyset$. For such m and n,*

$$\mathsf{D}^n(\Lambda_\Delta) = \bigoplus_{\delta \in DQ_{\leq n}} \mathsf{J}^n_\delta.$$

In particular $\mathsf{D}^0(\Lambda_\Delta) = 0$. Note also that

$$\mathsf{H}^n(\mathsf{D}^\bullet(\Lambda_\Delta)) = \bigoplus_{\delta \in DQ_{\leq n}} \mathsf{H}^n_\delta(\Delta).$$

We consider now the exact sequence of complexes of cochains, where Λ_Δ is omitted in the notation:

$$0 \to \mathsf{D}^\bullet \to \mathsf{J}^\bullet \to (\mathsf{J}/\mathsf{D})^\bullet \to 0.$$

Our next purpose is to describe the cohomology of the quotient complex $(\mathsf{J}/\mathsf{D})^\bullet$. Let CQ be the set of cycles of Q and let

$$\mathsf{C}^n = \bigoplus_{\gamma \in CQ} \mathsf{J}^n_\gamma.$$

Recall that for $\gamma \in CQ$,

$$\mathsf{J}^n_\gamma = \bigoplus_{\tau \in T_n(\gamma)} \mathsf{J}^n_\tau = \bigoplus_{\tau \in T_n(\gamma)} \mathsf{Hom}_k(\tau_\Delta,\ \Delta_\gamma)$$

where $\Delta_\gamma = A_{s(\gamma)} = A_{t(\gamma)}$. The set of paths of Q is the disjoint union of DQ and CQ, hence at each degree there is a vector space decomposition

$$\mathsf{J}^n = \mathsf{D}^n \oplus \mathsf{C}^n$$

which provides a vector space identification between C^n and $(\mathsf{J}/\mathsf{D})^n$.

Remark 4.11

- *The vector spaces $\{\mathsf{C}^n\}_{n\geq 0}$ are not in general a subcomplex of $\mathsf{J}^\bullet$. Indeed, let τ be a trajectory over a cycle γ, and let $f_\tau \in \mathsf{J}_\tau$. By Proposition 3.9, $df_\tau \in \bigoplus_{\sigma\in\tau^+} \mathsf{J}^{n+1}_\sigma$, where $\tau^+ = \tau^+_0 \cup \tau^+_1 \cup \tau^+_2$. Observe that a trajectory $\sigma \in \tau^+_1$ is not anymore over a cycle since it is obtained by adding an arrow either after the end of τ or before its beginning - recall that there are no loops in Q. Moreover, in general $(df_\tau)_\sigma$ is not zero: the image of f_τ lies in $A_{s(\omega)} = A_{t(\omega)}$, hence the first and the last summands may be not zero.*

- *Notice that if $\sigma \in \tau^+_2$, then $(df_\tau)_\sigma = 0$. Indeed, the proof of the last item of Proposition 4.3 is also valid for cycles.*

- *On the other hand, the trajectories in τ^+_0 are over the same cycle γ, since they are obtained by increasing by one a waiting time of τ.*

Let τ be a trajectory over a path ω, and let $f_\tau \in \mathsf{Hom}_k(\tau_\Delta,\ \Delta_\omega)$. Recall that

$$df_\tau = \sum_{\sigma\in\tau^+_0} (df_\tau)_\sigma + \sum_{\sigma\in\tau^+_1} (df_\tau)_\sigma + \sum_{\sigma\in\tau^+_2} (df_\tau)_\sigma\,.$$

We have proven that the last summand is zero for multi-extension zero algebras, from now on we will omit it.

Definition 4.12 *Let $\gamma \in CQ_m$, and let τ be a trajectory over γ of duration n - hence $n \geq m$. Let*

$$d' : \mathsf{C}^n \to \mathsf{C}^{n+1}$$

be the map defined by:

$$d' f_\tau = \sum_{\sigma\in\tau^+_0} (df_\tau)_\sigma.$$

Notice that the image of d' is indeed contained in C^{n+1} since the trajectories of τ_0^+ are over the cycle γ.

Theorem 4.13 *The complex of cochains* $(\mathsf{J}/\mathsf{D})^\bullet$ *with the induced differential* $\overline{d}$ *is isomorphic to* $(\mathsf{C}^\bullet, d')$.

Proof. Remark 4.11 shows that through the mentioned identification between $(\mathsf{J}/\mathsf{D})^\bullet$ and $\mathsf{C}^\bullet$, the coboundary $\overline{d}$ becomes d'. ◇

The subcomplex $\mathsf{D}^\bullet$ of $\mathsf{J}^\bullet$ is a direct sum of subcomplexes indexed by non cycles. For $\delta \in DQ$ we have defined its cohomology $\mathsf{H}^\bullet_\delta$.

We observe that a similar situation is in force for the quotient complex of cochains $((\mathsf{J}/\mathsf{D})^\bullet, \overline{d}) = (\mathsf{C}^\bullet, d')$, namely this quotient is the direct sum of subcomplexes indexed by CQ.

Moreover the differential d' has the same description than the one given in Proposition 4.5 for d. This enables us to define globally the cohomology along an arbitrary path as follows.

Definition 4.14 *Let Q be a simply laced quiver with a Q-data $\Delta = (A, M)$, and let $\omega \in Q_m$. For $n \geq m$, the* cohomology along ω of Δ *in degree n is denoted* $\mathsf{H}^n_\omega(\Delta)$ *and is the cohomology of the following complex of cochains* $(\mathsf{K}^\bullet_\omega, \mathsf{d})$

$$0 \to \mathsf{Hom}_k(M_{a_m} \otimes \cdots \otimes M_{a_1},\ \Delta_\omega) \xrightarrow{\mathsf{d}} \cdots$$
$$\xrightarrow{\mathsf{d}} \bigoplus_{\tau \in T_n(\omega)} \mathsf{Hom}_k(\tau_\Delta, \Delta_\omega) \xrightarrow{\mathsf{d}} \bigoplus_{\tau \in T_{n+1}(\omega)} \mathsf{Hom}_k(\tau_\Delta, \Delta_\omega) \xrightarrow{\mathsf{d}} \cdots$$

where
$$\mathsf{d} f_\tau = \sum_{\sigma \in \tau_0^+} (df_\tau)_\sigma.$$

Remark 4.15 *Let Q be a simply laced quiver with Q-data $\Delta = (A, M)$.*

- *If $\omega = \delta$ is a non cycle, then $(\mathsf{K}^\bullet_\delta, \mathsf{d}) = (\mathsf{J}^\bullet_\delta, d)$ and the previous definition agrees with Definition 4.4.*
- *If $\omega = \gamma$ is a cycle, then we have seen just before that $(\mathsf{K}^\bullet_\gamma, \mathsf{d})$ is a direct summand of the quotient complex $((\mathsf{J}/\mathsf{D})^\bullet, \overline{d}) = (\mathsf{C}^\bullet, d')$.*
- *If $\omega = x$ is a vertex, that is a cycle of length 0, then $\mathsf{K}^\bullet_x$ is the usual complex which computes the Hochschild cohomology of A_x. Hence*
$$\mathsf{H}^n_x(\Delta) = \mathsf{HH}^n(A_x).$$

The proof of the following result is clear.

Proposition 4.16 *Let Λ be a multi-extension zero algebra, with simply laced quiver Q and Q-data $\Delta = (A, M)$. Let $\mathsf{J}^\bullet$ be the complex of cochains of Lemma 3.1, which computes $\mathsf{HH}^\bullet(\Lambda_\Delta)$. Let $\mathsf{D}^\bullet$ be the non cycle subcomplex, let $(\mathsf{J}/\mathsf{D})^\bullet$ be the quotient, and let $\mathsf{H}^*(\mathsf{D}^\bullet)$ and $\mathsf{H}^*((\mathsf{J}/\mathsf{D})^\bullet)$ be their respective cohomologies.*

The following decompositions hold:

$$\mathsf{H}^n(\mathsf{D}^\bullet) = \bigoplus_{\delta \in DQ_{\leq n}} \mathsf{H}^n_\delta(\Delta),$$

$$\mathsf{H}^n((\mathsf{J}/\mathsf{D})^\bullet) = \bigoplus_{\gamma\in CQ_{\leq n}} \mathsf{H}^n_\gamma(\Delta).$$

Corollary 4.17 *For a multi-extension zero algebra as before,* $\mathsf{HH}^*(A)$ *is a direct summand of* $\mathsf{H}^*((\mathsf{J}/\mathsf{D})^\bullet)$.

As particular cases of the previous proposition, we obtain the following results in low degrees:

- $\mathsf{H}^0(\mathsf{D}^\bullet) = 0$, since there are no non cycles of length less or equal to zero.
- $\mathsf{H}^1(\mathsf{D}^\bullet) = \bigoplus_{a\in Q_1} \mathsf{H}^1_a(\Delta) = \bigoplus_{a\in Q_1} \mathsf{End}_{A_{t(a)}-A_{s(a)}} M_a = \mathsf{End}_{A-A} M$, by Lemma 4.7.
- $\mathsf{H}^0((\mathsf{J}/\mathsf{D})^\bullet) = \times_{x\in Q_0} \mathsf{Z}A_x = \mathsf{Z}A$, that is the center of A. The last item of the previous remark provides the equality.
- $\mathsf{H}^1((\mathsf{J}/\mathsf{D})^\bullet) = \bigoplus_{x\in Q_0} \mathsf{HH}^1(A_x) = \mathsf{HH}^1(A)$, since there are no loops in Q, that is $CQ_{\leq 1} = Q_0$, and as a consequence of the last item of the previous remark.

