AN ASYMPOTIC CELL CATEGORY FOR CYCLIC GROUPS

by

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In [**Ma**], Malle has associated to any spetsial imprimitive complex reflection group W a set of *unipotent characters* (which is in natural bijection with the set of *unipotent characters* of the associated finite reductive group whenever W is a Weyl group). In this generalization of Lusztig's work, he also obtained a partition of this set into *families* and, to each family, he associated a $\mathbb{Z}$-fusion datum: by a $\mathbb{Z}$-fusion datum, we mean that we have all the axioms of a classical fusion datum (which will be called a $\mathbb{Z}_+$*-fusion datum* in the present paper) except that the structure constants of the associated fusion ring might be negative.

It is a classical problem to find a categorification of a $\mathbb{Z}_+$-fusion datum by a tensor category with suitable extra-structures (pivot, twist). The aim of this paper is to provide an *ad hoc* categorification of the $\mathbb{Z}$-fusion datum associated with the non-trivial family of the cyclic complex reflection group of order d: it is provided by the stable category of the Drinfeld double of the *Taft algebra* of dimension d^2. It must be said that we have no theoretical explanation for this fact.

ACKNOWLEDGEMENTS - We wish to thank warmly Abel Lacabanne for the fruitful discussions we had with him and his very careful reading of a preliminary version of this work.

The first author is partly supported by the ANR (Project No ANR-16-CE40-0010-01 GeRepMod).
The second author was partly supported by the NSF (grant DMS-1161999 and DMS-1702305) and by a grant from the Simons Foundation (#376202).

1. The Drinfeld double of the Taft algebra

From now on, $\otimes$ will denote the tensor product $\otimes_{\mathbb{C}}$. We fix a natural number $d \geqslant 2$ as well as a primitive d-th root of unity $\zeta \in \mathbb{C}^\times$. We denote by $\boldsymbol{\mu}_d = \langle \zeta \rangle$ the group of d-th roots of unity. If $i \in \mathbb{Z}/d\mathbb{Z}$ and α is an element of a group such that $\alpha^d = 1$, we will denote by α^i the element $\alpha^{i'}$, where i' is any representative of i.

If $n \geqslant 1$ is a natural number and $\xi \in \mathbb{C}$, we set

$$(n)_\xi = 1 + \xi + \cdots + \xi^{n-1}$$

and

$$(n)!_\xi = \prod_{i=1}^{n} (i)_\xi.$$

We also set $(0)!_\xi = 1$.

1.A. The Taft algebra. — We denote by B the $\mathbb{C}$-algebra admitting the following presentation:

- Generators: K, E.
- Relations: $\begin{cases} K^d = 1, \\ E^d = 0, \\ KE = \zeta EK. \end{cases}$

It follows from [**Ka**, Proposition IX.6.1] that:

(Δ) There exists a unique morphism of algebras $\Delta : B \to B \otimes B$ such that

$$\Delta(K) = K \otimes K \qquad \text{and} \qquad \Delta(E) = (1 \otimes E) + (E \otimes K).$$

(ε) There exists a unique morphism of algebras $\varepsilon : B \to \mathbb{C}$ such that

$$\varepsilon(K) = 1 \qquad \text{and} \qquad \varepsilon(E) = 0.$$

(S) There exists a unique anti-automorphism S of B such that

$$S(K) = K^{-1} \qquad \text{and} \qquad S(E) = -EK^{-1}.$$

With Δ as a coproduct, ε as a counit and S as an (invertible) antipode, B becomes a Hopf algebra, called the *Taft algebra* [**EGNO**, Example 5.5.6]. It is easily checked that

$$B = \bigoplus_{i,j=0}^{d-1} \mathbb{C} K^i E^j = \bigoplus_{i,j=0}^{d-1} \mathbb{C} E^i K^j. \tag{1.1}$$

1.B. Dual algebra. — Let K^* and E^* denote the elements of B^* such that

$$K^*(E^i K^j) = \delta_{i,0}\zeta^j \qquad \text{and} \qquad E^*(E^i K^j) = \delta_{i,1}.$$

Recall that B^* is a Hopf algebra [**Ka**, Proposition III.3.3] and it follows from [**Ka**, Lemma IX.6.3] that

$$(E^{*i}K^{*j})(E^{i'}K^{j'}) = \delta_{i,i'}(i)!_\zeta \zeta^{j(i+j')}. \tag{1.2}$$

We deduce easily that $(E^{*i}K^{*j})_{0 \leqslant i,j \leqslant d-1}$ is a $\mathbb{C}$-basis of B^*.

We will give explicit formulas for the coproduct, the counit and the antipode in the next subsection. We will in fact use the Hopf algebra $(B^*)^{\mathrm{cop}}$, which is the Hopf algebra whose underlying space is B^*, whose product is the same as in B^* and whose coproduct is opposite to the one in B^*.

1.C. Drinfeld double. — We denote by $D(B)$ the *Drinfeld quantum double* of B, as defined for instance in [**Ka**, Definition IX.4.1] or [**EGNO**, Definition 7.14.1]. Recall that $D(B)$ contains B and $(B^*)^{\mathrm{cop}}$ as Hopf subalgebras and that the multiplication induces an isomorphism of vector spaces $(B^*)^{\mathrm{cop}} \otimes B \xrightarrow{\sim} D(B)$. A presentation of $D(B)$, with generators E, E^*, K, K^* is given for instance in [**Ka**, Proposition IX.6.4]. We shall slightly modify it by setting

$$z = K^{*-1}K \qquad \text{and} \qquad F = \zeta E^* K^{*-1}.$$

Then [**Ka**, Proposition IX.6.4] can we rewritten as follows:

Proposition 1.3. — *The $\mathbb{C}$-algebra $D(B)$ admits the following presentation:*

- *Generators: E, F, K, z;*
- *Relations:* $\begin{cases} K^d = z^d = 1, \\ E^d = F^d = 0, \\ [z,E] = [z,F] = [z,K] = 0, \\ KE = \zeta EK, \\ KF = \zeta^{-1}FK, \\ [E,F] = K - zK^{-1}. \end{cases}$

The next corollary follows from an easy induction argument:

Corollary 1.4. — *If $i \geqslant 1$, then*

$$[E, F^i] = (i)_\zeta F^{i-1}(\zeta^{1-i}K - zK^{-1})$$

and

$$[F, E^i] = (i)_\zeta E^{i-1}(\zeta^{1-i}zK^{-1} - K).$$

The algebra $D(B)$ is endowed with a structure of Hopf algebra, where the comultiplication, the counit and the antipode are still denoted by Δ, ε and S respectively (as they extend the corresponding objects for B). We have [**Ka**, Proposition IX.6.2]:

$$\text{(1.5)}\qquad \begin{cases} \Delta(K) = K\otimes K, \\ \Delta(z) = z\otimes z, \\ \Delta(E) = (1\otimes E)+(E\otimes K), \\ \Delta(F) = (F\otimes 1)+(zK^{-1}\otimes F), \end{cases} \qquad \begin{cases} S(K) = K^{-1}, \\ S(z) = z^{-1}, \\ S(E) = -EK^{-1}, \\ S(F) = -\zeta^{-1}FKz^{-1}, \end{cases}$$

$$\text{(1.6)}\qquad \varepsilon(K)=\varepsilon(z)=1 \qquad\text{and}\qquad \varepsilon(E)=\varepsilon(F)=0.$$

1.D. Morphisms to $\mathbb{C}$. — If $\xi\in\boldsymbol{\mu}_d$, we denote by $\varepsilon_\xi : D(B)\to\mathbb{C}$ the unique morphism of algebras such that

$$\varepsilon_\xi(K)=\xi,\quad \varepsilon_\xi(z)=\xi^2 \quad\text{and}\quad \varepsilon_\xi(E)=\varepsilon_\xi(F)=0.$$

It is easily checked that the ε_ξ's are the only morphisms of algebras $D(B)\to\mathbb{C}$. Note that $\varepsilon_1=\varepsilon$ is the counit.

1.E. Group-like elements. — It follows from (1.5) that K and z are group-like, so that K^iz^j is group-like for all $i,\ j\in\mathbb{Z}$. The converse also holds (and is certainly already well-known):

Lemma 1.7. — *If $g\in D(B)$ is group-like, then there exist $i,\ j\in\mathbb{Z}$ such that $g=K^iz^j$.*

Proof. — Let $g\in D(B)$ be a group-like element. Let us write

$$g=\sum_{i,j,k,l=0}^{d-1}\alpha_{i,j,k,l}K^iz^jE^kF^l.$$

We denote by (k_0,l_0) the biggest pair (for the lexicographic order) such that there exist $i,\ j\in\{0,1,\dots,d-1\}$ such that $\alpha_{i,j,k_0,l_0}\neq 0$. Then the coefficient of $K^iz^jE^{k_0}F^{l_0}\otimes K^iz^jE^{k_0}F^{l_0}$ in $g\otimes g$ is equal to α^2_{i,j,k_0,l_0}, so it is different from 0.

