ON THE COHOMOLOGY OF CALOGERO-MOSER SPACES

by

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Abstract. — We compute the equivariant cohomology of smooth Calogero-Moser spaces and some associated symplectic resolutions of symplectic quotient singularities.

1. Notation and main results

All along this note, we will abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$. By an algebraic variety, we mean a reduced scheme of finite type over $\mathbb{C}$.

1.A. Reflection group. — Let V be a $\mathbb{C}$-vector space of finite dimension n and let W be a finite subgroup of $\mathbf{GL}_{\mathbb{C}}(V)$. We set

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}} V^s = n-1\}$$

and we assume that

$$W = \langle \mathrm{Ref}(W) \rangle.$$

We set $\varepsilon : W \to \mathbb{C}^\times$, $w \mapsto \det(w)$.

If $s \in \mathrm{Ref}(W)$, we denote by $\alpha_s^\vee$ and α_s two elements of V and V^* respectively such that $V^s = \mathrm{Ker}(\alpha_s)$ and $V^{*s} = \mathrm{Ker}(\alpha_s^\vee)$, where $\alpha_s^\vee$ is viewed as a linear form on V^*.

If $w \in W$, we set

$$\mathrm{cod}(w) = \mathrm{codim}_{\mathbb{C}}(V^w)$$

and we define a filtration $\mathscr{F}_\bullet(\mathbb{C}W)$ of the group algebra of W as follows: let

$$\mathscr{F}_i(\mathbb{C}W) = \bigoplus_{\mathrm{cod}(w) \leqslant i} \mathbb{C}w.$$

Then

$$\mathbb{C}\,\mathrm{Id}_V = \mathscr{F}_0(\mathbb{C}W) \subset \mathscr{F}_1(\mathbb{C}W) \subset \cdots \subset \mathscr{F}_n(\mathbb{C}W) = \mathbb{C}W = \mathscr{F}_{n+1}(\mathbb{C}W) = \cdots$$

is a filtration of $\mathbb{C}W$. For any subalgebra A of $\mathbb{C}W$, we set $\mathscr{F}_i(A) = A \cap \mathscr{F}_i(\mathbb{C}W)$, so that

$$\mathbb{C}\,\mathrm{Id}_V = \mathbb{C}\mathscr{F}_0(A) \subset \mathscr{F}_1(A) \subset \cdots \subset \mathscr{F}_n(A) = A = \mathscr{F}_{n+1}(A) = \cdots$$

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is also a filtration of A. Let $\hbar$ be a formal variable, and write

$$\begin{aligned}\mathrm{Rees}^\bullet_{\mathscr{F}}(A) &= \bigoplus_{i\geqslant 0} \hbar^i \mathscr{F}_i(A) \subset \mathbb{C}[\hbar]\otimes A \qquad \text{(the \textit{Rees algebra})},\\ \mathrm{gr}^\bullet_{\mathscr{F}}(A) &= \bigoplus_{i\geqslant 0} \mathscr{F}_i(A)/\mathscr{F}_{i-1}(A).\end{aligned}$$

Recall that $\mathrm{gr}^\bullet_{\mathscr{F}}(A) \simeq \mathrm{Rees}^\bullet_{\mathscr{F}}(A)/\hbar\,\mathrm{Rees}^\bullet_{\mathscr{F}}(A)$.

1.B. Rational Cherednik algebra at $t = 0$. — All along this note, we fix a function c : $\mathrm{Ref}(W) \to \mathbb{C}$ which is invariant under conjugacy. We define the rational Cherednik algebra $\mathbf{H}_c$ to be the quotient of the algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $\mathrm{T}(V\oplus V^*)$ with the group W) by the relations

$$(\mathscr{H}_c) \qquad \begin{cases} [x,x'] = [y,y'] = 0, \\ [y,x] = \displaystyle\sum_{s\in\mathrm{Ref}(W)} (\varepsilon(s)-1)c_s \frac{\langle y,\alpha_s\rangle\langle\alpha_s^\vee,x\rangle}{\langle\alpha_s^\vee,\alpha_s\rangle} s, \end{cases}$$

for all x ,$x' \in V^*$, y, $y' \in V$. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \to \mathbf{H}_c$ and $\mathbb{C}[V^*] \to \mathbf{H}_c$. Recall [**4**, Theorem 1.3] that we have an isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C}[V]\otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_c \tag{1.1}$$

induced by multiplication (this is the so-called *PBW-decomposition*).