Theorem 4.18 *Let* Λ_Δ *be a multi-extension zero algebra with simply laced quiver* Q *and* Q*-data* $\Delta = (A, M)$. *There is a cohomology long exact sequence as follows:*

$$\begin{array}{llllllll}
0 & \to & 0 & \to & \mathsf{HH}^0(\Lambda_\Delta) & \to & \mathsf{HH}^0(A) & \to \\
 & & \mathsf{End}_{A-A} M & \to & \mathsf{HH}^1(\Lambda_\Delta) & \to & \mathsf{HH}^1(A) & \to \\
 & & \bigoplus_{\delta\in DQ_{\leq 2}} \mathsf{H}^2_\delta(\Delta) & \to & \mathsf{HH}^2(\Lambda_\Delta) & \to & \mathsf{HH}^2(A) \oplus \bigoplus_{\gamma\in CQ_2} \mathsf{H}^2_\gamma(\Delta) & \to \\
 & & \dots & & & & & \\
 & & \bigoplus_{\delta\in DQ_{\leq n}} \mathsf{H}^n_\delta(\Delta) & \to & \mathsf{HH}^n(\Lambda_\Delta) & \to & \mathsf{HH}^n(A) \oplus \bigoplus_{\substack{\gamma\in CQ_{\leq n}\\ \gamma\notin Q_0}} \mathsf{H}^n_\gamma(\Delta) & \to \\
 & & \dots & & & & &
\end{array}$$

Corollary 4.19 *If* Q *has no cycles - that is if* $CQ = Q_0$ *- the cohomology long exact sequence is as follows:*

$$\begin{array}{llllllll}
0 & \to & 0 & \to & \mathsf{HH}^0(\Lambda_\Delta) & \to & \mathsf{HH}^0(A) & \to \\
 & & \mathsf{End}_{A-A} M & \to & \mathsf{HH}^1(\Lambda_\Delta) & \to & \mathsf{HH}^1(A) & \to \\
 & & \bigoplus_{\delta\in DQ_{\leq 2}} \mathsf{H}^2_\delta(\Delta) & \to & \mathsf{HH}^2(\Lambda_\Delta) & \to & \mathsf{HH}^2(A) & \to \\
 & & \dots & & & & & \\
 & & \bigoplus_{\delta\in DQ_{\leq n}} \mathsf{H}^n_\delta(\Delta) & \to & \mathsf{HH}^n(\Lambda_\Delta) & \to & \mathsf{HH}^n(A) & \to \\
 & & \dots & & & & &
\end{array}$$

In Lemma 4.7 we have computed the cohomology along $\delta \in Q_m$ in its lowest degree:

$$\mathsf{H}^m_\delta = \mathsf{Hom}_{A_{t(\omega)}-A_{s(\omega)}}(M_\omega, \Delta_\omega).$$

In particular for an arrow a,

$$\mathsf{H}^1_a(\Delta) = \mathsf{End}_{A_{t(a)}-A_{s(a)}} M_a.$$

In the next section we will show that

$$\mathsf{H}_a^{r+1}(\Delta) = \mathsf{Ext}^r_{A_{t(a)}-A_{s(a)}}(M_a, M_a).$$

For longer paths, we will compute the cohomology under some additional hypotheses, regardless if the path is a non cycle or a cycle.

5 Cohomology along paths

Let Q be a simply laced quiver. Let Δ be a Q-data consisting in a set of algebras $\{A_x\}_{x\in Q_0}$ attached to the vertices of Q, and a bimodule M_a for each arrow $a \in Q_1$, where M_a is an $A_{t(a)} - A_{s(a)}$-bimodule.

Our aim is to compute the cohomology $\mathsf{H}^*_\omega(\Delta)$ along a path $\omega \in Q_m$, see Definition 4.14. In case $m \geq 2$, we will need extra assumptions in order to perform the computation. If ω is an arrow, no additional assumption is required.

Let $a \in Q_1$ and $n \geq 1$. A trajectory of duration n over a is $\tau_{q,p} = t(a)^q, a, s(a)^p$, where $p \geq 0$ and $q \geq 0$ verify $q + 1 + p = n$.

For simplicity, we set $B = A_{t(a)}$, $A = A_{s(a)}$ and $M = M_a$. Moreover, we omit tensor product symbols over k between vector spaces, and we replace tensors between elements by commas. Recall that

$$(\tau_{q,p})_\Delta = B^q M A^p.$$

Observe that $(\tau_{q,p})_0^+$ has two $(n+1)$-trajectories:

$$\tau_{q+1,p} = t(a)^{q+1}, a, s(a)^p \quad \text{and} \quad \tau_{q,p+1} = t(a)^q, a, s(a)^{p+1}.$$

Note that $T_0(a)$ is empty, $T_1(a)$ has one trajectory $t(a)^0, a, s(a)^0$, while $T_2(a)$ has two trajectories, $t(a)^1, a, s(a)^0$ and $t(a)^0, a, s(a)^1$.

These observations lead to the following explicit description of the complex which provides the cohomology along an arrow:

Lemma 5.1 *Given an arrow a, the complex of cochains $(\mathsf{K}_a^\bullet, \mathsf{d})$ which computes $\mathsf{H}^*_a(\Delta)$ is:*

$$\begin{aligned}0 \longrightarrow\ & \mathsf{Hom}_k(M,M) \xrightarrow{\mathsf{d}_1} \mathsf{Hom}_k(BM,M) \oplus \mathsf{Hom}_k(MA,M) \xrightarrow{\mathsf{d}_2} \\ & \mathsf{Hom}_k(B^2M,M) \oplus \mathsf{Hom}_k(BMA,M) \oplus \mathsf{Hom}_k(MA^2,M) \xrightarrow{\mathsf{d}_3} \cdots \xrightarrow{\mathsf{d}_{n-1}} \\ & \oplus_{p+q+1=n} \mathsf{Hom}_k(B^qMA^p,M) \xrightarrow{\mathsf{d}_n} \oplus_{p+q+1=n+1} \mathsf{Hom}_k(B^qMA^p,M) \xrightarrow{\mathsf{d}_{n+1}} \\ & \cdots\end{aligned}$$

where for $q+1+p=n$ and $f \in \mathsf{Hom}_k(B^qMA^p,M)$

$$\begin{aligned}(\mathsf{d}_n f)_{\tau_{q+1,p}}(b_1,\dots,b_{q+1},m,a_1,\dots,a_p) = \\ & b_1 f(b_2,\dots,b_{q+1},m,a_1,\dots,a_p) + \\ & \textstyle\sum_1^q (-1)^i f(b_1,\dots,b_ib_{i+1},\dots,b_{q+1},m,a_1,\dots,a_p) + \\ & (-1)^{q+1} f(b_1,\dots,b_{q+1}m,a_1,\dots,a_p)\end{aligned}$$

and

$$\begin{aligned}(\mathsf{d}_n f)_{\tau_{q,p+1}}(b_1,\dots,b_q,m,a_1,\dots,a_{p+1}) = \\ & (-1)^q[f(b_1,\dots,b_q,ma_1,\dots,a_{p+1}) + \\ & \textstyle\sum_1^p (-1)^i f(b_1,\dots,b_q,m,a_1,\dots,a_ia_{i+1},\dots,a_{p+1}) + \\ & (-1)^{p+1} f(b_1,\dots,b_q,m,a_1,\dots,a_p)a_{p+1}].\end{aligned}$$

In particular for $f \in \mathsf{Hom}_k(M, M)$:

$$(\mathsf{d}_1 f)_{\tau_{1,0}}(b, m) = bf(m) - f(bm) \quad \textit{and} \quad (\mathsf{d}_1 f)_{\tau_{0,1}}(m, a) = f(ma) - f(m)a.$$

We recall the bar resolution over an arbitrary algebra R of a left R-module X, by free R-modules.

$$\cdots \xrightarrow{\beta} R^n X \xrightarrow{\beta} R^{n-1}X \xrightarrow{\beta} \cdots \xrightarrow{\beta} R^2 X \xrightarrow{\beta} RX \xrightarrow{\beta} X \to 0$$

where

$$\beta(r_1, \dots, r_n, x) = \begin{array}{l} \sum_1^{n-1}(-1)^{i+1}(r_1, \dots, r_i r_{i+1}, \dots, r_n, x) + \\ (-1)^{n+1}(r_1 \dots, r_{n-1}, r_n x). \end{array}$$

Note that if $X = R$, then the bar resolution is also a resolution of R as R-bimodule.

The next two results have been obtained in [8], see also [9]. We recall their proof for further use. The first one provides a canonical resolution of M as $B - A$-bimodule, which is not its bar resolution over $B \otimes A^{\mathsf{op}}$. It is suitable in order to obtain the complex of cochains $\mathsf{K}_a^\bullet$ described above, which in turn will enable us to obtain the second result.