But, if we compute the coefficient of $K^iz^jE^{k_0}F^{l_0}\otimes K^iz^jE^{k_0}F^{l_0}$ in

$$g\otimes g=\Delta(g)=\sum_{i,j,k,l=0}^{d-1}\alpha_{i,j,k,l}\Delta(K)^i\Delta(z)^j\Delta(E)^k\Delta(F)^l$$

using the formulas (1.5), we see that it is equal to 0 if $(k_0,l_0)\neq(0,0)$. Therefore $(k_0,l_0)=(0,0)$, and so g belongs to the linear span of the family $(K^iz^j)_{i,j\in\mathbb{Z}}$.

Now the result follows from the linear independence of group-like elements. □

1.F. Braiding. — For $0 \leqslant i, j \leqslant d-1$, we set

$$\beta_{i,j} = \frac{E^{*i}}{d \cdot (i)!_\zeta} \sum_{k=0}^{d-1} \zeta^{-k(i+j)} K^{*k}.$$

It follows from (1.2) that $(\beta_{i,j})_{0 \leqslant i,j \leqslant d-1}$ is a dual basis to $(E^i K^j)_{0 \leqslant i,j \leqslant d-1}$. We then set

$$R = \sum_{i,j=0}^{d-1} E^i K^j \otimes \beta_{i,j} \in D(B) \otimes D(B).$$

Then R is a universal R-matrix for $D(B)$ which endows $D(B)$ with a structure of braided Hopf algebra [**Ka**, Theorem IX.4.4]). Using our generators E, F, K, z, we have:

$$R = \frac{1}{d} \sum_{i,j,k=0}^{d-1} \frac{\zeta^{(i-k)(i+j)-i(i+1)/2}}{(i)!_\zeta} E^i K^j \otimes z^{-k} F^i K^k. \tag{1.8}$$

1.G. Twist. — Let us define

$$\begin{array}{rccc} \tau : & D(B) \otimes D(B) & \longrightarrow & D(B) \otimes D(B) \\ & a \otimes b & \longmapsto & b \otimes a. \end{array}$$

Following [**Ka**, §VIII.4], we set

$$u = \sum_{i,j=0}^{d-1} S(\beta_{ij}) E^i K^j \in D(B).$$

Recall that u is called the *Drinfeld element* of $D(B)$. It satisfies several properties (see for instance [**Ka**, Proposition VIII.4.5]). For instance, u is invertible and we will recall only three equalities:

$$\varepsilon(u) = 1, \quad \Delta(u) = (\tau(R)R)^{-1}(u \otimes u) \quad \text{and} \quad S^2(b) = u\,b\,u^{-1} \tag{1.9}$$

for all $b \in D(B)$. A straightforward computation shows that

$$S^2(b) = K\,b\,K^{-1} \tag{1.10}$$

for all $b \in D(B)$. We now set

$$\theta = K^{-1} u.$$

Then it follows from (1.9) and (1.10) that:

Proposition 1.11. — *The element θ is central and invertible in $D(B)$ and satisfies*

$$\varepsilon(\theta)=1 \qquad \textit{and} \qquad \Delta(\theta)=(\tau(R)R)^{-1}(\theta\otimes\theta).$$

Let us give a formula for θ:

$$\theta=\frac{1}{d}\sum_{i,j,k=0}^{d-1}(-1)^i\frac{\zeta^{(i-k)(i+j)-i}}{(i)!_\zeta}z^{k-i}F^iE^iK^{i+j-k-1}. \tag{1.12}$$

Corollary 1.13. — $S(\theta)=z\theta$.

Proof. — Let $g=S(\theta)\theta^{-1}$. Since $\Delta\circ S=\tau\circ(S\otimes S)\circ\Delta$ and $(S\otimes S)(R)=R$ (see for instance [**Ka**, Theorems III.3.4 and VIII.2.4], it follows from Proposition 1.11 that g is central and group-like. Hence, by Lemma 1.7, there exists $l\in\mathbb{Z}$ such that $S(\theta)=\theta z^l$. So, by (1.12), we have

$$S(\theta)E^{d-1}=\theta z^lE^{d-1}=\frac{1}{d}\sum_{j,k\in\mathbb{Z}/d\mathbb{Z}}^{d-1}\zeta^{-jk}z^{k+l}K^{j-k-1}E^{d-1}. \tag{$\sharp$}$$

Let us now compute $S(\theta)E^{d-1}$ by using directly (1.12). We get

$$\begin{aligned}S(\theta)E^{d-1}=E^{d-1}S(\theta) &= \frac{1}{d}\sum_{j,k\in\mathbb{Z}/d\mathbb{Z}}\zeta^{-jk}E^{d-1}z^{-k}K^{1+k-j}\\ &= \frac{1}{d}\sum_{j,k\in\mathbb{Z}/d\mathbb{Z}}\zeta^{-jk}\zeta^{1+k-j}z^{-k}K^{1+k-j}E^{d-1}\\ &= \frac{1}{d}\sum_{j,k\in\mathbb{Z}/d\mathbb{Z}}\zeta^{(1-j)(1+k)}z^{-k}K^{1+k-j}E^{d-1}.\end{aligned}$$

So, if we set $j'=1-j$ and $k'=-1-k$, we get

$$S(\theta)E^{d-1}=\frac{1}{d}\sum_{j',k'\in\mathbb{Z}/d\mathbb{Z}}\zeta^{-j'k'}z^{k'+1}K^{j'-k'-1}E^{d-1}.$$

Comparing with ($\sharp$), we get that $z^l=z$. $\square$

2. $D(B)$-modules

Most of the result of this section are due to Chen **[Ch1]** or Edrmann, Green, Snashall and Taillefer **[EGST1]**, **[EGST2]**. By a $D(B)$-module, we mean a finite dimensional left $D(B)$-module. We denote by $D(B)$-mod the category of (finite dimensional left) $D(B)$-modules. If $\alpha_1,\dots,\alpha_{l-1} \in \mathbb{C}$, we set

$$J_l^+(\alpha_1,\dots,\alpha_{l-1}) = \begin{pmatrix} 0 & \alpha_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & \alpha_{l-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

and
$$J_l^-(\alpha_1,\dots,\alpha_{l-1}) = {}^t J_l^+(\alpha_1,\dots,\alpha_{l-1}).$$

If M is a $D(B)$-module and $b \in D(B)$, we denote by $b|_M$ the endomorphism of M induced by b. For instance, $E|_M$ and $F|_M$ are nilpotent and $K|_M$ and $z|_M$ are semisimple.

2.A. Simple modules. — If $1 \leqslant l \leqslant d$ and $p \in \mathbb{Z}/d\mathbb{Z}$, we denote by $M_{l,p}$ the $D(B)$-module with $\mathbb{C}$-basis $\mathscr{M}^{(l,p)} = (e_i^{(l,p)})_{1 \leqslant i \leqslant l}$ and such that the action of z, K, E and F in the basis $\mathscr{M}^{(l,p)}$ are given by the following matrices:

$$z|_{M_{l,p}} = \zeta^{2p+l-1}\,\mathrm{Id}_{M_{l,p}},$$

$$K|_{M_{l,p}} = \zeta^p\,\mathrm{diag}(\zeta^{l-1},\zeta^{l-2},\dots,\zeta,1),$$

$$E|_{M_{l,p}} = \zeta^p J_l^+((1)_\zeta(\zeta^{l-1}-1),(2)_\zeta(\zeta^{l-2}-1),\dots,(l-1)_\zeta(\zeta-1)),$$

$$F|_{M_{l,p}} = J_l^-(1,\dots,1).$$

It is readily checked from the relations given in Proposition 1.3 that this defines a $D(B)$-module of dimension l. The next result is proved in **[Ch1**, Theorem 2.5].

***Theorem 2.1* (Chen)**. — *The map*

$$\begin{array}{ccc} \{1,2,\dots,d\}\times\mathbb{Z}/d\mathbb{Z} & \longrightarrow & \mathrm{Irr}(D(B)) \\ (l,p) & \longmapsto & M_{l,p} \end{array}$$

is bijective.

2.B. Blocks. — We denote by $\Lambda(d)$ the set $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$. Then $\Lambda(d)$ is in canonical bijection with the set $\{1,2,\dots,d\} \times \mathbb{Z}/d\mathbb{Z}$, which parametrizes the simple $D(B)$-modules. So if $\lambda \in \Lambda(d)$, we will denote by M_λ the corresponding simple $D(B)$-module. We also set $(\mathbb{Z}/d\mathbb{Z})^\# = (\mathbb{Z}/d\mathbb{Z}) \setminus \{0\}$ and $\Lambda^\#(d) = (\mathbb{Z}/d\mathbb{Z})^\# \times \mathbb{Z}/d\mathbb{Z}$. Finally, let $\Lambda^0(d) = \{0\} \times \mathbb{Z}/d\mathbb{Z}$ be the complement of $\Lambda^\#(d)$ in $\Lambda(d)$.