We denote by $\mathbf{Z}_c$ the center of $\mathbf{H}_c$: it is well-known [**4**] that $\mathbf{Z}_c$ is an integral domain, which is integrally closed and contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras (so it contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$), and that it is a free $\mathbf{P}$-module of rank $|W|$. We denote by $\mathscr{Z}_c$ the algebraic variety whose ring of regular functions $\mathbb{C}[\mathscr{Z}_c]$ is $\mathbf{Z}_c$: this is the *Calogero-Moser space* associated with the datum (V,W,c).

Using the PBW-decomposition, we define a $\mathbb{C}$-linear map $\mathbf{\Omega}^c_{\mathbf{H}} : \mathbf{H}_c \longrightarrow \mathbb{C}W$ by

$$\mathbf{\Omega}^c_{\mathbf{H}}(f\,w\,g) = f(0)g(0)w$$

for all $f \in \mathbb{C}[V]$, $g \in \mathbb{C}[V^*]$ and $w \in \mathbb{C}W$. This map is W-equivariant for the action on both sides by conjugacy, so it induces a well-defined $\mathbb{C}$-linear map

$$\mathbf{\Omega}^c : \mathbf{Z}_c \longrightarrow \mathrm{Z}(\mathbb{C}W).$$

Recall from [**3**, Corollary 4.2.11] that $\mathbf{\Omega}^c$ is a morphism of algebras, and that

(1.2) *$\mathscr{Z}_c$ is smooth if and only if $\mathbf{\Omega}^c$ is surjective.*

The "only if" part is essentially due to Gordon [**6**, Corollary 5.8] (see also [**3**, Proposition 9.6.6 and (16.1.2)]) while the "if" part follows from the work of Bellamy, Schedler and Thiel [**13**, Corollary 1.4].

1.C. Grading. — The algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ can be $\mathbb{Z}$-graded in such a way that the generators have the following degrees

$$\begin{cases} \deg(y) = -1 & \text{if } y \in V, \\ \deg(x) = 1 & \text{if } x \in V^*, \\ \deg(w) = 0 & \text{if } w \in W. \end{cases}$$

This descends to a $\mathbb{Z}$-grading on $\mathbf{H}_c$ since the defining relations $(\mathscr{H}_c)$ are homogeneous. Since the center of a graded algebra is always graded, it follows that $\mathbf{Z}_c$ also inherits a

$\mathbb{Z}$-grading. By definition $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of $\mathbf{Z}_c$. So the Calogero-Moser space $\mathscr{Z}_c$ inherits a regular $\mathbb{C}^\times$-action.

1.D. Main results. — For a complex algebraic variety $\mathscr{X}$ (equipped with its classical topology), we denote by $\mathrm{H}^i(\mathscr{X})$ its i-th singular cohomology group with coefficients in $\mathbb{C}$. If $\mathscr{X}$ carries a regular action of a torus $\mathbf{T}$, we denote by $\mathrm{H}^i_{\mathbf{T}}(\mathscr{X})$ its i-th $\mathbf{T}$-equivariant cohomology group (still with coefficients in $\mathbb{C}$). Note that $\mathrm{H}^{2\bullet}(\mathscr{X}) = \bigoplus_{i \geqslant 0} \mathrm{H}^{2i}(\mathscr{X})$ is a graded $\mathbb{C}$-algebra and $\mathrm{H}^{2\bullet}_{\mathbf{T}}(\mathscr{X}) = \bigoplus_{i \geqslant 0} \mathrm{H}^{2i}_{\mathbf{T}}(\mathscr{X})$ is a graded $\mathrm{H}^{2\bullet}_{\mathbf{T}}(\mathbf{pt})$-algebra. The following result [**4**, Theorem 1.8] describes the algebra structure on the cohomology of $\mathscr{Z}_c$ (with coefficients in $\mathbb{C}$):