Lemma 5.2 *Let A and B be algebras, and let M be a $B - A$-bimodule. The following complex $\mathsf{C}^\bullet(M)$ is a free-resolution of M as bimodule,*

$$\cdots \to \bigoplus_{\substack{p+q=n+2 \\ p,q>0}} B^q M A^p \xrightarrow{\beta_n} \bigoplus_{\substack{p+q=n+1 \\ p,q>0}} B^q M A^p \to$$
$$\cdots \to BMA^2 \oplus B^2 MA \xrightarrow{\beta_1} BMA \xrightarrow{\beta_0} M \to 0$$

where $\beta_0(b, m, a) = bma$. For $n > 0$, the differential β_n is the sum of

$$\beta_n^{q,p} : B^q M A^p \longrightarrow B^{q-1} M A^p \ \oplus \ B^q M A^{p-1}$$

where $p + q = n + 2$ and

- *if $q \geq 2$ and $p \geq 2$, then $\beta_n^{q,p}$ can be written as the sum of two components ${}_{(1,0)}\beta_n^{q,p}$ and ${}_{(0,1)}\beta_n^{q,p}$ given by*

  $${}_{(1,0)}\beta_n^{q,p}(b_1, \dots, b_q, m, a_1, \dots, a_p) = \begin{array}{l} \sum_1^{q-1}(-1)^{i+1}(b_1, \dots, b_i b_{i+1}, \dots, b_q, m, a_1, \dots, a_p) + \\ (-1)^{q+1}(b_1, \dots, b_{q-1}, b_q m, a_1, \dots, a_p) \end{array}$$

  and analogously for ${}_{(0,1)}\beta_n^{q,p}$,

- *if $p = 1$ and $q \geq 2$, then*

  $$\beta_n^{q,1}(b_1, \dots, b_q, m, a_1) = \begin{array}{l} \sum_1^{q-1}(-1)^{i+1}(b_1, \dots, b_i b_{i+1}, \dots, b_q, m, a_1) + \\ (-1)^{q+1}(b_1, \dots, b_{q-1}, b_q m, a_1), \end{array}$$

- *and if $p \geq 2$ and $q = 1$, the definition of $\beta_n^{1,p}$ is analogous to that of $\beta_n^{q,1}$ in the previous item.*

Proof. Consider the bar resolution of M as a left B-module, and the bar resolution of A as A-bimodule:

$$\cdots \to BBM \to BM \to 0 \quad \text{and} \quad \cdots \to AAA \to AA \to 0.$$

Their tensor product over A provides the complex described in the statement. In order to use the Künneth formula (see for instance [25]), we first observe that the cycles in each degree of the bar resolution of A are projective left A-modules, since the resolution splits as a sequence of left A-modules. Hence the tensor product of the bar resolutions has zero homology in positive degrees, while in degree zero its homology is $M \otimes_A A = M$. $\diamond$

Theorem 5.3 *Let Q be a simply laced quiver with a Q-data Δ, and let $a \in Q_1$. The cohomology of Δ along a is as follows:*

$$\mathsf{H}_a^{1+r}(\Delta) = \mathsf{Ext}^r_{B-A}(M, M).$$

Proof. First we apply the functor $\mathsf{Hom}_{B-A}(-, M)$ to the previous resolution. Let Y and X be $B-A$-bimodules. The canonical isomorphism

$$\mathsf{Hom}_{B-A}(BYA, X) = \mathsf{Hom}_k(Y, X)$$

provides the complex of cochains $\mathsf{K}_a^\bullet$. $\diamond$

As a consequence, in case there are no paths of length greater than or equal to 2, the long exact sequence of Corollary 4.19 is simpler, as we prove in Corollary 5.4. Note that for this sort of quivers a multi-extension algebra is automatically a multi-extension zero algebra.

Corollary 5.4 *Let Q be a simply laced quiver with Q_2 empty. Let*

$$\Delta = (A = \times_{x \in Q_0} A_x,\ M = \oplus_{a \in Q_1} M_a)$$

be a Q-data and let Λ_Δ be the corresponding multi-extension algebra. The cohomology long exact sequence is:

$$\begin{array}{llllllll}
0 & \longrightarrow & 0 & \longrightarrow & \mathsf{HH}^0(\Lambda_\Delta) & \longrightarrow & \mathsf{HH}^0(A) & \longrightarrow \\
 & & \mathsf{End}_{A-A} M & \longrightarrow & \mathsf{HH}^1(\Lambda_\Delta) & \longrightarrow & \mathsf{HH}^1(A) & \longrightarrow \\
 & & \mathsf{Ext}^1_{A-A}(M, M) & \longrightarrow & \mathsf{HH}^2(\Lambda_\Delta) & \longrightarrow & \mathsf{HH}^2(A) & \longrightarrow \\
 & & \cdots & & & & & \\
 & & \mathsf{Ext}^{n-1}_{A-A}(M, M) & \longrightarrow & \mathsf{HH}^n(\Lambda_\Delta) & \longrightarrow & \mathsf{HH}^n(A) & \longrightarrow \\
 & & \cdots & & & & &
\end{array}$$

Proof. There are no cycles of positive length in Q, so we consider the long exact sequence of Corollary 4.19. For $n \geq 1$, we have $DQ_{\leq n} = Q_1$. Moreover we have just proven that the cohomology along an arrow a is $\mathsf{H}_a^{1+r}(\Delta) = \mathsf{Ext}^r_{A_{t(a)}-A_{s(a)}}(M_a, M_a)$. Finally notice that $\bigoplus_{a \in Q_1} \mathsf{Ext}^r_{A_{t(a)}-A_{s(a)}}(M_a, M_a) = \mathsf{Ext}^r_{A-A}(M, M)$. $\diamond$

Example 5.5 *Let $A_y[M]$ be a one point extension, where A_y is an algebra and M is a left A_y-module. By definition $A_y[M] = \begin{pmatrix} k & 0 \\ M & A_y \end{pmatrix} = \begin{pmatrix} A_y & M \\ 0 & k \end{pmatrix}$. A one point extension is a multi-extension algebra for the quiver $x\cdot \xrightarrow{a} \cdot y$ and the Q-data (A, M) where $A_x = k$ and $M_a = M$. Since there are no paths of length two, the long exact sequence of Corollary 5.4 considered for $A_y[M]$ is in force. It coincides with the long exact sequence obtained by D. Happel in [16]. Indeed, for a field k we have $\mathsf{HH}^0(k) = k$, while $\mathsf{HH}^n(k) = 0$ if $n > 0$. The long exact sequence is then as follows:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathsf{HH}^0(A_y[M]) & \longrightarrow & \mathsf{HH}^0(A_y) \oplus k & \longrightarrow \\
 & & \mathsf{End}_{A_y} M & \longrightarrow & \mathsf{HH}^1(A_y[M]) & \longrightarrow & \mathsf{HH}^1(A_y) & \longrightarrow \\
 & & \mathsf{Ext}^1_{A_y}(M, M) & \longrightarrow & \mathsf{HH}^2(A_y[M]) & \longrightarrow & \mathsf{HH}^2(A_y) & \longrightarrow \\
 & & \dots & & & & & \\
 & & \mathsf{Ext}^{n-1}_{A_y}(M, M) & \longrightarrow & \mathsf{HH}^n(A_y[M]) & \longrightarrow & \mathsf{HH}^n(A_y) & \longrightarrow \\
 & & \dots & & & & &
\end{array}
$$

Example 5.6 *Let again Q be the quiver $x\cdot \xrightarrow{a} \cdot y$. Let Δ be a Q-data. The corresponding multi-extension algebra is the corner algebra*

$$
\begin{pmatrix} A_x & 0 \\ M_a & A_y \end{pmatrix} = \begin{pmatrix} A_y & M_a \\ 0 & A_x \end{pmatrix}.
$$

The long exact sequence of cohomology has been obtained in this case independently in [8], [22] and [15].

Next we consider a simply laced quiver Q and a path $a_2a_1 \in Q_2$. We set $C = A_{t(a_2)}$, $B = A_{s(a_2)} = A_{t(a_1)}$ and $A = A_{s(a_1)}$.

Lemma 5.7 *The complex of cochains $(\mathsf{K}^\bullet_{a_2a_1}, \mathsf{d})$ which computes $\mathsf{H}^*_{a_2a_1}(\Delta)$ is as follows:*

$$
\begin{aligned}
&0 \longrightarrow \mathsf{Hom}_k(M_{a_2}M_{a_1}, X) \xrightarrow{\mathsf{d}_1} \\
&\mathsf{Hom}_k(CM_{a_2}M_{a_1}, X) \oplus \mathsf{Hom}_k(M_{a_2}BM_{a_1}, X) \oplus \mathsf{Hom}_k(M_{a_2}M_{a_1}A, X) \xrightarrow{\mathsf{d}_2} \\
&\dots \\
&\xrightarrow{\mathsf{d}_{n-2}} \bigoplus\nolimits_{p+q+r+2=n} \mathsf{Hom}_k(C^rM_{a_2}B^qM_{a_1}A^p, X) \xrightarrow{\mathsf{d}_{n-1}} \\
&\qquad\qquad \bigoplus\nolimits_{p+q+r+2=n+1} \mathsf{Hom}_k(C^rM_{a_2}B^qM_{a_1}A^p, X) \xrightarrow{\mathsf{d}_n} \\
&\dots
\end{aligned}
$$

Proof. Note that a trajectory $\tau_{r,q,p} \in T_n(a_2a_1)$ is

$$
\tau_{r,q,p} = t(a_2)^r, a_2, s(a_2)^q, a_1, s(a_1)^p
$$

such that $r + q + p + 2 = n$. Moreover

$$
(\tau_{r,q,p})_\Delta = C^rM_{a_2}B^qM_{a_1}A^p.
$$

The stated complex equals the one of Definition 4.14 with $\omega = a_2a_1$. $\diamond$

Theorem 5.8 *Let M_{a_1} be a $B-A$-bimodule and let M_{a_2} be a $C-B$-bimodule, corresponding to $\omega = a_2a_1 \in Q_2$ as previously. If $\mathsf{Tor}_B^n(M_{a_2}, M_{a_1}) = 0$ for $n > 0$, then*

$$\mathsf{H}^{2+r}_{a_2a_1}(\Delta) = \mathsf{Ext}^r_{C-A}(M_\omega, \Delta_\omega)$$

where M_ω is the bimodule defined in 4.6.