Let

$$\begin{array}{rccc} \iota: & \Lambda(d) & \longrightarrow & \Lambda(d) \\ & (l,p) & \longmapsto & (-l,p+l). \end{array}$$

Then $\iota^2 = \mathrm{Id}_{\Lambda(d)}$ and $\Lambda^0(d)$ is the set of fixed points of ι. If $\mathscr{L}$ is a ι-stable subset of $\Lambda(d)$, we denote by $[\mathscr{L}/\iota]$ a set of representatives of ι-orbits in $\mathscr{L}$. The next result is proved in [**EGST1**, Theorem 2.26].

***Theorem* 2.2 (Erdmann-Green-Snashall-Taillefer).** — *Let λ, $\lambda' \in \Lambda(d)$. Then M_λ and $M_{\lambda'}$ belong to the same block of $D(B)$ if and only if λ and λ' are in the same ι-orbit.*

We have constructed in §1.G a central element, namely θ. Note that

(2.3) *The element θ acts on $M_{l,p}$ by multiplication by $\zeta^{(p-1)(l+p-1)}$.*

Proof. — It is sufficient to compute the action of θ on $e_1^{(l,p)}$. Note that $E^i e_1^{(l,p)} = 0$ as soon as $i \geqslant 1$. Therefore, for computing $\theta e_1^{(l,p)}$ using the formula (1.12), only the terms corresponding to $i=0$ remain. Consequently,

$$\begin{aligned} \omega_{l,p}(\theta) &= \frac{1}{d} \sum_{j,k=0}^{d-1} \zeta^{-jk} \zeta^{(2p+l-1)k} \zeta^{(p+l-1)(j-k-1)} \\ &= \frac{\zeta^{1-l-p}}{d} \sum_{k=0}^{d-1} \zeta^{pk} \Big(\sum_{j=0}^{d-1} \zeta^{(p+l-1-k)j} \Big) \end{aligned}$$

The term inside the big parenthesis is equal to d if $p+l-1-k \equiv 0 \mod d$, and is equal to 0 otherwise. The result follows. □

2.C. Projective modules. — If $\lambda \in \Lambda(d)$, we denote by P_λ an indecomposable projective cover of M_λ. The next result is proved in [**EGST1**, Corollary 2.25].

Theorem 2.4 **(Erdmann-Green-Snashall-Taillefer)**. — *Let* $\lambda \in \Lambda(d)$*. Then:*

(a) *If* $\lambda \in \Lambda^\#(d)$*, then* $\dim_{\mathbb{C}}(P_\lambda) = 2d$*,* $\mathrm{Rad}^3(P_\lambda) = 0$ *and the Loewy structure of* P_λ *is given by:*

$$\begin{array}{rcc} P_\lambda/\mathrm{Rad}(P_\lambda) & \simeq & M_\lambda \\ \mathrm{Rad}(P_\lambda)/\mathrm{Rad}^2(P_\lambda) & \simeq & M_{\iota(\lambda)} \oplus M_{\iota(\lambda)} \\ \mathrm{Rad}^2(P_\lambda) & \simeq & M_\lambda \end{array}$$

(b) $P_{d,p} = M_{d,p}$ *has dimension* d*.*

3. Tensor structure

We mainly refer here to the work of Erdmann, Green, Snashall and Taillefer [**EGST1**], [**EGST2**]. Since $D(B)$ is a finite dimensional Hopf algebra, the category $D(B)$-mod inherits a structure of a tensor category. We will compute here some tensor products between simple modules. For simplifying, we will denote by M_l the simple module $M_{l,0}$.

3.A. Invertible modules. — We denote by $V_\xi = \mathbb{C} v_\xi$ the one-dimensional $D(B)$-module associated with the morphism $\varepsilon_\xi : D(B) \to \mathbb{C}$ defined in §1.D:

$$b\, v_\xi = \varepsilon_\xi(b) v_\xi$$

for all $b \in D(B)$. We have

$$V_{\zeta^p} \simeq M_{1,p}. \tag{3.1}$$

An immediate computation using the comultiplication Δ shows that

$$M_{l,p} \otimes V_{\zeta^q} \simeq V_{\zeta^q} \otimes M_{l,p} \simeq M_{l,p+q} \tag{3.2}$$

as $D(B)$-modules. The V_ξ's are (up to isomorphism) the only invertible objets in the tensor category $D(B)$-mod.

3.B. Tensor product with M_2. — For simplifying, we set $e_i = e_i^{(2,0)}$ for $i \in \{1,2\}$, so that (e_1, e_2) is the standard basis of M_2. The next result is a particular case of [**EGST1**, Theorem 4.1].

***Theorem 3.3* (Edrmann-Green-Snashall-Taillefer).** — *Let $\lambda \in \Lambda(d)$ and let (l,p) be a representative of λ in $\{1,2,\dots,d\} \times \mathbb{Z}/d\mathbb{Z}$. Then:*

(a) *If $l \leqslant d-1$ (i.e. if $\lambda \in \Lambda^{\#}(d)$), then $M_2 \otimes M_{l,p} \simeq M_{l+1,p} \oplus M_{l-1,p+1}$.*
(b) *$M_2 \otimes M_{d,p} \simeq P_{d-1,p}$.*

4. Grothendieck rings

We denote by $\mathrm{Gr}(D(B))$ the Grothendieck ring of the category of (left) $D(B)$-modules.

4.A. Structure. — Since $D(B)$ is a braided Hopf algebra (with universal R-matrix R),

(4.1) *The ring* $\mathrm{Gr}(D(B))$ *is commutative.*

If M is a $D(B)$-module, we denote by $[\,M\,]$ the class of M in $\mathrm{Gr}(D(B))$. We set

$$\mathbf{m}_\lambda = [\,M_\lambda\,], \qquad \mathbf{m}_l = [\,M_{l,0}\,] \qquad \text{and} \qquad \mathbf{v}_\xi = [\,V_\xi\,] \in \mathrm{Gr}(D(B)),$$

Recall that $\mathbf{v}_{\zeta^p} = \mathbf{m}_{1,p}$.Then it follows from (3.2) and Theorem 3.3 that

$$\text{(4.2)} \qquad \mathbf{v}_{\zeta^q}\mathbf{m}_{l,p} = \mathbf{m}_{l,p+q} \quad \text{and} \quad \mathbf{m}_2\mathbf{m}_{l,p} = \begin{cases} \mathbf{m}_{l+1,p} + \mathbf{m}_{l-1,p+1} & \text{if } l \leqslant d-1, \\ 2(\mathbf{m}_{d-1,p} + \mathbf{m}_{1,p-1}) & \text{if } l = d. \end{cases}$$

***Proposition 4.3*.** — *The Grothendieck ring* $\mathrm{Gr}(D(B))$ *is generated by* $\mathbf{v}_\zeta$ *and* $\mathbf{m}_2$.

Proof. — We will prove by induction on l that $\mathbf{m}_{l,p} \in \mathbb{Z}[\mathbf{v}_\zeta, \mathbf{m}_2]$. Since $\mathbf{m}_{1,p} = (\mathbf{v}_\zeta)^p$, this is true for $l = 1$. Since $\mathbf{m}_{2,p} = (\mathbf{v}_\zeta)^p\mathbf{m}_2$, this is also true for $l = 2$. Now the induction proceeds easily by using (4.2). □

4.B. Some characters. — If $b \in D(B)$ is *group-like*, then the map

$$\begin{array}{ccc} \mathrm{Gr}(D(B)) & \longrightarrow & \mathbb{C} \\ [M] & \longmapsto & \mathbf{Tr}(b|_M) \end{array}$$

is a morphism of rings. Here, **Tr** denotes the usual trace (not the quantum trace) of an endomorphism of a finite dimensional vector space. Recall from Lemma 1.7 that the only group-like elements of $D(B)$ are the $K^i z^j$, where $(i,j) \in \Lambda(d)$. We set

$$\begin{array}{cccc} \chi_{i,j}: & \mathrm{Gr}(D(B)) & \longrightarrow & \mathbb{C} \\ & [M] & \longmapsto & \mathbf{Tr}(K^i z^j|_M). \end{array}$$

An easy computation yields

$$\chi_{i,j}(\mathbf{m}_{l,p}) = \zeta^{pi+(2p+l-1)j} \cdot (l)_{\zeta^i}. \tag{4.4}$$

Note that the $\chi'_{i,j}s$ are not necessarily distinct:

Lemma 4.5. — *Let λ and λ' be two elements of $\Lambda(d)$. Then $\chi_\lambda = \chi_{\lambda'}$ if and only if λ and λ' are in the same ι-orbit.*

Proof. — Let us write $\lambda = (i,j)$ and $\lambda' = (i',j')$. The "if" part follows directly from (4.4). Conversely, assume that $\chi_{i,j} = \chi_{i',j'}$. By applying these two characters to $\mathbf{v}_\zeta$ and $\mathbf{m}_2$, we get:

$$\begin{cases} \zeta^{i+2j} = \zeta^{i'+2j'}, \\ \zeta^j(1+\zeta^i) = \zeta^{j'}(1+\zeta^{i'}). \end{cases}$$

So the result follows by applying exactly the same argument as in the proof of Theorem 2.2. □

4.C. Stable category. — As B is a Hopf algebra, it is Frobenius and so its stable category $B\,\mathrm{st}$ (namely the quotient of B-mod by the full subcategory B-proj consisting of projective objects) is triangulated. Similarly, the category $D(B)$-stab is triangulated. Note also that a B-module (resp. a $D(B)$-module) is projective if and only if it is injective. Since the tensor product of a projective $D(B)$-module by any $D(B)$-module is still projective [**EGNO**, Proposition 4.2.12], it inherits a structure of monoidal category (such that the canonical functor $D(B)$-mod $\to D(B)$-stab is monoidal). In particular, its Grothendieck group (as a triangulated category), which will be denoted by $\mathrm{Gr}^{\mathrm{st}}(D(B))$, is a ring and the natural map

$$\begin{array}{ccc} \mathrm{Gr}(D(B)) & \longrightarrow & \mathrm{Gr}^{\mathrm{st}}(D(B)) \\ \mathbf{m} & \longmapsto & \mathbf{m}^{\mathrm{st}} \end{array}$$

is a morphism of rings. If M is a $D(B)$-module, we denote by $[M]_{\mathrm{st}}$ its class in $\mathrm{Gr}^{\mathrm{st}}(D(B))$.