***Theorem 1.3* (Etingof-Ginzburg).** — *Assume that $\mathscr{Z}_c$ is smooth. Then:*

(a) $\mathrm{H}^{2i+1}(\mathscr{Z}_c) = 0$ *for all i.*

(b) *There is an isomorphism of graded $\mathbb{C}$-algebras* $\mathrm{H}^{2\bullet}(\mathscr{Z}_c) \simeq \mathrm{gr}^\bullet_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W))$.

In this note, we prove an equivariant version of this statement (we identify $R = \mathrm{H}^*_{\mathbb{C}^\times}(\mathbf{pt})$ with $\mathbb{C}[\hbar]$ in the usual way):

Theorem A. — *Assume that $\mathscr{Z}_c$ is smooth. Then:*

(a) $\mathrm{H}^{2i+1}_{\mathbb{C}^\times}(\mathscr{Z}_c) = 0$ *for all i.*

(b) *There is an isomorphism of graded R-algebras* $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{Z}_c) \simeq \mathrm{Rees}^\bullet_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W))$.

Note that Theorem A(a) just follows from statement (a) of Etingof-Ginzburg Theorem by Proposition 2.4(a) below. As a partial consequence of Theorem A, we also obtain the following application to the equivariant cohomology of some symplectic resolutions.

Theorem B. — *Assume that the symplectic quotient singularity $(V \times V^*)/W$ admits a symplectic resolution $\pi : \mathscr{X} \longrightarrow (V \times V^*)/W$. Recall that the $\mathbb{C}^\times$-action on $(V \times V^*)/W$ lifts (uniquely) to $\mathscr{X}$ (see* [**11**, Theorem 1.3(ii)]*). Then:*

(a) $\mathrm{H}^{2i+1}_{\mathbb{C}^\times}(\mathscr{X}) = 0$ *for all i.*

(b) *There is an isomorphism of graded R-algebras* $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{X}) \simeq \mathrm{Rees}^\bullet_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W))$.

Note that for $W = \mathfrak{S}_n$ acting on $\mathbb{C}^n$, Theorem B describes the equivariant cohomology of the Hilbert scheme of n points in $\mathbb{C}^2$: this was already proved (by different methods) by Vasserot [**14**]. In [**5**, Conjecture 1.3], Ginzburg-Kaledin proposed a conjecture for the equivariant cohomology of a symplectic resolution of a symplectic quotient singularity E/G, where E is a finite dimensional symplectic vector space and G is a finite subgroup of $\mathbf{Sp}(E)$. However, their conjecture cannot hold as stated, because they considered the $\mathbb{C}^\times$-action by dilatation, which is contractible. Theorem B shows that the correct equivariant cohomological realization of the Rees algebra is provided by the symplectic $\mathbb{C}^\times$-action, which exists only when G stabilizes a Lagrangian subspace of E.

Remark 1.4. — Recall from the works of Etingof-Ginzburg [**4**], Ginzburg-Kaledin [**5**], Gordon [**6**] and Bellamy [**1**] that the existence of a symplectic resolution of $(V \times V^*)/W$ is equivalent to the existence of a parameter c such that $\mathscr{Z}_c$ is smooth, and that it can only occurs if all the irreducible components of W are of type $G(d,1,n)$ (for some d, $n \geqslant 1$) or G_4 in Shephard-Todd classification. ■

1.E. Conjectures. — In [**3**, Chapter 16, Conjectures COH and ECOH], Rouquier and the first author proposed the following conjecture which aims to generalize the above Etingof-Ginzburg Theorem 1.3 into two directions: it includes singular Calogero-Moser spaces and it extends to equivariant cohomology.