Proof. Let $\mathsf{C}^\bullet(M_{a_2})$ and $\mathsf{C}^\bullet(M_{a_1})$ be the free resolutions of Lemma 5.2. In particular, they are right and left B-projective resolutions of M_{a_2} and M_{a_1} respectively. Consequently, the homology of the complex $\mathsf{C}^\bullet(M_{a_2}) \otimes_B \mathsf{C}^\bullet(M_{a_1})$ is $\mathsf{Tor}^\bullet_B(M_{a_2}, M_{a_1})$, see for instance [21, Theorem 9.3]. Our assumption insures that this complex is a free $C-A$ resolution of $M_{a_2a_1}$. Applying the functor $\mathsf{Hom}_{C-A}(-, \Delta_{a_2a_1})$ to it, and using the canonical isomorphism $\mathsf{Hom}_{C-A}(CYA, X) = \mathsf{Hom}_k(Y, X)$, yields the complex of cochains $\mathsf{K}^\bullet_{a_2a_1}$. $\diamond$

Theorem 5.9 *Let $\omega = a_3a_2a_1 \in Q_3$. Let $A = A_{s(a_1)}$, $B = A_{s(a_2)}$ $C = A_{s(a_3)}$ and $D = A_{t(a_3)}$. If $\mathsf{Tor}_n^B(M_{a_2}, M_{a_1}) = \mathsf{Tor}_n^C(M_{a_3}, M_{a_2a_1}) = 0$ for $n > 0$, then*

$$\mathsf{H}^{3+r}_\omega(\Delta) = \mathsf{Ext}^r_{D-A}(M_\omega, \Delta_\omega).$$

Proof. The tensor product of the resolutions of M_{a_2} and M_{a_1} provides as before a free $C-A$ resolution of $M_{a_2a_1}$. In turn, we tensorize it with the resolution of M_{a_3}, obtaining a free $D-A$ resolution of $M_{a_3a_2a_1}$. Applying the appropriate functor yields $\mathsf{K}^\bullet_{a_3a_2a_1}$. $\diamond$

Now, we consider the general situation.

Definition 5.10 *Let Q be a simply laced quiver and let Δ be a Q-data. A path $\omega = a_m \dots a_1$ of length $m \geq 2$ is* Tor-vanishing *if*

$$\mathsf{Tor}_n^{A_{s(a_i)}}(M_{a_i}, M_{a_{i-1}\dots a_1}) = 0$$

for $i = 2, \dots, m$ and for all $n > 0$.

The following result is a straightforward generalization of the previous theorem.

Theorem 5.11 *Let Q be a simply laced quiver, let Δ be a Q-data and let $\omega \in Q_m$ with $m \geq 2$, be a Tor vanishing path. For $r \geq 0$, the following holds*

$$\mathsf{H}^{m+r}_\omega(\Delta) = \mathsf{Ext}^r_{A_{t(w)}-A_{s(w)}}(M_\omega, \Delta_\omega).$$

Definition 5.12 *A Q-data is* Tor-vanishing *if all the paths of length $m \geq 2$ are Tor vanishing.*

Observe that if Q is a quiver without cycles and if its maximal paths of length $m \geq 2$ are Tor vanishing, then the Q-data is Tor vanishing.

Corollary 5.13 *Let Q be a simply laced quiver without cycles. Let $\Delta = (A, M)$ be a Q-data, and suppose that the maximal paths - excepting arrows - are Tor vanishing. Let Λ_Δ be the corresponding multi-extension zero algebra. There is a cohomology long exact sequence as follows*

$$
\begin{array}{rcccccc}
0 & \to & \mathsf{HH}^0(\Lambda_\Delta) & \to & \mathsf{HH}^0(A) & \to \\
\mathsf{End}_A M & \to & \mathsf{HH}^1(\Lambda_\Delta) & \to & \mathsf{HH}^1(A) & \to \\
\mathsf{Hom}_{A-A}(M^{\otimes_A 2}, M) \oplus \mathsf{Ext}^1_{A-A}(M,M) & \to & \mathsf{HH}^2(\Lambda_\Delta) & \to & \mathsf{HH}^2(A) & \to \\
\cdots & & & & & \\
\bigoplus_{r+s=n} \mathsf{Ext}^r_{A-A}(M^{\otimes_A s}, M) & \to & \mathsf{HH}^n(\Lambda_\Delta) & \to & \mathsf{HH}^n(A) & \to \\
\cdots & & & & &
\end{array}
$$

Proof. Since $CQ_m = \emptyset$ for $m \neq 0$, we consider the long exact sequence of Corollary 4.19. For a path ω of length $m \geq 2$, Theorem 5.11 provides

$$
\mathsf{H}^{m+r}_\omega(\Delta) = \mathsf{Ext}^r_{A_{t(w)}-A_{s(w)}}(M_\omega, \Delta_\omega).
$$

Observe that $\bigoplus_{\omega \in Q_m} M_\omega = M^{\otimes_A m}$. Moreover if a is an arrow such that $M_a \neq \Delta_\omega$, then $\mathsf{Ext}^r_{A_{t(w)}-A_{s(w)}}(M_\omega, M_a) = 0$. We infer

$$
\bigoplus_{\omega \in Q_m} \mathsf{Ext}^*_{A_{t(\omega)}-A_{s(\omega)}}(M_\omega, \Delta_\omega) = \mathsf{Ext}^*_{A-A}(M^{\otimes_A m}, M). \tag{5.1}
$$

$\diamond$

In case Q has oriented cycles, we denote

$$
(M^{\otimes_A m})_D = \bigoplus_{\delta \in DQ_m} M_\delta \quad \text{and} \quad (M^{\otimes_A m})_C = \bigoplus_{\gamma \in CQ_m} M_\gamma.
$$

There is a decomposition

$$
M^{\otimes_A m} = (M^{\otimes_A m})_D \oplus (M^{\otimes_A m})_C. \tag{5.2}
$$

Corollary 5.14 *Let Q be a simply laced quiver and $\Delta = (A, M)$ a Tor vanishing Q-data, and let Λ_Δ be the corresponding multi-extension zero algebra. There is a cohomology long exact sequence as follows*

$$
\begin{array}{rcccccc}
0 & \to & \mathsf{HH}^0(\Lambda_\Delta) & \to & \mathsf{HH}^0(A) & \to \\
\mathsf{End}_{A-A} M & \to & \mathsf{HH}^1(\Lambda_\Delta) & \to & \mathsf{HH}^1(A) & \to \\
\mathsf{Hom}_{A-A}((M^{\otimes_A 2})_D, M) \oplus \mathsf{Ext}^1_{A-A}(M,M) & \to & \mathsf{HH}^2(\Lambda_\Delta) & \to & \mathsf{HH}^2(A) \oplus \mathsf{Hom}_{A-A}((M^{\otimes_A 2})_C, A) & \to \\
\cdots & & & & & \\
\bigoplus^n_{r+s=n} \mathsf{Ext}^r_{A-A}((M^{\otimes_A s})_D, M) & \to & \mathsf{HH}^n(\Lambda_\Delta) & \to & \mathsf{HH}^n(A) \oplus \bigoplus_{r+s=n} \mathsf{Ext}^r_{A-A}((M^{\otimes_A s})_C, A) & \to \\
\cdots & & & & &
\end{array}
$$

Corollary 5.15 *Let Q be a simply laced quiver and $\Delta = (A, M)$ a Q-data where M is projective as A-bimodule. Let Λ_Δ be the corresponding multi-extension zero algebra. The cohomology long exact sequence is as follows*

$$
\begin{array}{rcccccc}
0 & \to & \mathsf{HH}^0(\Lambda_\Delta) & \to & \mathsf{HH}^0(A) & \to \\
\mathsf{End}_{A-A} M & \to & \mathsf{HH}^1(\Lambda_\Delta) & \to & \mathsf{HH}^1(A) & \to \\
\mathsf{Hom}_{A-A}((M^{\otimes_A 2})_D, M) & \to & \mathsf{HH}^2(\Lambda_\Delta) & \to & \mathsf{HH}^2(A) \oplus \mathsf{Hom}_{A-A}((M^{\otimes_A 2})_C, A) & \to \\
\cdots & & & & & \\
\mathsf{Hom}_{A-A}((M^{\otimes_A n})_D, M) & \to & \mathsf{HH}^n(\Lambda_\Delta) & \to & \mathsf{HH}^n(A) \oplus \mathsf{Hom}_{A-A}((M^{\otimes_A n})_C, A) & \to \\
\cdots & & & & &
\end{array}
$$

Proof. Recall that $A = \times_{x\in Q_0} A_x$ and $M = \oplus_{a\in Q_1} M_a$ where M_a is an $A_{t(a)} - A_{s(a)}$-bimodule. Hence M is projective as A-bimodule if and only if for every $a \in Q_1$ the bimodule M_a is projective. In turn this implies that the Q-data is Tor vanishing. The previous corollary therefore gives the long exact sequence. $\diamond$

6 Multiplicative structures

The Hochschild cohomology of an algebra Λ is an associative algebra with the cup product. Gerstenhaber in [12] proved that it is graded commutative. We recall that if f' and f are cochains of degrees n' and n of the complex of Lemma 3.1 - for $Z = \Lambda$, their *cup product* $f' \smile f$ is the composition

$$\Lambda^{\otimes_D (n'+n)} \cong \Lambda^{\otimes_D n'} \otimes_D \Lambda^{\otimes_D n} \xrightarrow{f'\otimes_D f} \Lambda \otimes_D \Lambda \longrightarrow \Lambda$$

where the last map is the product of Λ. The graded Leibniz rule

$$d(f' \smile f) = d(f') \smile f + (-1)^{n'} f' \smile d(f)$$

holds, so the cup product is well defined in cohomology.