Also, it follows Theorem 2.4 that

$$\mathbf{m}^{\mathrm{st}}_{d,p}=0 \qquad \text{and} \qquad 2(\mathbf{m}^{\mathrm{st}}_{l,p}+\mathbf{m}^{\mathrm{st}}_{d-l,p+l})=0 \tag{4.6}$$

if $l \leqslant d-1$.

4.D. A further quotient. — We denote by $D(B)$-PROJ the full subcategory of $D(B)$-mod whose objects are the $D(B)$-modules M such that $\mathrm{Res}_B^{D(B)} M$ is a projective B-module. Since $D(B)$ is a free B-module (of rank d^2), $D(B)$-proj is a full subcategory of $D(B)$-PROJ. We denote by $D(B)$-STAB the quotient of the category $D(B)$-mod by the full subcategory $D(B)$-PROJ: it is also the quotient of $D(B)$-stab by the image of $D(B)$-PROJ in $D(B)$-stab.

Lemma 4.7. — *The image of* $D(B)$-PROJ *in* $D(B)$-stab *is a thick triangulated subcategory. In particular,* $D(B)$-STAB *is triangulated.*

Proof. — If M is a $D(B)$-module, we denote by $\pi_M : P(M) \twoheadrightarrow M$ (resp. $i_M : M \hookrightarrow I(M)$) a projective cover (resp. injective hull) of M. We just need to prove the following facts:

(a) If M belongs to $D(B)$-PROJ, then $\mathrm{Ker}(\pi_M)$ and $I(M)/\mathrm{Im}(i_M)$ also belong to $D(B)$-PROJ.
(b) If $M \oplus N$ belongs to $D(B)$-PROJ, then M and N also belong to $D(B)$-PROJ.
(c) If M and N belong to $D(B)$-PROJ and $f : M \to N$ is a morphism of $D(B)$-modules, then the cone of f also belong to $D(B)$-PROJ.

(a) Assume that M belongs to $D(B)$-PROJ. Since M is a projective B-module, there exists a morphism of B-modules $f : M \to P(M)$ such that $\pi_M \circ f = \mathrm{Id}_M$. In particular, $P(M) \simeq \mathrm{Ker}(\pi_M) \oplus M$, as a B-module. So $\mathrm{Ker}(\pi_M)$ is a projective B-module.

On the other hand, $I(M)$ is a projective $D(B)$-module since $D(B)$ so it is a projective B-module and so it is an injective B-module. So, again, $I(M) \simeq M \oplus I(M)/\mathrm{Im}(i_M)$, so $I(M)/\mathrm{Im}(i_M)$ is a projective B-module. This proves (a).

(b) is obvious.

(c) Let M and N belong to $D(B)$-PROJ and $f : M \to N$ be a morphism of $D(B)$-modules. Let $\Delta_f : M \to I(M) \oplus N$, $m \mapsto (i_M(m), f(m))$. Then the cone of f is isomorphic in $D(B)$-stab to $(I(M) \oplus N)/\mathrm{Im}(\Delta_f)$. But Δ_f is injective, M is an injective B-module and so $I(M) \simeq M \oplus (I(M) \oplus N)/\mathrm{Im}(\Delta_f)$ as a B-module, which shows that $(I(M) \oplus N)/\mathrm{Im}(\Delta_f)$ is a projective B-module. □

Since $D(B)$-STAB is triangulated, we can define its Grothendieck group, which will be denoted by $\mathrm{Gr}^{\mathrm{ST}}(D(B))$. Also, if M belongs to $D(B)$-PROJ and N is any $D(B)$-module, then $M\otimes N$ and $N\otimes M$ are projective as B-modules [**EGNO**, Proposition 4.2.12], so $D(B)$-STAB inherits a structure of monoidal category, compatible with the triangulated structure. This endows $\mathrm{Gr}^{\mathrm{ST}}(D(B))$ with a ring structure. The natural map $\mathrm{Gr}(D(B)) \to \mathrm{Gr}^{\mathrm{ST}}(D(B))$ will be denoted by $\mathbf{m} \mapsto \mathbf{m}^{\mathrm{ST}}$: it is a morphism of rings. Of course, this morphism factors through $\mathrm{Gr}^{\mathrm{st}}(D(B))$.

If $\lambda \in \Lambda^{\#}(d)$, then it follows from [**EGST2**, Property 1.4] that there exists a $D(B)$-module P_λ^B which is projective as a B-module and such that there is an exact sequence

$$0 \longrightarrow M_{\iota(\lambda)} \longrightarrow P_\lambda^B \longrightarrow M_\lambda \longrightarrow 0.$$

It then follows that

$$\mathbf{m}_\lambda^{\mathrm{ST}} + \mathbf{m}_{\iota(\lambda)}^{\mathrm{ST}} = 0. \tag{4.8}$$

Also, we still have

$$\mathbf{m}_{d,p}^{\mathrm{ST}} = 0. \tag{4.9}$$

The next Theorem follows from (4.8), (4.9) and Proposition 4.3:

Theorem 4.10. — *The ring* $\mathrm{Gr}^{\mathrm{ST}}(D(B))$ *is generated by* $\mathbf{v}_\zeta^{\mathrm{ST}}$ *and* $\mathbf{m}_2^{\mathrm{ST}}$. *Moreover,*

$$\mathrm{Gr}^{\mathrm{ST}}(D(B)) = \bigoplus_{\lambda \in [\Lambda^{\#}(d)/\iota]} \mathbb{Z}\mathbf{m}_\lambda^{\mathrm{ST}}$$

and $\mathrm{Gr}^{\mathrm{ST}}(D(B))$ *is a free* $\mathbb{Z}$*-module of rank* $d(d-1)/2$.

Recall that Lemma 4.5 shows that, through the $\chi_{i,j}$'s, only $d(d+1)/2$ different characters of the ring $\mathrm{Gr}(D(B))$ have been defined. It is not clear if $\mathbb{C}\,\mathrm{Gr}(D(B))$ is semisimple in general but, for $d=2$, it can be checked that it is semisimple (of dimension 4), so that there is a fourth character $\mathrm{Gr}(D(B)) \to \mathbb{C}$ which is not obtained through the $\chi_{i,j}$'s.

Now, a character $\chi : \mathrm{Gr}(D(B)) \to \mathbb{C}$ factors through $\mathrm{Gr}^{\mathrm{ST}}(D(B))$ if and only if its kernel contains the $\mathbf{m}_{\lambda_0}$'s (where λ_0 runs over $\Lambda^0(d)$) and the $\mathbf{m}_\lambda + \mathbf{m}_{\iota(\lambda)}$'s (where λ runs over $\Lambda^{\#}(d)$). This implies the following result:

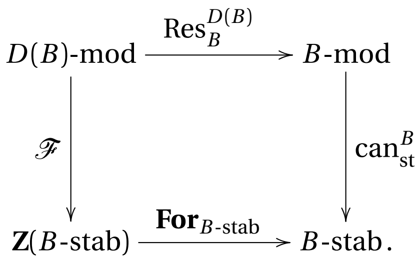
Theorem 4.11. — *The character* $\chi_\lambda : \mathrm{Gr}(D(B)) \to \mathbb{C}$ *factors through* $\mathrm{Gr}^{\mathrm{ST}}(D(B))$ *if and only if* $\lambda \in \Lambda^{\#}(d)$. *So the* $(\chi_\lambda)_{\lambda \in [\Lambda^{\#}(d)/\iota]}$ *are all the characters of* $\mathrm{Gr}^{\mathrm{ST}}(D(B))$ *and the* $\mathbb{C}$*-algebra* $\mathbb{C}\,\mathrm{Gr}^{\mathrm{ST}}(D(B))$ *is semisimple.*

4.E. Complements. — If $\mathscr{C}$ is a monoidal category, we denote by $\mathbf{Z}(\mathscr{C})$ its *Drinfeld center* (see [**Ka**, §XIII.4]) and we denote by $\mathbf{For}_{\mathscr{C}} : \mathbf{Z}(\mathscr{C}) \to \mathscr{C}$ the forgetful functor.