Conjecture 1.5. — *With the above notation, we have:*

(1) $\mathrm{H}^{2i+1}(\mathscr{Z}_c)=0$ *for all* i.
(2) *We have an isomorphism of graded* $\mathbb{C}$*-algebras* $\mathrm{H}^{2\bullet}(\mathscr{Z}_c)\simeq \mathrm{gr}_{\mathscr{F}}(\mathrm{Im}(\boldsymbol{\Omega}^c))$.
(3) *We have an isomorphism of graded* R*-algebras* $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{Z}_c)\simeq \mathrm{Rees}_{\mathscr{F}}(\mathrm{Im}(\boldsymbol{\Omega}^c))$.

By (1.2), when $\mathscr{Z}_c$ is smooth, the image of $\boldsymbol{\Omega}^c$ coincide with the center of $\mathbb{C}W$. So Theorem A proves this conjecture for smooth $\mathscr{Z}_c$.

We will also prove in Example 3.6 the following result:

Proposition 1.6. — *If* $\dim_{\mathbb{C}}(V)=1$, *then Conjecture 1.5 holds.*

1.F. Structure of the paper. — The paper is organized as follows. The proof of Theorem A relies on classical theorems about restriction to fixed points. In the next Section 2, we first recall basic properties on equivariant cohomology and equivariant K-theory and restriction to fixed points. Section 3 explains how these general principles can be applied to Calogero-Moser spaces. Theorem A will be proved in Section 4. The cyclic group case (Proposition 1.6) will be handled in Example 3.6. The proof of Theorem B will be given in Section 5.

2. Equivariant cohomology, K-theory and fixed points

2.A. Equivariant cohomology. — Let $\mathscr{X}$ be a complex algebraic variety equipped with a regular action of a torus $\mathbf{T}$. Recall that the *equivariant cohomology* of $\mathscr{X}$ is defined by

$$\mathrm{H}^\bullet_{\mathbf{T}}(\mathscr{X})=\mathrm{H}^\bullet(E_{\mathbf{T}}\times_{B_{\mathbf{T}}} X),$$

where $E_{\mathbf{T}}\to B_{\mathbf{T}}$ is a universal $\mathbf{T}$-bundle. The pullback of the structural morphism $x:\mathscr{X}\to$ **pt** yields a ring homomorphism $\mathrm{H}^\bullet_{\mathbf{T}}(\mathbf{pt})\to \mathrm{H}^\bullet_{\mathbf{T}}(\mathscr{X})$, which makes $\mathrm{H}^\bullet_{\mathbf{T}}(\mathscr{X})$ a graded $\mathrm{H}^\bullet_{\mathbf{T}}(\mathbf{pt})$-algebra.

Denote by $X(\mathbf{T})$ the character lattice of $\mathbf{T}$. The for each $\chi\in X(\mathbf{T})$, denote by $\mathbb{C}_\chi$ the one dimensional $\mathbf{T}$-module of character χ, the first Chern class c_χ of the line bundle $E_{\mathbf{T}}\times_{\mathbf{T}}\mathbb{C}_\chi$ on $B_{\mathbf{T}}$ is an element in $\mathrm{H}^2(B_{\mathbf{T}})$. Identify the vector space $\mathbb{C}\otimes_{\mathbb{Z}} X(\mathbf{T})$ with the dual $\mathbf{t}^*$ of the Lie algebra $\mathbf{t}$ of $\mathbf{T}$ via $\chi\mapsto d\chi$. Then the assignment $\chi\mapsto c_\chi$ yields an isomorphism of graded $\mathbb{C}$-algebras $S(\mathbf{t}^*)=\mathrm{H}^{2\bullet}_{\mathbf{T}}(\mathbf{pt})$.