Our next purpose is to consider a cup product on the cohomology along paths of a simply laced quiver Q provided with a data Δ.

Definition 6.1 *Let ω' and ω be paths of Q and let τ' and τ be trajectories over them, of durations n' and n respectively. Let $f'_{\tau'} : \tau'_\Delta \to \Delta_{\omega'}$ and $f_\tau : \tau_\Delta \to \Delta_\omega$ be cochains of $(\mathsf{K}^\bullet_{\omega'}, \mathsf{d})$ and $(\mathsf{K}^\bullet_{\omega}, \mathsf{d})$ respectively - see Definition 4.14. Their* cup product *$f'_{\tau'} \smile f_\tau$ is as follows:*

- *If ω' and ω are not concatenable - that is if $s(\omega') \neq t(\omega)$, then $f'_{\tau'} \smile f_\tau = 0$.*
- *If ω' and ω are concatenable, let $\tau'\tau$ be the obvious trajectory of duration $n' + n$ over the concatenated path $\omega'\omega$. The* cup product *$f'_{\tau'} \smile f_\tau$ is the composition*

$$(\tau'\tau)_\Delta = \tau'_\Delta \otimes \tau_\Delta \xrightarrow{f'_{\tau'}\otimes f_\tau} \Delta_{\omega'} \otimes \Delta_\omega \longrightarrow \Delta_{\omega'\omega}$$

where the last map is the product in the multi-extension zero algebra Λ_Δ.

Remark 6.2 *If both ω' and ω are not cycles, then the above product map is zero.*

Proposition 6.3 *The graded Leibniz rule*

$$\mathsf{d}(f' \smile f) = \mathsf{d}(f') \smile f + (-1)^{n'} f' \smile \mathsf{d}(f)$$

holds, and there is a well defined cup product

$$\mathsf{H}^{n'}_{\omega'} \otimes \mathsf{H}^{n}_{\omega} \longrightarrow \mathsf{H}^{n'+n}_{\omega'\omega},$$

which is zero if either ω' and ω are not concatenable or if both of them are not cycles.

The proof of the proposition is the usual one, taking into account the different cases which occur in this setting.

Remark 6.4 *Let xQx be the set of cycles of Q at the vertex x. The cup product defined above provides an algebra structure on*

$$\bigoplus_{\omega \in xQx} \mathsf{H}^{\bullet}_{\omega}.$$

Moreover $\mathsf{HH}^{\bullet}(A_x)$ is a subalgebra and its complement in the direct sum is a two-sided ideal.

The proof of the following result is straightforward.

Theorem 6.5 *Let Q be a simply laced quiver, let Δ be a Q-data and let Λ_Δ be the corresponding multi-extension zero algebra.*

- *The cup product in $\mathsf{HH}^n(\Lambda_\Delta)$, restricted to the images of the maps*

$$\bigoplus_{\delta \in DQ_{\leq n}} \mathsf{H}^n_{\delta}(\Delta) \longrightarrow \mathsf{HH}^n(\Lambda_\Delta)$$

  of the cohomology long exact sequence of Theorem 4.18 is zero. In other words, the cup product annihilates on cocycles which belong to $\mathsf{D}^{\bullet}$.

- *The family of maps*

$$\mathsf{HH}^n(\Lambda_\Delta) \longrightarrow \mathsf{HH}^n(A) \oplus \bigoplus_{\substack{\gamma \in CQ_{\leq n} \\ \gamma \notin Q_0}} \mathsf{H}^n_{\gamma}(\Delta)$$

  of the cohomology long exact sequence of Theorem 4.18 provide a graded algebra map.

Our next purpose is to give a formula for the connecting homomorphism

$$\mathsf{HH}^n(A) \oplus \bigoplus_{\substack{\gamma \in CQ_{\leq n} \\ \gamma \notin Q_0}} \mathsf{H}^n_{\gamma}(\Delta) \xrightarrow{\nabla_n} \bigoplus_{\delta \in DQ_{\leq n+1}} \mathsf{H}^{n+1}_{\delta}(\Delta)$$

of the cohomology long exact sequence of Theorem 4.18.

Observe that for each arrow a the identity map 1_{M_a} is a canonical element of $\mathsf{H}^1_a(\Delta) = \mathsf{End}_{A_{t(a)}-A_{s(a)}} M_a$. Moreover, $1_M = \sum_{a \in Q_1} 1_{M_a} \in \mathsf{End}_{A-A} M$.

Theorem 6.6 *For $m \leq n$, let $\gamma \in CQ_m$ and let $f \in \mathsf{H}^n_{\gamma}(\Delta)$. The following holds:*

$$\nabla_n(f) = 1_M \smile f + (-1)^{n+1} f \smile 1_M.$$

Moreover, ∇_n is of degree 1 with respect to the length of the paths, that is

$$\nabla_n(f) \in \bigoplus_{\delta \in DQ_{m+1}} \mathsf{H}^{n+1}_{\delta}(\Delta).$$

Proof. Let τ be a trajectory over γ of duration n. We consider a cocycle f_τ in $\mathsf{C}^n = (\mathsf{J}/\mathsf{D})^n$. Note that f_τ has its image contained in A_x, where $x = s(\gamma) = t(\gamma)$. In order to compute $\nabla_n(f_\tau)$, we view f_τ as a cochain in J^n, thus the formula 4.1 provides the equalities

$$df_\tau = \sum_{\sigma \in \tau^+} (df_\tau)_\sigma = \sum_{\sigma \in \tau_1^+} (df_\tau)_\sigma.$$

Recall that τ_1^+ is the set of the $n+1$-trajectories

$$t(c)^0, c, s(c)^{p_{m+1}}, a_m, s(a_m)^{p_m}, \cdots, s(a_2)^{p_2}, a_1, s(a_1)^{p_1}$$

where $c \in Q_1$ is any arrow after ω, joint with the analogous set of trajectories obtained for any arrow c before ω. Consequently

$$df_\tau = \sum_{c \in xQ_1} (1_{M_c} \smile f_\tau) + (-1)^{n+1} \sum_{c \in Q_1 x} (f_\tau \smile 1_{M_c}).$$

We know that $1_M = \sum_{a \in Q_1} 1_{M_a}$. Moreover, if $s(\gamma) \neq t(a)$, the map $1_{M_a} \smile f_\tau$ is zero already at the cochain level. These observations lead to the formula. $\diamond$

Remark 6.7 *Let $A_y[M]$ be a one point extension corresponding to the quiver $x\cdot \longrightarrow \cdot y$, see Example 5.5. For $n > 0$, the connecting homomorphism*

$$\nabla_n : \mathsf{HH}^n(A_y) \longrightarrow \mathsf{Ext}^n_{A_y}(M, M)$$

is as follows. If $f \in \mathsf{HH}^n(A_y)$, then

$$\nabla_n(f) = (-1)^{n+1}(f \smile 1_M).$$

Indeed, $1_M \smile f = 0$ for no-concatenation reasons. Moreover, since $A_x = k$, the cohomology $\mathsf{HH}^\bullet(A_x)$ is concentrated in degree 0, where its value is k.

Through the previous remark, we end this section by linking our work to some of the results of [14]. In that paper, an algebra map is constructed and it is shown that it coincides with the connecting homomorphism of the long exact sequence. Consequently this map is the above ∇.

It is straightforward to check that the family of maps ∇_n for one-point extensions given by $\nabla_n(f) = (-1)^{n+1}(f \smile 1_M)$ is indeed a graded algebra map.

Another way of considering the same fact is as follows. Let A be an algebra and let M be a left A-module. It is well known that the Hochschild cohomology of the A-bimodule $\mathsf{End}_k\, M$ verifies

$$\mathsf{H}^n(A, \mathsf{End}_k\, M) = \mathsf{Ext}^n_A(M, M).$$

Moreover, $\mathsf{End}_k\, M$ being an A-bimodule, $\mathsf{H}^\bullet(A, \mathsf{End}_k\, M)$ is a $\mathsf{HH}^\bullet(A)$-graded bimodule. The point is that for this specific bimodule $\mathsf{End}_k\, M$ there is a canonical element

$$1_M \in \mathsf{H}^0(A, \mathsf{End}_k\, M) = \mathsf{End}_A\, M.$$

In fact $\mathsf{End}_k\, M$ is more than an A-bimodule, it is also an A-module algebra for the composition, that is the structural product map is A-balanced and is an A-bimodule map. An immediate consequence of this is that $\mathsf{H}^n(A, \mathsf{End}_k\, M)$ is an algebra, which is canonically isomorphic to the algebra $\mathsf{Ext}^n_A(M, M)$ with the Yoneda product.

Proposition 6.8 *The morphism*

$$\mathsf{HH}^n(A) \longrightarrow \mathsf{H}^n(A, \mathsf{End}_k M) = \mathsf{Ext}^n_A(M, M)$$

given by the (left or right) action on the canonical element 1_M is a morphism of algebras.

7 Square projective algebras

In this section we focus on null-square projective algebras, see for instance [10].