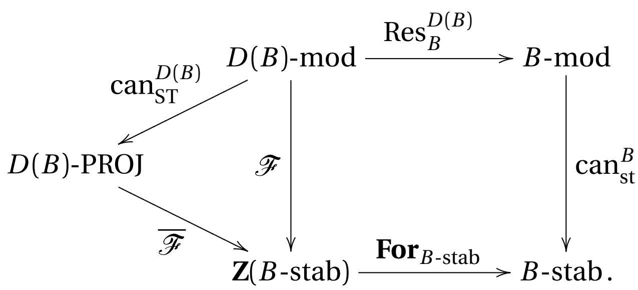
It turns out that the category $\mathbf{Z}(B\text{-mod})$ is naturally equivalent to $D(B)$-mod and that, through this equivalence, the forgetful functor just becomes the restriction functor $\mathrm{Res}_B^{D(B)}$. We will denote by $\mathrm{can}_{\mathrm{st}}^{D(B)} : D(B)\text{-mod} \to D(B)\text{-stab}$, $\mathrm{can}_{\mathrm{st}}^{B} : B\text{-mod} \to B\text{-stab}$ and $\mathrm{can}_{\mathrm{ST}} : D(B)\text{-mod} \to D(B)\text{-STAB}$ the canonical functors. Then the functor

$$\mathrm{can}_{\mathrm{st}}^{B} \circ \mathrm{Res}_B^{D(B)} : D(B)\text{-mod} \longrightarrow B\text{-stab}$$

factors through $\mathbf{Z}(B\text{-stab})$ so that we get a commutative diagram of functors



But any $D(B)$-module which is projective as a B-module is sent to the zero object of $\mathbf{Z}(B\text{-stab})$ through $\mathscr{F}$, so $\mathscr{F}$ factors through $D(B)$-PROJ and we get a commutative diagram of functors



Question. *Is* $\overline{\mathscr{F}} : D(B)\text{-PROJ} \to \mathbf{Z}(B\text{-stab})$ *an equivalence of categories?*

5. Fusion datum

5.A. Quantum traces. — The element $R \in D(B) \otimes D(B)$ defined in §1.F is a universal R-matrix which endows $D(B)$ with a structure of braided Hopf algebra, the category $D(B)$-mod is braided as follows: if M and N are two $D(B)$-modules, then the braiding $c_{M,N} : M \otimes N \xrightarrow{\sim} N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = \tau(R)(n \otimes m).$$

Recall that $\tau : D(B) \otimes D(B) \xrightarrow{\sim} D(B) \otimes D(B)$ is given by $\tau(a \otimes b) = b \otimes a$. In particular,

(5.1) $\qquad c_{N,M}\, c_{M,N} M \otimes N \xrightarrow{\sim} M \otimes N$ *is given by the action by* $\tau(R)R$.

For each $i \in \mathbb{Z}$, we have $S^2(b) = (z^{-i}K)b(z^{-i}K)^{-1}$ for all $b \in D(B)$ and $z^{-i}K$ is group-like, so the algebra $D(B)$ is *pivotal* with pivot $z^{-i}K$. This endows the tensor category $D(B)$-mod with a structure of pivotal category (see Section A) whose associated traces $\mathrm{Tr}_+^{(i)}$ and $\mathrm{Tr}_-^{(i)}$ are given as follows: if M is a $D(B)$-module and $f \in \mathrm{End}_{D(B)}(M)$, then

$$\mathrm{Tr}_+^{(i)}(f) = \mathbf{Tr}(z^{-i}Kf) \qquad \text{and} \qquad \mathrm{Tr}_-^{(i)}(f) = \mathbf{Tr}(fK^{-1}z^i).$$

Recall that $\mathbf{Tr}$ denotes the "classical" trace for endomorphisms of a finite dimensional vector space. So the pivotal structure depends on the choice of i (modulo d). The corresponding twist is $\theta_i = z^i\theta$, which endows $D(B)$-mod with a structure of balanced braided category (depending on i).

Hypothesis and notation. *From now on, and until the end of this paper, we assume that the Hopf algebra $D(B)$ is endowed with the pivotal structure whose pivot is $z^{-1}K$. The structure of balanced braided category is given by $\theta_1 = z\theta$ and the associated quantum traces $\mathrm{Tr}_\pm^{(1)}$ are denoted by $\mathrm{Tr}_\pm$.*

If M is a $D(B)$-module, we set $\dim_\pm(M) = \mathrm{Tr}_\pm(\mathrm{Id}_M)$.

We define

$$\dim(D(B)) = \sum_{M \in \mathrm{Irr}\, D(B)} \dim_-(M)\dim_+(M).$$

Then

$$\dim(D(B)) = \frac{2d^2}{(1-\zeta)(1-\zeta^{-1})}. \tag{5.2}$$

This follows easily from the fact that

$$\dim_+ M_{l,p} = \zeta^{1-l-p}(l)_\zeta \quad \text{and} \quad \dim_- M_{l,p} = \zeta^{p+l-1}(l)_{\zeta^{-1}} = \zeta^p(l)_\zeta. \tag{5.3}$$

5.B. Characters of $\mathrm{Gr}(D(B))$ via the pivotal structure. — As in Section A, these structures (braiding, pivot) allow to define characters of $\mathrm{Gr}(D(B))$ associated with (endo)simple modules. So, if $\lambda \in \Lambda(d)$, then we set

$$\begin{array}{rccc} s^+_{M_\lambda}: & \mathrm{Gr}(D(B)) & \longrightarrow & \mathbb{C} \\ & [M] & \longmapsto & (\mathrm{Id}_{M_\lambda} \otimes \mathrm{Tr}^M_+)(c_{M,M_\lambda} c_{M_\lambda,M}) \end{array}$$

and

$$\begin{array}{rccc} s^-_{M_\lambda}: & \mathrm{Gr}(D(B)) & \longrightarrow & \mathbb{C} \\ & [M] & \longmapsto & (\mathrm{Id}_{M_\lambda} \otimes \mathrm{Tr}^M_-)(c_{M,M_\lambda} c_{M_\lambda,M}) \end{array}$$

These are morphism of rings (see Proposition A.4). The main result of this section is the following:

Theorem 5.4. — *If $\lambda \in \Lambda(d)$, then*

$$s^+_{M_\lambda} = \chi_{-\lambda} \qquad \textit{and} \qquad s^-_{M_\lambda} = \chi_{(0,1)-\lambda}.$$

Proof. — Write $\lambda = (l,p)$. For simplification, we set

$$\boldsymbol{\Upsilon}_{i,j,k} = \frac{\zeta^{(i-k)(i+j)-i(i+1)/2}}{(i)!_\zeta}.$$

Then

$$\tau(R)R = \frac{1}{d^2} \sum_{i,i',j,j',k,k'=0}^{d-1} \boldsymbol{\Upsilon}_{i,j,k} \boldsymbol{\Upsilon}_{i',j',k'} \zeta^{i(k'-j')} (z^{-k'} F^{i'} E^i K^{k'+j}) \otimes (z^{-k} E^{i'} F^i K^{j'+k}).$$

We need to compute the endomorphism of $M_{l,p}$ equal to $(\mathrm{Id}_{M_{l,p}} \otimes \mathrm{Tr}^M_+)(\tau(R)R|_{M_{l,p}\otimes M})$. Since $M_{l,p}$ is simple, this endomorphism is the multiplication by a scalar ϖ, and so it is sufficient to compute the action on $e_1^{(l,p)} \in M_{l,p}$. Therefore, all the terms (in the big sum giving $\tau(R)R$) corresponding to $i \neq 0$ disappear (because $E e_1^{(l,p)} = 0$). Also, since we are only interested in the coefficient on $e_1^{(l,p)}$ of the result (because the coefficients on other vectors will be zero), all the terms corresponding to $i' \neq 0$ also disappear. Therefore,

$$\varpi = \frac{1}{d^2} \sum_{j,j',k,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-kj-k'j'} (\zeta^{2p+l-1})^{-k'} (\zeta^{p+l-1})^{k'+j} \mathbf{Tr}(z^{-1} K z^{-k} K^{j'+k}|_M).$$

So it remains to compute the element

$$b = \frac{1}{d^2} \sum_{j,j',k,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{-kj-k'j'} (\zeta^{2p+l-1})^{-k'} (\zeta^{p+l-1})^{k'+j} z^{-k-1} K^{j'+k+1}$$

of $D(B)$. But,

$$b = \frac{1}{d^2} \sum_{j',k \in \mathbb{Z}/d\mathbb{Z}} \Big(\sum_{j,k' \in \mathbb{Z}/d\mathbb{Z}} \zeta^{l(p+l-1-k)} \zeta^{k'(-p-j')} \Big) z^{-k-1} K^{j'+k+1}.$$

So, only the terms corresponding to $k = l + p - 1$ and $j' = -p$ remain and so

$$b = z^{-l-p} K^l,$$

as expected.