2.B. Equivariant K-theory. — We denote by $\mathrm{K}_{\mathbf{T}}(\mathscr{X})$ the Grothendieck ring of the category of $\mathbf{T}$-equivariant vector bundles on $\mathscr{X}$. Note that when $\mathscr{X}$ is a point, a $\mathbf{T}$-equivariant vector bundles is the same as a finite dimensional $\mathbf{T}$-module. We have a canonical isomorphism $\mathrm{K}_{\mathbf{T}}(\mathbf{pt}) = \mathbb{Z}[X(\mathbf{T})]$ which sends the class of a $\mathbf{T}$-module M to

$$\dim^{\mathbf{T}}(M) = \sum_{\chi \in X(\mathbf{T})} \dim_{\mathbb{C}}(M_\chi)\chi, \tag{2.1}$$

where M_χ is the χ-weight space in M.

Let $\hat{\mathrm{H}}_{\mathbf{T}}^{2\bullet}(\mathscr{X})$ be the completion of $\mathrm{H}_{\mathbf{T}}^{2\bullet}(\mathscr{X})$ with respect to the ideal $\bigoplus_{i>0} \mathrm{H}_{\mathbf{T}}^{2i}(\mathscr{X})$. The *equivariant Chern character* provides a ring homomorphism

$$\mathrm{ch}_{\mathscr{X}} : \mathrm{K}_{\mathbf{T}}(\mathscr{X}) \longrightarrow \hat{\mathrm{H}}_{\mathbf{T}}^{2\bullet}(\mathscr{X})$$

with the following properties. First, when $\mathscr{X}$ is a point $\mathbf{pt}$, we have

$$\begin{array}{rccc} \mathrm{ch}_{\mathbf{pt}} : & \mathrm{K}_{\mathbf{T}}(\mathbf{pt}) = \mathbb{Z}[X(\mathbf{T})] & \longrightarrow & \hat{\mathrm{H}}_{\mathbf{T}}^{2\bullet}(\mathbf{pt}) = \hat{\mathrm{S}}(\mathbf{t}^*) \\ & \chi & \longmapsto & \exp(d\chi). \end{array} \tag{2.2}$$

Next the Chern character commutes with pullback. More precisely, if $\mathscr{Y}$ is another variety with a regular action of the same torus $\mathbf{T}$ and if $\varphi : \mathscr{X} \to \mathscr{Y}$ is a $\mathbf{T}$-equivariant morphism, then the following diagram commutes

$$\begin{array}{ccc} \mathrm{K}_{\mathbf{T}}(\mathscr{Y}) & \xrightarrow{\ \varphi^*\ } & \mathrm{K}_{\mathbf{T}}(\mathscr{X}) \\ \Big\downarrow \mathrm{ch}_{\mathscr{Y}} & & \Big\downarrow \mathrm{ch}_{\mathscr{X}} \\ \hat{\mathrm{H}}_{\mathbf{T}}^{2\bullet}(\mathscr{Y}) & \xrightarrow{\ \hat{\varphi}^*\ } & \hat{\mathrm{H}}_{\mathbf{T}}^{2\bullet}(\mathscr{X}). \end{array} \tag{2.3}$$

Here φ^* denotes both the pullback map in K-theory or in equivariant cohomology, and $\hat{\varphi}^*$ is the map induced after completion by the pullback map in equivariant cohomology. In particular, by applying the above diagram to $\mathscr{Y} = \mathbf{pt}$ and the structural morphism $\mathscr{X} \to \mathbf{pt}$, we may view $\mathrm{ch}_{\mathscr{X}}$ as a morphism of algebras over $\mathrm{K}_{\mathbf{T}}(\mathbf{pt})$, with the $\mathrm{K}_{\mathbf{T}}(\mathbf{pt})$-algebra structure on $\hat{\mathrm{H}}_{\mathbf{T}}^{2\bullet}(\mathscr{X})$ provided by the embedding (2.2).