A *square algebra* is a multi-extension algebra built on the following:

- the round trip quiver
$$Q = {}_x\cdot \rightleftarrows \cdot_y,$$
where the arrow from x to y is denoted a and the reverse one is denoted b,
- a Q-data $(\mathbb{A}, \mathbb{M})$ given by
  - the algebras $A_x = A$, $A_y = B$ and the *corner bimodules* $M_a = M$ and $M_b = N$, hence $\mathbb{A} = A \times B$ and $\mathbb{M} = M \oplus N$,
  - the bimodule maps
  $$\alpha : N \otimes_B M \to A \quad \text{and} \quad \beta : M \otimes_A N \to B$$
  verifying *associativity constraints*, that is the following diagrams commute:

$$\begin{tikzcd} M \otimes_A N \otimes_B M \arrow[r, "1\otimes\alpha"] \arrow[d, "\beta\otimes 1"'] & M \otimes_A A \arrow[d] \\ B \otimes_B M \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} N \otimes_B M \otimes_A N \arrow[r, "1\otimes\beta"] \arrow[d, "\alpha\otimes 1"'] & N \otimes_B B \arrow[d] \\ A \otimes_A N \arrow[r] & N \end{tikzcd} \tag{7.1}$$

The square algebra is $\begin{pmatrix} A & N \\ M & B \end{pmatrix}$ with matrix multiplication such that if $m \in M$ and $n \in N$, then

$$\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix}\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \beta(m \otimes n) \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} = \begin{pmatrix} \alpha(n \otimes m) & 0 \\ 0 & 0 \end{pmatrix}.$$

We note that an algebra Λ with a chosen idempotent e provides a square algebra $\Lambda = \begin{pmatrix} e\Lambda e & e\Lambda f \\ f\Lambda e & f\Lambda f \end{pmatrix}$, where $f = 1 - e$.

Definition 7.1

- *A* square projective algebra *is a square algebra such that $\mathbb{M}$ is a projective $\mathbb{A}$-bimodule, or, equivalently, the corner bimodules are projective.*
- *A* null-square algebra *is a square algebra such that $\alpha = \beta = 0$.*

- *A* null-square projective algebra *is a square algebra verifying both previous requirements.*

Proposition 7.2 *Let Λ be a null-square projective algebra. For $m>0$, there is a five-term exact sequence as follows:*

$$0 \longrightarrow \mathsf{HH}^{2m}(\Lambda) \longrightarrow \mathsf{HH}^{2m}(\mathbb{A}) \oplus \mathsf{Hom}_{\mathbb{A}-\mathbb{A}}\left(\mathbb{M}^{\otimes_{\mathbb{A}} 2m}, \mathbb{A}\right) \xrightarrow{\nabla_{2m}}$$
$$\mathsf{Hom}_{\mathbb{A}-\mathbb{A}}\left(\mathbb{M}^{\otimes_{\mathbb{A}} 2m+1}, \mathbb{M}\right) \longrightarrow \mathsf{HH}^{2m+1}(\Lambda) \longrightarrow \mathsf{HH}^{2m+1}(\mathbb{A}) \longrightarrow 0.$$

For $m=0$ the exact sequence:

$$0 \longrightarrow \mathsf{HH}^{0}(\Lambda) \longrightarrow \mathsf{HH}^{0}(\mathbb{A}) \xrightarrow{\nabla_0} \mathsf{End}_{\mathbb{A}-\mathbb{A}}\,\mathbb{M} \longrightarrow \mathsf{HH}^{1}(\Lambda) \longrightarrow \mathsf{HH}^{1}(\mathbb{A}) \longrightarrow 0.$$

Proof. A null-square algebra is built on the round trip quiver

$$Q = \cdot \rightleftarrows \cdot$$

where cycles are precisely the paths of even length. If n is odd, then $(\mathbb{M}^{\otimes_{\mathbb{A}} n})_C = 0$, and $(\mathbb{M}^{\otimes_{\mathbb{A}} n})_D = \mathbb{M}^{\otimes_{\mathbb{A}} n}$, while if $n>0$ is even, then $(\mathbb{M}^{\otimes_{\mathbb{A}} n})_D = 0$ and $(\mathbb{M}^{\otimes_{\mathbb{A}} n})_C = \mathbb{M}^{\otimes_{\mathbb{A}} n}$. Corollary 5.15 applies since $\mathbb{M}$ is a projective bimodule. The previous observations show that the cohomology long exact sequence of Corollary 5.15 splits into five-term exact sequences. $\diamond$

Let $\nabla'_{2m} : \mathsf{Hom}_{\mathbb{A}-\mathbb{A}}\left(\mathbb{M}^{\otimes_{\mathbb{A}} 2m}, \mathbb{A}\right) \longrightarrow \mathsf{Hom}_{\mathbb{A}-\mathbb{A}}\left(\mathbb{M}^{\otimes_{\mathbb{A}} 2m+1}, \mathbb{M}\right)$ be the restriction of ∇_{2m} to the second direct summand.

Theorem 7.3 *Let Λ be a null-square projective algebra as before. For $m>0$, the following holds:*

- $\mathsf{HH}^{2m}(\Lambda) = \mathsf{H}^{2m}(\mathbb{A}) \oplus \mathsf{Ker}\nabla'_{2m}$.
- *There is a short exact sequence*

$$0 \to \mathsf{Coker}\nabla'_{2m} \to \mathsf{H}^{2m+1}(\Lambda) \to \mathsf{H}^{2m+1}(\mathbb{A}) \to 0.$$

Proof. We assert that ∇_{2m} restricted to $\mathsf{HH}^{2m}(\mathbb{A})$ is zero, for $m>0$. Indeed, we know that ∇_{2m} is of degree 1 with respect to the length of paths, see Theorem 6.6. Hence

$$\nabla_{2m}(\mathsf{HH}^{2m}(\mathbb{A})) \ \subset\ \mathsf{HH}_a^{2m+1}(\Delta) \oplus \mathsf{HH}_b^{2m+1}(\Delta) = \mathsf{Ext}^{2m}_{\mathbb{A}-\mathbb{A}}(\mathbb{M},\mathbb{M}) = 0$$

since $\mathbb{M}$ is projective. Consequently, $\mathsf{Ker}\nabla_{2m} = \mathsf{H}^{2m}(\mathbb{A}) \oplus \mathsf{Ker}\nabla'_{2m}$. This provides the required decomposition of $\mathsf{HH}^{2m}(\Lambda)$ since the first map of the previous five-term exact sequence is injective.

Moreover $\mathsf{Coker}\nabla_{2m} = \mathsf{Coker}\nabla'_{2m}$, and the exact sequence of the statement follows. $\diamond$

Corollary 7.4 *Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a null-square projective algebra.*

- *If $n>0$ and Λ is finite dimensional, then*

$$\dim \mathsf{HH}^n(\Lambda) \geq \dim \mathsf{HH}^n(A) + \dim \mathsf{HH}^n(B).$$

- *If there is a positive integer h such that $\mathbb{M}^{\otimes_{\mathbb{A}} h} = 0$, then for $n \geq h$*

$$\mathsf{HH}^n(\Lambda) = \mathsf{HH}^n(A) \oplus \mathsf{HH}^n(B).$$

Proof. Since $\mathbb{A} = A \times B$, we have $\mathsf{HH}^n(\mathbb{A}) = \mathsf{HH}^n(A) \oplus \mathsf{HH}^n(B)$. The previous theorem provides the results. $\diamond$

Next we will consider square projective algebras where the corner bimodules are free bimodules of rank one, that is $M = BA$ and $N = AB$ - recall that we drop tensor product symbols over k. We will first prove that the bimodule morphisms α and β are necessarily zero, except if $A = B = k$. Next we will show that ∇'_{2m} is injective for $m > 0$.

Theorem 7.5 *Let $\Lambda = \begin{pmatrix} A & AB \\ BA & B \end{pmatrix}$ be a square algebra where the corner bimodules are free of rank one. The algebra Λ is a null-square algebra, except if $A = B = k$.*

Proof. Let $\alpha : N \otimes_B M \to A$ and $\beta : M \otimes_A N \to B$ be the bimodule maps of the data verifying the associativity constraints (7.1) for $M = BA$ and $N = AB$. We will prove that $\alpha = \beta = 0$. Notice that

$$\mathsf{Hom}_{A-A}(N \otimes_B M, A) = \mathsf{Hom}_{A-A}(AB \otimes_B BA, A) = \mathsf{Hom}_{A-A}(ABA, A) = \mathsf{Hom}_k(B, A).$$

Let $\overline{\alpha} \in \mathsf{Hom}_k(B, A)$ be the linear map corresponding to α through the composition of the previous canonical isomorphisms. Similarly, let $\overline{\beta} \in \mathsf{Hom}_k(A, B)$ be the linear map corresponding to β. A simple computation shows that the associativity constraints (7.1) are equivalent to the following for every $a \in A$ and $b \in B$:

$$1 \otimes a\overline{\alpha}(b) = \overline{\beta}(a)b \otimes 1 \ \in BA \tag{7.2}$$
$$1 \otimes b\overline{\beta}(a) = \overline{\alpha}(b)a \otimes 1 \ \in AB. \tag{7.3}$$

Let A' and B' be vector subspaces of A and B, such that $A = k \oplus A'$ and $B = k \oplus B'$. For $a \in A$ and $b \in B$, let $a = a_1 + a'$ and $b = b_1 + b'$ be the corresponding decompositions.