The other formula is obtained through a similar computation. □

We denote by $\mathbb{S}^{\pm} = (\mathbb{S}^{\pm}_{\lambda,\lambda'})_{\lambda,\lambda' \in \Lambda(d)}$ the square matrix defined by

$$\mathbb{S}^{\pm}_{\lambda,\lambda'} = \mathrm{Tr}_{\pm}(c_{M_{\lambda'},M_{\lambda}} \circ c_{M_{\lambda},M_{\lambda'}}).$$

Similary, we define $\mathbb{T}^{\pm}$ to be the diagonal matrix (whose rows and columns are indexed by $\lambda \in \Lambda(d)$) and whose λ-entry is

$$\mathbb{T}^{\pm}_{\lambda} = \omega_{\lambda}(\theta_1^{\pm 1}).$$

Let us first give a formula for $\mathbb{S}^{\pm}_{\lambda,\lambda'}$ and $\mathbb{T}^{\pm}_{\lambda}$.

Corollary 5.5. — *Let (l,p), $(l',p') \in \Lambda(d)$. Then*

$$\mathbb{S}^{+}_{(l,p),(l',p')} = \frac{\zeta}{1-\zeta} \zeta^{-ll'-lp'-pl'-2pp'}(1-\zeta^{ll'}), \qquad \mathbb{T}^{+}_{(l,p)} = \zeta^{p(p+l)},$$

$$\mathbb{S}^{-}_{(l,p),(l',p')} = \frac{\zeta^{2p+l+2p'+l'-1}}{1-\zeta} \zeta^{-ll'-lp'-pl'-2pp'}(1-\zeta^{ll'}) \qquad \text{and} \qquad \mathbb{T}^{-}_{(l,p)} = \zeta^{-p(p+l)}.$$

Proof. — This follows immediately from (2.3), (5.3), (4.4) and Theorem 5.4. □

5.C. Fusion datum associated with $D(B)$-STAB. — Let $\mathscr{E}$ denote a set of representatives of ι-orbits in $\{1,2,\dots,d-1\}\times\mathbb{Z}/d\mathbb{Z}$. We then define arbitrarily

$$\dim^{\mathrm{ST}}(D(B))=\sum_{(l,p)\in\mathscr{E}}\dim_-(M_{l,p})\dim_+(M_{l,p}).$$

We have no theoretical justification for this definition... Then

$$\dim^{\mathrm{ST}}(D(B))=\frac{1}{2}\dim(D(B))=\frac{d^2}{(1-\zeta)(1-\zeta^{-1})}.$$

So $\dim^{\mathrm{ST}} D(B)$ is a positive real number and we denote by $\sqrt{\dim^{\mathrm{ST}} D(B)}$ its positive square root. Since $1-\zeta^{-1}=-\zeta^{-1}(1-\zeta)$, there exists a unique square root $\sqrt{-\zeta}$ of $-\zeta$ such that

$$\sqrt{\dim^{\mathrm{ST}}(D(B))}=\frac{d\sqrt{-\zeta}}{1-\zeta}.$$

We denote by $\mathbb{S}^{\mathrm{ST}}=(\mathbb{S}_{\lambda,\lambda'})_{\lambda,\lambda'\in\mathscr{E}}$ the square matrix defined by

$$\mathbb{S}^{\mathrm{ST}}_{\lambda,\lambda'}=\frac{\mathbb{S}^{+}_{\lambda,\lambda'}}{\sqrt{\dim^{\mathrm{ST}}(D(B))}}.$$

Then it follows from Corollary 5.5 that

$$\mathbb{S}^{\mathrm{ST}}_{(l,p),(l',p')}=\frac{\sqrt{-\zeta}}{d}\zeta^{-ll'-lp'-pl'-2pp'}(\zeta^{ll'}-1). \tag{5.6}$$

6. Comparison with Malle $\mathbb{Z}$-fusion datum

We refer to **[Ma]** and **[Cu]** for most of the material of this section. We denote by $\mathscr{E}(d)$ the set of pairs (i,j) of natural numbers such that $0\leqslant i<j\leqslant d-1$.

6.A. Set-up. — Let $Y=\{0,1,\dots,d\}$ and let $\pi: Y\to\{0,1\}$ be the map defined by

$$\pi(i)=\begin{cases}1 & \text{if } i\in\{0,1\},\\ 0 & \text{if } i\geqslant 2.\end{cases}$$

We denote by $\Psi(Y,\pi)$ the set of maps $f: Y\to\{0,1,\dots,d-1\}$ such that f is strictly increasing on $\pi^{-1}(0)=\{2,3,\dots,d\}$ and strictly increasing on $\pi^{-1}(1)=\{0,1\}$. Since f is injective on $\{2,3,\dots,d\}$, there exists a unique element $\mathbf{k}(f)\in\{0,1,\dots,d-1\}$ which does not belong to $f(\{2,3,\dots,d\})$. Note that, since f

is strictly increasing on $\{2,3,\ldots,d\}$, $\mathbf{k}(f)$ determines the restriction of f to $\{2,3,\ldots,d\}$. So the map

$$\begin{array}{ccc} \Psi(Y,\pi) & \longrightarrow & \mathscr{E}(d)\times\{0,1,\ldots,d-1\} \\ f & \longmapsto & (f(0),f(1),\mathbf{k}(f)) \end{array} \tag{6.1}$$

is bijective. For $f\in\Psi(Y,\pi)$, we set

$$\varepsilon(f)=(-1)^{|\{(y,y')\in Y\times Y\ \mid\ y<y' \text{ and } f(y)<f(y')\}|}.$$

We denote by $V=\bigoplus_{i=0}^{d-1}\mathbb{C}v_i$ a vector space of dimension d endowed with a basis $(v_i)_{0\leqslant i\leqslant d-1}$ and we denote by $\mathscr{S}$ the square matrix $(\zeta^{ij})_{0\leqslant i,j\leqslant d-1}$, which will be viewed as an automorphism of V. Note that $\mathscr{S}$ is the character stable of the cyclic group $\boldsymbol{\mu}_d$. We set $\delta(d)=(-1)^{d(d-1)/2}\det(\mathscr{S})=\prod_{0\leqslant i<j\leqslant d-1}(\zeta^i-\zeta^j)$. Recall that

$$\delta(d)^2=(-1)^{(d-1)(d-2)/2}d^d.$$

If $f\in\Psi(Y,\pi)$, let

$$\mathbf{v}_f=(v_{f(0)}\wedge v_{f(1)})\otimes(v_{f(2)}\wedge v_{f(3)}\wedge\cdots\wedge v_{f(d)})\in\big(\textstyle\bigwedge^2 V\big)\otimes\big(\textstyle\bigwedge^{d-1} V\big).$$

Then $(\mathbf{v}_f)_{f\in\Psi(Y,\pi)}$ is a $\mathbb{C}$-basis of $\big(\bigwedge^2 V\big)\otimes\big(\bigwedge^{d-1} V\big)$. If $f'\in\Psi(Y,\pi)$, we write

$$\Big(\big(\textstyle\bigwedge^2\mathscr{S}\big)\otimes\big(\textstyle\bigwedge^{d-1}\mathscr{S}\big)\Big)(\mathbf{v}_{f'})=\sum_{f\in\Psi(Y,\pi)}\mathbf{S}_{f,f'}\mathbf{v}_f.$$

In other words, $(\mathbf{S}_{f,f'})_{f,f'\in\Psi(Y,\pi)}$ is the matrix of the automorphism $\big(\bigwedge^2\mathscr{S}\big)\otimes\big(\bigwedge^{d-1}\mathscr{S}\big)$ of $\big(\bigwedge^2 V\big)\otimes\big(\bigwedge^{d-1} V\big)$ in the basis $(\mathbf{v}_f)_{f\in\Psi(Y,\pi)}$.