The equality (7.2) for $a = b = 1$ gives $1 \otimes \overline{\alpha}(1) = \overline{\beta}(1) \otimes 1$, then

$$1 \otimes \overline{\alpha}(1)_1 + 1 \otimes \overline{\alpha}(1)' = \overline{\beta}(1)_1 \otimes 1 + \overline{\beta}(1)' \otimes 1.$$

The tensors $1 \otimes \overline{\alpha}(1)_1$ and $\overline{\beta}(1)_1 \otimes 1$ are both in $k \otimes_k k$, while $1 \otimes \overline{\alpha}(1)' \in k \otimes_k A'$ and $\overline{\beta}(1)' \otimes 1 \in B' \otimes_k k$ belong to different direct summands of BA, which implies that they are both zero. Moreover $\overline{\alpha}(1)' = 0 = \overline{\beta}(1)'$. Consequently there is $\lambda \in k$ such that

$$\overline{\alpha}(1) = \overline{\alpha}(1)_1 = \lambda = \overline{\beta}(1)_1 = \overline{\beta}(1).$$

For all $a \in A$ and $b = 1$, the equality (7.2) gives $1 \otimes a\overline{\alpha}(1) = \overline{\beta}(a) \otimes 1$, hence

$$1 \otimes \lambda a = \overline{\beta}(a) \otimes 1 \tag{7.4}$$

that is

$$1 \otimes \lambda a_1 + 1 \otimes \lambda a' = \overline{\beta}(a)_1 \otimes 1 + \overline{\beta}(a)' \otimes 1.$$

If $\lambda \neq 0$, then $a' = 0$ for every $a \in A$; the same computation for the other associativity constraints provides $b' = 0$ for every $b \in B$, that is $A = B = k$. If $\lambda = 0$ we infer from (7.4) that $\overline{\beta} = 0$; similarly $\overline{\alpha} = 0$. $\diamond$

Proposition 7.6 *Let Λ be as in Theorem 7.5. The morphism ∇'_{2m} is injective for $m > 0$, except if $A = k$ and $B = k$.*

Proof. From the previous result, Λ is a null-square projective algebra. We first pay attention to the general case where the corner projective bimodules M and N are not necessarily free of rank one. Consider

$$\nabla'_{2m} : \mathsf{Hom}_{\mathbb{A}-\mathbb{A}}\left(\mathbb{M}^{\otimes_{\mathbb{A}} 2m}, \mathbb{A}\right) \to \mathsf{Hom}_{\mathbb{A}-\mathbb{A}}\left(\mathbb{M}^{\otimes_{\mathbb{A}} 2m+1}, \mathbb{M}\right)$$

and the vector space decomposition

$$\mathsf{Hom}_{A-A}\left((N \otimes_B M)^{\otimes_A m}, A\right) \oplus \mathsf{Hom}_{B-B}\left((M \otimes_A N)^{\otimes_B m}, B\right) \xrightarrow{\nabla'_{2m}}$$

$$\mathsf{Hom}_{B-A}\left(M \otimes_A (N \otimes_B M)^{\otimes_A m}, M\right) \oplus \mathsf{Hom}_{A-B}\left((M \otimes_A N)^{\otimes_B m} \otimes_B M, N\right).$$

Let $[\nabla'_{2m}]_M$ and $[\nabla'_{2m}]_N$ be the components of ∇'_{2m} with values in the first and in the second target summands. Hence

$$\mathsf{Ker}\nabla'_{2m} = \mathsf{Ker}\,[\nabla'_{2m}]_M \cap \mathsf{Ker}\,[\nabla'_{2m}]_N\,.$$

Moreover, for (φ, ψ) in the source vector space, Theorem 6.6 provides

$$[\nabla'_{2m}]_M\,(\varphi, \psi) = 1_M \otimes \varphi - \psi \otimes 1_N.$$

Then

$$\mathsf{Ker}\,[\nabla'_{2m}]_M = \{(\varphi, \psi) \;\mid\; 1_M \otimes \varphi = \psi \otimes 1_N\}.$$

In other words, let

$$\mu : \mathsf{Hom}_{A-A}\left((N \otimes_B M)^{\otimes_A m}, A\right) \to \mathsf{Hom}_{B-A}\left(M \otimes_A (N \otimes_B M)^{\otimes_A m}, M\right)$$

$$\nu : \mathsf{Hom}_{B-B}\left((M \otimes_A N)^{\otimes_B m}, B\right) \to \mathsf{Hom}_{B-A}\left(M \otimes_A (N \otimes_B M)^{\otimes_A m}, M\right)$$

be defined by $\mu(\varphi) = 1_M \otimes \varphi$ and $\nu(\psi) = \psi \otimes 1_N$. Notice that $\mathsf{Ker}\,[\nabla'_{2m}]_M$ is the pullback of μ and ν.

For $M = BA$ and $N = AB$ there are canonical identifications:

- $(N \otimes_B M)^{\otimes_A m} = A(BA)^m$
- $M \otimes_A (N \otimes_B M)^{\otimes_A m} = (BA)^{m+1}$
- $\mathsf{Hom}_{A-A}\,((N \otimes_B M)^{\otimes_A m}, A) = \mathsf{Hom}_k((BA)^{m-1}B, A)$.
- $\mathsf{Hom}_{B-A}\,(M \otimes_A (N \otimes_B M)^{\otimes_A m}, M) = \mathsf{Hom}_k(A(BA)^{m-1}B, BA)$.

For simplicity we set $X = (BA)^{m-1}$. Through the previous identifications, μ corresponds to a linear map:

$$\overline{\mu} : \mathsf{Hom}_k(XB, A) \to \mathsf{Hom}_k(AXB, BA)$$

$$\overline{\mu}(f)(a \otimes x \otimes b) = 1 \otimes af(x \otimes b).$$

Similarly we obtain:

$$\overline{\nu} : \mathsf{Hom}_k(AX, B) \to \mathsf{Hom}_k(AXB, BA)$$

$$[\overline{\nu}(g)](a \otimes x \otimes b) = g(a \otimes x)b \otimes 1.$$

We suppose that for all $a \in A$, $x \in X$ and $b \in B$

$$1 \otimes af(x \otimes b) = g(a \otimes x)b \otimes 1$$

which is similar to the equality (7.2). Computing Ker $[\nabla'_{2m}]_N$ leads to the analogous result, which is similar to (7.3). Computations equal to those in the proof of Theorem 7.5 show that $f = g = 0$, except when $A = k$ and $B = k$. $\diamond$

Corollary 7.7 *Let A and B be finite dimensional algebras, and let $\Lambda = \begin{pmatrix} A & AB \\ BA & B \end{pmatrix}$ be the square algebra where the corner bimodules are free of rank one.*

Except if $A = B = k$, the following hold

- $\mathsf{HH}^0(\Lambda) = k \times k$,
- $\dim \mathsf{HH}^1(\Lambda) = \dim \mathsf{HH}^1(A) + \dim \mathsf{HH}^1(B) - (\dim \mathsf{HH}^0 A + \dim \mathsf{HH}^0 B) + 2(\dim A \dim B + 1)$,
- $\mathsf{HH}^{2m}(\Lambda) = \mathsf{HH}^{2m}(A) \oplus \mathsf{HH}^{2m}(B)$ *for* $m > 0$,
- $\dim \mathsf{HH}^{2m+1}(\Lambda) = \dim \mathsf{HH}^{2m+1}(A) + \dim \mathsf{HH}^{2m+1}(B) + 2(\dim A \dim B)^m(\dim A \dim B - 1)$ *for* $m > 0$.

Proof. The center of a square algebra $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ is

$$\{(a, b) \in \mathsf{HH}^0(A) \times \mathsf{HH}^0(B) \mid \text{ for all } m \in M \ n \in N, \ bm = ma \text{ and } an = nb\}.$$

If $M = BA$ and $N = AB$, we infer $\mathsf{HH}^0(\Lambda) = k \times k$.

Note that, as vector spaces, $\mathsf{End}_{B-A}\, BA = BA$ and $\mathsf{End}_{A-B}\, AB = AB$. The five-term exact sequence of Theorem 7.3 for $m = 0$ is

$$0 \to k \times k \to \mathsf{HH}^0(A) \oplus \mathsf{HH}^0(B) \stackrel{\nabla_0}{\to} BA \oplus AB \to \mathsf{HH}^1(\Lambda) \to \mathsf{HH}^1(A) \oplus \mathsf{HH}^1(B) \to 0.$$

Hence

$$2 - \dim \mathsf{HH}^0 A - \dim \mathsf{HH}^0 B + 2 \dim A \dim B - \dim \mathsf{HH}^1(\Lambda) + \dim \mathsf{HH}^1(A) + \dim \mathsf{HH}^1(B) = 0.$$

The two equalities for $m > 0$ are a consequence of Theorem 7.3 and of Proposition 7.6. Indeed,

$$\begin{aligned} \dim \mathsf{Hom}_{B-A}\left(M \otimes_A (N \otimes_B M)^{\otimes_A m}, M\right) &= \dim \mathsf{Hom}_k(A(BA)^{m-1}B, BA) = \\ &\quad (\dim A \dim B)^{m+1} \\ \dim \mathsf{Hom}_{A-B}\left(N \otimes_B (M \otimes_A N)^{\otimes_B m}, N\right) &= \dim \mathsf{Hom}_k(B(AB)^{m-1}A, AB) = \\ &\quad (\dim A \dim B)^{m+1} \end{aligned}$$

and

$$\begin{aligned} \dim \mathsf{Hom}_{A-A}\left((N \otimes_B M)^{\otimes_A m}, A\right) &= \dim \mathsf{Hom}_k((BA)^{m-1}B, A) = \\ &\quad (\dim A \dim B)^m \\ \dim \mathsf{Hom}_{B-B}\left((M \otimes_A N)^{\otimes_B m}, B\right) &= \dim \mathsf{Hom}_k((AB)^{m-1}A, B) = \\ &\quad (\dim A \dim B)^m. \end{aligned}$$

Since ∇'_{2m} is injective, we obtain:

$$\dim \operatorname{Coker} \nabla'_{2m} = 2(\dim A \dim B)^{m+1} - 2(\dim A \dim B)^{m}.$$

$\diamond$

Corollary 7.8 *Let Λ be an algebra as in the previous result.*

- $\mathsf{HH}^{2m+1}(\Lambda) \neq 0$ *for all m.*
- *For $m > 0$, $\mathsf{HH}^{2m}(\Lambda) = 0$ if and only if $\mathsf{HH}^{2m}(A) = 0 = \mathsf{HH}^{2m}(B)$.*

Remark 7.9 *Let $A = kQ_A/I$ and $B = kQ_B/J$ be finite dimensional algebras where Q_A and Q_B are finite quivers, and I and J are admissible ideals - hence Q_A and Q_B are the Gabriel's quivers of A and B.*

Let $\Lambda = \begin{pmatrix} A & AB \\ BA & B \end{pmatrix}$ be a square algebra where the corner bimodules are free of rank one.