Lemma 6.2. — *Let f, $f'\in\Psi(Y,\pi)$. We define*

$$i=f(0),\quad j=f(1),\quad k=\mathbf{k}(f),$$
$$i'=f'(0),\quad j'=f'(1),\quad k'=\mathbf{k}(f').$$

Then

$$\mathbf{S}_{f,f'}=(-1)^{k+k'}\frac{\delta(d)}{d}\zeta^{-kk'}(\zeta^{ii'+jj'}-\zeta^{ij'+ji'}).$$

Proof. — The computation of the action of $\bigwedge^2\mathscr{S}$ is easy, and gives the term $\zeta^{ii'+jj'}-\zeta^{ij'+ji'}$. So it remains to show that the determinant of the matrix $\mathscr{S}(k,k')$ obtained from $\mathscr{S}$ by removing the k-th row and the k'-th column is equal to $(-1)^{k+k'}\zeta^{-kk'}\delta(d)/d$. For this, let $\mathscr{S}'(k)$ denote the matrix whose k-th row is equal to $(1,t,t^2,\ldots,t^{d-1})$ (where t is an indeterminate) and whose other rows coincide with those of $\mathscr{S}$. Then $(-1)^{k+k'}\det(\mathscr{S}(k,k'))$ is equal to the coefficient of t^k in the polynomial $\det(\mathscr{S}'(k))$. This is a Vandermonde determinant and

$$\begin{aligned}\det(\mathscr{S}'(k)) &= \prod_{\substack{0 \leqslant i<j \leqslant d-1 \\ i\neq k, j\neq k}} (\zeta^j - \zeta^i) \cdot \prod_{i=0}^{k-1} (t-\zeta^i) \cdot \prod_{i=k+1}^{d-1} (\zeta^i - t) \\ &= \delta(d) \prod_{\substack{i=0 \\ i\neq k}}^{d-1} \frac{(t-\zeta^i)}{(\zeta^k - \zeta^i)}.\end{aligned}$$

But

$$\prod_{\substack{i=0 \\ i\neq k}}^{d-1} (t-\zeta^i) = \frac{t^d - 1}{t-\zeta^k} = \sum_{i=0}^{d-1} t^i \zeta^{(d-1-i)k}.$$

Therefore,

$$\det(\mathscr{S}(k,k')) = (-1)^{k+k'} \delta(d) \frac{\zeta^{(d-1-k')k}}{d\zeta^{(d-1)k}} = (-1)^{k+k'} \frac{\delta(d)}{d} \zeta^{-kk'},$$

as desired. □

6.B. Malle $\mathbb{Z}$-fusion datum. — Let

$$\Psi^{\#}(Y,\pi) = \{f \in \Psi(Y,\pi) \mid \sum_{y \in Y} f(y) \equiv \frac{d(d-1)}{2} \mod d\}$$

and, if $f \in \Psi^{\#}(Y,\pi)$, we define

$$\mathrm{Fr}(f) = \zeta_*^{d(1-d^2)} \prod_{y \in Y} \zeta_*^{-6(f(y)^2 + df(y))},$$

where ζ_* is a primitive $12d$-th root of unity such that $\zeta_*^{12} = \zeta$.

We denote by $\mathbb{T}$ diagonal matrix (whose rows and column are indexed by $\Psi^{\#}(Y,\pi)$) equal to $\mathrm{diag}(\mathrm{Fr}(f))_{f \in \Psi^{\#}(Y,\pi)}$. We denote by $\mathbb{S} = (\mathbb{S}_{f,g})_{f,g \in \Psi^{\#}(Y,\pi)}$ the square matrix defined by

$$\mathbb{S}_{f,g} = \frac{(-1)^{d-1}}{\delta(d)} \varepsilon(f)\varepsilon(g)\, \mathbf{S}_{f,g}.$$

Note that $\mathbb{S}_{f,f_{0,1}} \neq 0$ for all $f \in \Psi^{\#}(Y,\pi)$ (see Lemma 6.2).

Proposition 6.3 **(Malle [Ma]).** — *With the previous notation, we have:*

(a) $\mathbb{S}^4 = (\mathbb{S}\mathbb{T})^3 = [\mathbb{S}^2, \mathbb{T}] = 1$.
(b) ${}^t\mathbb{S} = \mathbb{S}$ *and* ${}^t\overline{\mathbb{S}}\,\mathbb{S} = 1$.

(c) *For all $f, g, h \in \Psi^{\#}(Y,\pi)$, the number*

$$N_{f,g}^{h} = \sum_{i\in\Psi^{\#}(Y,\pi)} \frac{\mathbb{S}_{i,f}\mathbb{S}_{i,g}\overline{\mathbb{S}}_{i,h}}{\mathbb{S}_{i,f_{0,1}}}$$

belongs to $\mathbb{Z}$.

The pair $(\mathbb{S},\mathbb{T})$ is called the *Malle $\mathbb{Z}$-fusion datum*.

6.C. Comparison. — We wish to compare the $\mathbb{Z}$-fusion datum $(\mathbb{S},\mathbb{T})$ with the ones obtained from the tensor categories $D(B)$-mod and $D(B)$-stab. For this, we will use the bijection (6.1) to characterize elements of $\Psi^{\#}(Y,\pi)$ (here, if $k\in\mathbb{Z}$, then k^{res} denotes the unique element in $\{0,1,\dots,d-1\}$ such that $k\equiv k^{\mathrm{res}} \bmod d$):

Lemma 6.4. — *Let $f \in \Psi(Y,\pi)$. Then $f \in \Psi^{\#}(Y,\pi)$ if and only if $\mathbf{k}(f) = (f(0)+f(1))^{\mathrm{res}}$. Consequently, the map*

$$\begin{array}{ccc} \Psi^{\#}(Y,\pi) & \longrightarrow & \mathscr{E}(d) \\ f & \longmapsto & (f(0),f(1)) \end{array}$$

is bijective

Proof. — We have

$$\sum_{y\in Y} f(y) = f(0)+f(1)+\frac{d(d-1)}{2}-\mathbf{k}(f)$$

and the result follows. □

If $(i,j)\in\Lambda(d)$, we denote by $f_{i,j}$ the unique element of $\Psi^{\#}(Y,\pi)$ such that $f_{i,j}(0)=i$ and $f_{i,j}(1)=j$. We have

$$\mathrm{Fr}(f_{i,j}) = \zeta^{ij} \tag{6.5}$$

and, if $(i,j), (i',j')\in\Lambda(d)$, then

$$\mathbb{S}_{f_{i,j},f_{i',j'}} = \frac{(-1)^{(i+j)^{\mathrm{res}}+(i'+j')^{\mathrm{res}}}}{d}\varepsilon(f_{i,j})\varepsilon(f_{i',j'})(\zeta^{-ij'-ji'}-\zeta^{-ii'-jj'}). \tag{6.6}$$

Proof. — The second equality follows immediately from Lemmas 6.2 and 6.4. Let us prove the first one. By definition, $\mathrm{Fr}(f_{i,j})=\zeta_*^{\alpha}$, where

$$\alpha=d(1-d^2)-6\sum_{y\in Y}(f_{i,j}(y)^2+d\,f_{i,j}(y)).$$

The construction of $f_{i,j}$ shows that

$$\alpha=d(1-d^2)-6(i^2+di)-6(j^2+dj)-6\sum_{k=0}^{d-1}(k^2+dk)+6(((i+j)^{\mathrm{res}})^2+d(i+j)^{\mathrm{res}}).$$

Write $i+j=(i+j)^{\mathrm{res}}+\eta d$, with $\eta\in\{0,1\}$. Then $\eta^2=\eta$ and so

$$\begin{aligned}(i+j)^2+d(i+j) &= ((i+j)^{\mathrm{res}})^2+d(i+j)^{\mathrm{res}}+2\eta d(i+j)+2\eta d^2\\ &\equiv ((i+j)^{\mathrm{res}})^2+d(i+j)^{\mathrm{res}} \mod 2d.\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha &\equiv 12ij+d(1-d^2)-6\sum_{k=0}^{d-1}(k^2+dk) \mod 12d\\ &\equiv 12ij \mod 12d.\end{aligned}$$

So $\mathrm{Fr}(f_{i,j})=\zeta_*^{12ij}=\zeta^{ij}$. □

Let

$$\begin{array}{rccc}\varphi: & \mathscr{E}(d) & \longrightarrow & \Lambda^{\#}(d)\\ & (i,j) & \longmapsto & (j-i,i).\end{array}$$

The $\pi(\mathscr{E}(d))$ is a set of representatives of ι-orbits in $\Lambda^{\#}(d)$ and the pairs of matrices $(\mathbb{S}^{\mathrm{ST}},\mathbb{T}^{\mathrm{ST}})$ and $(\mathbb{S},\mathbb{T})$ are related by the following equality (which follows immediately from Corollary 5.5, (6.5) and (6.6)):

(6.7) $$\mathbb{S}_{f_{i,j},f_{i',j'}}=\sqrt{-\zeta}\;\mathbb{S}^{\mathrm{ST}}_{\varphi(i,j),\varphi(i',j')}\qquad\text{and}\qquad\mathbb{T}_{f_{i,j}}=\mathbb{T}^{\mathrm{ST}}_{\varphi(i,j)}.$$

Therefore:

Theorem 6.8. — *Malle $\mathbb{Z}$-fusion datum $(\mathbb{S},\mathbb{T})$ can be categorified by the monoidal category $D(B)$-*STAB*, endowed with the pivotal structure induced by the pivot $z^{-1}K$ and the balanced structure induced by $z\theta$.*

Remark 6.9. — M. Broué, G. Malle & J. Michel have associated to a class of exceptional reflection groups (the *spetsial* ones) a set of *"unipotent characters"* and a partition of these unipotent characters into families. To each family, they have also associated a $\mathbb{Z}$-fusion datum (S,T). It turns out that some of these $\mathbb{Z}$-fusion data can by categorified by Hopf quotients of the algebra $D(B)$ (this is investigated by A. Lacabanne in his Ph.D. Thesis). ■

A. Appendix. Recollection of S-matrices

We follow closely [**EGNO**, Chapters 4 and 8].