The Gabriel quiver Q_Λ of Λ is the disjoint union of Q_A and Q_B, with in addition new arrows as follows: one arrow from each vertex of Q_A to each vertex of Q_B, and conversely. Let K be the admissible ideal of kQ_Λ generated by I, J, and all the paths containing two new arrows. Then $\Lambda = kQ_\Lambda/K$.

Example 7.10 *Let $A = kQ_A$ and $B = kQ_B$ be finite dimensional path algebras, where Q_A and Q_B are connected quivers without oriented cycles which are not both reduced to one vertex without arrows - that is $A = B = k$ is not considered.*

Let Q be the quiver as in the previous remark, that is Q is the disjoint union of Q_A and Q_B, with in addition new arrows joining all the vertices of Q_A to all the vertices of Q_B, and vice versa. *Let K be the two sided ideal of kQ generated by the paths of the form $v\omega u$ where ω is a path of Q_A or of Q_B, and u and v are new arrows. Let $\Lambda = kQ/K$. Then*

- $\dim \mathsf{HH}^0(\Lambda) = 2,$
- $\dim \mathsf{HH}^1(\Lambda) = \dim \mathsf{HH}^1(A) + \dim \mathsf{HH}^1(B) + 2 \dim A \dim B,$
- $\dim \mathsf{HH}^{2m}(\Lambda) = 0,$ *for $m > 0$,*
- $\dim \mathsf{HH}^{2m+1}(\Lambda) = 2(\dim A \dim B)^m(\dim A \dim B - 1),$ *for $m > 0$.*

Indeed, the Hochschild cohomology of a path algebra vanishes in degrees 2 and higher.

8 Square projective algebras via Peirce quivers

The main purpose of this section is to describe square algebras $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ which have the property that

$$\mathsf{HH}^n(\Lambda) = \mathsf{HH}^n(A) \oplus \mathsf{HH}^n(B)$$

for n large enough.

Recall that a system E of an algebra Λ is a finite set of complete orthogonal idempotents - not necessarily primitive.

Let Λ be a multi-extension algebra built on a simply laced quiver with a Q-data Δ. As already mentioned, the Peirce Q_0-quiver of Λ_Δ is precisely Q.

In this section we will consider null-square projective algebras $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ where the projective corner bimodules will be given through systems of A and B. Let E and F be systems of the algebras A and B, respectively. If $e \in E$ and $f \in F$, then $Bf \otimes eA$ is a projective $B-A$-bimodule. Hence for integers ${}_fm_e \geq 0$, the bimodule

$$M = \bigoplus_{e\in E, f\in F} {}_fm_e(Bf \otimes eA)$$

is also projective. Note that M is free of rank one if and only if all the integers ${}_fm_e$ are 1.

The proof of the following is straightforward.

Lemma 8.1 *Let A and B be algebras provided with systems E and F, and let Q_E and Q_F be their corresponding Peirce quivers.*

Let $M = \bigoplus_{e\in E,f\in F} {}_fm_e(Bf \otimes eA)$ and $N = \bigoplus_{e\in E,f\in F} {}_en_f(Ae \otimes fB)$ be projective bimodules and let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be the corresponding null-square projective algebra, that is the multi-extension zero algebra built on the round trip quiver with data the algebras A and B on the vertices and the bimodules M and N on the arrows. We set $\mathbb{A} = A \times B$ and $\mathbb{M} = M \oplus N$.

The set $E \cup F$ is a system of Λ, and the Pierce $E \cup F$-quiver $Q_{E\cup F}$ of Λ is the disjoint union of Q_E and Q_F - we view these quivers as located in horizontal up and down plans. In addition there is a (vertical down) *arrow from e to f if ${}_fm_e \neq 0$ and a* (vertical up) *arrow from f to e if ${}_en_f \neq 0$, see Figure 1 below.*

Definition 8.2 *In the situation of the previous lemma, an* efficient path *of $Q_{E\cup F}$ is a path of $Q_{E\cup F}$ which does not contain two successive arrows of Q_E, nor of Q_F.*

Theorem 8.3 *In the setting of Lemma 8.1, there exists h such that $\mathbb{M}^{\otimes_{\mathbb{A}} h} = 0$ if and only if $Q_{E\cup F}$ has no efficient cycles.*

Proof. We first assert that for $h \geq 2$, we have $\mathbb{M}^{\otimes_{\mathbb{A}} h} \neq 0$ if and only if there exists an efficient path which has h vertical arrows.

The following decomposition holds:

$$N \otimes_B M = \bigoplus_{\substack{e,e'\in E \\ f,f'\in F}} {}_{e'}n_{f'}\ {}_fm_e \left(Ae' \otimes f'B \otimes_B Bf \otimes eA\right) =$$

$$\bigoplus_{\substack{e,e'\in E \\ f,f'\in F}} {}_{e'}n_{f'}\ {}_fm_e \left(Ae' \otimes f'Bf \otimes eA\right).$$

Note that if the direct summand $(Ae' \otimes f'Bf \otimes eA)$ is not zero, then there is an efficient path from e to e' which contains two vertical arrows, namely from e to f and from f' to e'. If $f \neq f'$, then there is an arrow in Q_B corresponding to $f'Bf \neq 0$, see Figure 1. If $f = f'$, then the vertical arrows are concatenated.

Conversely, if there is an efficient path starting at a vertex $e_- \in E$ and ending at a vertex e'_+ of E which has two vertical arrows, then there is a direct summand of $N \otimes_B M$ which is non zero.

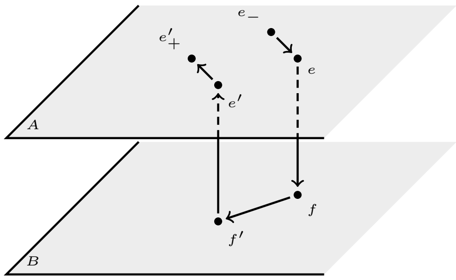

Fig. 1

Hence, $N \otimes_B M \neq 0$ if and only if there exists an efficient path starting and ending at vertices of E, and containing two vertical arrows. The analogous statement holds for $M \otimes_A N$.

Since $\mathbb{M} \otimes_{\mathbb{A}} \mathbb{M} = (N \otimes_B M) \oplus (M \otimes_A N)$, the assertion is proved for $h = 2$. For arbitrary h, the proof follows by the same type of considerations.

If there are no efficient cycles, then the length of the efficient paths is bounded, since the number of vertical arrows is finite. Hence there exists h such that $\mathbb{M}^{\otimes_{\mathbb{A}} h} = 0$. Conversely, if $\mathbb{M}^{\otimes_{\mathbb{A}} h} = 0$ for some h, then $\mathbb{M}^{\otimes_{\mathbb{A}} n} = 0$ for all $n \geq h$. However, if there is an efficient cycle with l vertical arrows, then there are efficient paths with rl vertical arrows for any positive integer r, and $\mathbb{M}^{\otimes_{\mathbb{A}} rl} \neq 0$ for all positive integers r. $\diamond$

Theorem 8.4 *Let A and B be algebras provided with systems E and F respectively. Let $M = \bigoplus_{e\in E, f\in F}\, {}_f m_e (Bf \otimes eA)$ and $N = \bigoplus_{e\in E, f\in F}\, {}_e n_f (Ae \otimes fB)$ be projective bimodules and let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be the corresponding null-square projective algebra. We set $\mathbb{A} = A \times B$ and $\mathbb{M} = M \oplus N$. Suppose there are no efficient cycles in the $E \cup F$-Peirce quiver of Λ. There exists a positive integer h such that for $n \geq h$*

$$\mathsf{HH}^n(\Lambda) = \mathsf{HH}^n(A) \oplus \mathsf{HH}^n(B).$$

Proof. According to the previous result, there exists h such that $\mathbb{M}^{\otimes_{\mathbb{A}} h} = 0$, and in this situation Corollary 7.4 provides the result. $\diamond$

Example 8.5 *Let $A = kQ_A$ and $B = kQ_B$ be path algebras, where Q_A and Q_B are finite quivers, possibly with oriented cycles. We view these quivers as situated in horizontal plans, up and down. A new quiver Q is obtained by adding chosen sets of vertical up and down arrows which join vertices of Q_A and Q_B. Let K be the two sided ideal of kQ generated by the paths containing two vertical arrows, and let $\Lambda = kQ/K$.*

Suppose there are no efficient cycles with respect to the $(Q_A)_0 \cup (Q_B)_0$-Peirce quiver of Λ. Equivalently, suppose there are no cycles in Q of the form $\delta v \dots v\delta v\delta v$ or $v \dots v\delta v\delta v$ which have at least one vertical arrow, and where the v's belong to the set of vertical arrows and the δ's to the set of paths of Q_A or of Q_B. Therefore $\mathsf{HH}^n(\Lambda) = 0$ for n large enough.

Indeed, by Theorem 8.4 there exists a positive integer h such that $\mathsf{HH}^n(\Lambda) = \mathsf{HH}^n(A) \oplus \mathsf{HH}^n(B)$ *if* $n \geq h$*. Moreover, Hochschild cohomology of a path algebra vanishes in degrees larger or equal to* 2*, see e.g. [7, p. 98].*

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