Let $\mathscr{C}$ be a *tensor category* over $\mathbb{C}$, as defined in [**EGNO**, Definition 4.1.1]: $\mathscr{C}$ is a locally finite $\mathbb{C}$-linear rigid monoidal category (whose unit object is denoted by $\mathbf{1}$) such that the bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is $\mathbb{C}$-bilinear on morphisms and $\mathrm{End}_{\mathscr{C}}(\mathbf{1}) = \mathbb{C}$. If X is an object in $\mathscr{C}$, its left (respectively right) dual is denoted by X^* (respectively ${}^*X$) and we denote by

$$\mathrm{coev}_X : \mathbf{1} \longrightarrow X \otimes X^* \qquad \text{and} \qquad \mathrm{ev}_X : X^* \otimes X \longrightarrow \mathbf{1}$$

the *coevealuation* and *evaluation* morphisms respectively.

We assume that $\mathscr{C}$ is *braided*, namely that it is endowed with a bifunctorial family of isomorphisms $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ such that

$$c_{X,Y\otimes Y'} = (\mathrm{Id}_Y \otimes c_{X,Y'}) \circ (c_{X,Y} \otimes \mathrm{Id}_{Y'}) \tag{A.1}$$

and

$$c_{X\otimes X',Y} = (c_{X,Y} \otimes \mathrm{Id}_{X'}) \circ (\mathrm{Id}_X \otimes c_{X',Y}). \tag{A.2}$$

for all objects X, X', Y and Y' in $\mathscr{C}$ (we have omitted the associativity constraints).

Finally, we also assume that $\mathscr{C}$ is *pivotal* [**EGNO**, Definition 4.7.8], i.e. that it is equipped with a family of functorial isomorphisms $a_X : X \to X^{**}$ (for X running over the objects of $\mathscr{C}$) such that $a_{X\otimes Y} = a_X \otimes a_Y$. If $f \in \mathrm{End}_{\mathscr{C}}(X)$, the pivotal structure allows to define two *traces*:

$$\mathrm{Tr}_+(f) = \mathrm{ev}_{X^*} \circ (a_X f \otimes \mathrm{Id}_{X^*}) \circ \mathrm{coev}_X \in \mathrm{End}_{\mathscr{C}}(\mathbf{1}) = \mathbb{C}$$

and $$\mathrm{Tr}_-(f) = \mathrm{ev}_X \circ (\mathrm{Id}_{X^*} \otimes f\, a_X^{-1}) \circ \mathrm{coev}_{X^*} \in \mathrm{End}_{\mathscr{C}}(\mathbf{1}) = \mathbb{C}.$$

We will sometimes write $\mathrm{Tr}_+^X(f)$ or $\mathrm{Tr}_-^X(f)$ for $\mathrm{Tr}_+(f)$ and $\mathrm{Tr}_-(f)$. We define two *dimensions*

$$\dim_+(X) = \mathrm{Tr}_+(\mathrm{Id}_X) \qquad \text{and} \qquad \dim_-(X) = \mathrm{Tr}_-(\mathrm{Id}_X).$$

To summarize, we will work under the following hypothesis:

Hypothesis and notation. *We fix in this section a braided pivotal tensor category $\mathscr{C}$ as above. We denote by* $\mathrm{Gr}(\mathscr{C})$ *its Grothendieck ring. If X is an object in $\mathscr{C}$, we denote by $[X]$ its class in* $\mathrm{Gr}(\mathscr{C})$*. The set of isomorphism classes of simple objects in $\mathscr{C}$ will be denoted by* $\mathrm{Irr}(\mathscr{C})$*. If $X \in \mathrm{Irr}(\mathscr{C})$ and Y is any object in $\mathscr{C}$, we denote by $[Y : X]$ the multiplicity of X in a Jordan-Hölder series of Y.*

A.A. S-matrices. — If X, Y are objects in $\mathscr{C}$, we set

$$s^+_{X,Y} = (\mathrm{Id}_X \otimes \mathrm{Tr}^Y_+)(c_{Y,X}\, c_{X,Y}) \in \mathrm{End}_{\mathscr{C}}(X).$$

and
$$s^-_{X,Y} = (\mathrm{Id}_X \otimes \mathrm{Tr}^Y_-)(c_{Y,X}\, c_{X,Y}) \in \mathrm{End}_{\mathscr{C}}(X).$$

It is clear that they induce two morphisms of abelian groups

$$\begin{array}{rccc} s^+_X : & \mathrm{Gr}(\mathscr{C}) & \longrightarrow & \mathrm{End}_{\mathscr{C}}(X) \\ & [\,Y\,] & \longmapsto & s^+_{X,Y} \end{array} \qquad \text{and} \qquad \begin{array}{rccc} s^-_X : & \mathrm{Gr}(\mathscr{C}) & \longrightarrow & \mathrm{End}_{\mathscr{C}}(X) \\ & [\,Y\,] & \longmapsto & s^-_{X,Y}. \end{array}$$

Definition A.3. — *An object X in $\mathscr{C}$ is said **endominimal** if* $\mathrm{End}_{\mathscr{C}}(X) = \mathbb{C}$.

For instance, a simple object is endominimal (and **1** is also endominimal, but **1** is simple in a tensor category [**EGNO**, Theorem 4.3.1]). Note also that an endominimal module is indecomposable. So if $\mathscr{C}$ is moreover semisimple, then an object is endominimal if and only if it is simple.

If X is endominimal, then we will view $s^+_{X,Y}$ and $s^-_{X,Y}$ as elements of $\mathbb{C} = \mathrm{End}_{\mathscr{C}}(X)$.

Proposition A.4. — *If X is endominimal, then $s^+_X : \mathrm{Gr}(\mathscr{C}) \to \mathbb{C}$ and $s^-_X : \mathrm{Gr}(\mathscr{C}) \to \mathbb{C}$ are morphisms of rings.*

Proof. — Assume that X is endominimal. We will only prove the result for s^+_X, which amounts to show that

$$(*) \qquad s^+_{X,Y\otimes Y'} = s^+_{X,Y}\, s^+_{X,Y'}.$$

First, note that the following equality

$$c_{Y\otimes Y',X}\, c_{X,Y\otimes Y'} = (c_{Y,X} \otimes \mathrm{Id}_{Y'}) \circ (\mathrm{Id}_Y \otimes c_{Y',X}\, c_{X,Y'}) \circ (c_{X,Y} \otimes \mathrm{Id}_{Y'})$$

holds by (A.1) and (A.2). Taking $\mathrm{Id}_X \otimes \mathrm{Id}_Y \otimes \mathrm{Tr}^{Y'}_+$ on the right-hand side, one gets $s^+_{X,Y'}\, c_{Y,X}\, c_{X,Y} \in \mathrm{End}_{\mathscr{C}}(X \otimes Y)$ (because X is endominimal). Applying now $\mathrm{Id}_X \otimes \mathrm{Tr}^Y_+$, one get $s^+_{X,Y'} s^+_{X,Y}\, \mathrm{Id}_X$. Since

$$(\mathrm{Id}_X \otimes \mathrm{Tr}^Y_+) \circ (\mathrm{Id}_X \otimes \mathrm{Id}_Y \otimes \mathrm{Tr}^{Y'}_+) = \mathrm{Id}_X \otimes \mathrm{Tr}^{Y\otimes Y'}_+,$$

this proves $(*)$. □

Proposition A.5. — *Let X be endominimal and let X' be an endominimal subquotient of X. Then*

$$s^+_X = s^+_{X'} \qquad \textit{and} \qquad s^-_X = s^-_{X'}.$$

Proof. — This is just because the endomorphism $(\mathrm{Id}_X \otimes \mathrm{Tr}_+^Y)(c_{Y,X}c_{X,Y})$ of X is a multiplication by a scalar, and this scalar can be computed on any non-trivial subquotient of X. □

Corollary A.6. — *Let X and X' be two endominimal objects in $\mathscr{C}$ belonging to the same block. Then*

$$s_X^+ = s_{X'}^+ \qquad \textit{and} \qquad s_X^- = s_{X'}^-.$$

Proof. — By Proposition A.5, we may asume that X and X' are simple. We may also assume that X is not isomorphic to X' and that $\mathrm{Ext}^1_{\mathscr{C}}(X,X') = 0$. So let $\mathbf{X} \in \mathscr{C}$ be such that there exists a non-split exact sequence

$$0 \longrightarrow X' \longrightarrow \mathbf{X} \longrightarrow X \longrightarrow 0.$$

Since $X \not\simeq X'$, we have $\mathrm{End}_{\mathscr{C}}(\mathbf{X}) = \mathbb{C}$ and so $\mathbf{X}$ is endominimal. Then it follows from Proposition A.5 that

$$s_{\mathbf{X}}^+ = s_X^+ = s_{X'}^+ \qquad \text{and} \qquad s_{\mathbf{X}}^- = s_X^- = s_{X'}^-,$$

as desired. □

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September 4, 2017

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