2.C. Fixed points. — We denote by $\mathscr{X}^{\mathbf{T}}$ the (reduced) variety consisting of fixed points of $\mathbf{T}$ in $\mathscr{X}$. Let $i_{\mathscr{X}} : \mathscr{X}^{\mathbf{T}} \hookrightarrow \mathscr{X}$ be the natural closed immersion. Since $\mathbf{T}$ acts trivially on $\mathscr{X}^{\mathbf{T}}$, we have $\mathrm{H}_{\mathbf{T}}^{\bullet}(\mathscr{X}^{\mathbf{T}}) = \mathrm{H}_{\mathbf{T}}^{\bullet}(\mathbf{pt}) \otimes \mathrm{H}^{\bullet}(\mathscr{X}^{\mathbf{T}})$ as $\mathrm{H}_{\mathbf{T}}^{\bullet}(\mathbf{pt})$-algebras. Recall that the $\mathbf{T}$-action on $\mathscr{X}$ is called *equivariantly formal* if the Leray-Serre spectral sequence

$$\mathrm{E}_2^{pq} = \mathrm{H}^p(B_{\mathbf{T}}; \mathrm{H}^q(X)) \Longrightarrow \mathrm{H}_{\mathbf{T}}^{p+q}(X)$$

for the fibration $E_{\mathbf{T}} \times_{\mathbf{T}} X \to B_{\mathbf{T}}$ degenerates at E_2. We have the following standard result on equivariant cohomology (see for instance [**9**, Proposition 2.1]):

Proposition 2.4. — *Assume that* $\mathrm{H}^{2i+1}(\mathscr{X}) = 0$ *for all* i. *Then* $\mathscr{X}$ *is equivariantly formal, and:*

(a) *There is an isomorphism of graded* $\mathrm{H}_{\mathbf{T}}^{\bullet}(\mathbf{pt})$*-modules* $\mathrm{H}_{\mathbf{T}}^{\bullet}(\mathscr{X}) \simeq \mathrm{H}_{\mathbf{T}}^{\bullet}(\mathbf{pt}) \otimes \mathrm{H}^{\bullet}(\mathscr{X})$. *In particular* $\mathrm{H}_{\mathbf{T}}^{2i+1}(\mathscr{X}) = 0$ *for all* i.

(b) *The pullback map* $i_{\mathscr{X}}^* : \mathrm{H}_{\mathbf{T}}^{\bullet}(\mathscr{X}) \to \mathrm{H}_{\mathbf{T}}^{\bullet}(\mathscr{X}^{\mathbf{T}})$ *is injective.*

(c) *Let* $\mathfrak{m}$ *be the unique graded maximal ideal of* $\mathrm{H}_{\mathbf{T}}^{\bullet}(\mathbf{pt})$. *Then we have an isomorphism of algebras* $\mathrm{H}^{\bullet}(\mathscr{X}) \simeq \mathrm{H}_{\mathbf{T}}^{\bullet}(\mathscr{X}) / \mathfrak{m}\mathrm{H}_{\mathbf{T}}^{\bullet}(\mathscr{X})$.

In particular, this shows that Conjectures 1.5(1) and (3) imply Conjecture 1.5(2).

Example 2.5 **(Blowing-up)**. — Let $\mathscr{Y}$ be an affine variety with a $\mathbf{T}$-action and let $\mathscr{C}$ be a $\mathbf{T}$-stable closed subvariety (not necessarily reduced) of $\mathscr{Y}$. Let $\mathscr{X}$ be the blowing-up of $\mathscr{Y}$ along $\mathscr{C}$. Write $\pi : \mathscr{X} \to \mathscr{Y}$ for the natural morphism, and we equip $\mathscr{X}$ with the unique $\mathbf{T}$-action such that π is equivariant. ***We assume that $\mathscr{X}^{\mathbf{T}}$ is finite***. We write

$$\mathrm{H}_{\mathbf{T}}^{2\bullet}(\mathscr{X}^{\mathbf{T}}) = \bigoplus_{x \in \mathscr{X}^{\mathbf{T}}} \mathrm{S}(\mathfrak{t}^*)\mathbf{e}_x,$$

where $\mathbf{e}_x \in \mathrm{H}_{\mathbf{T}}^{0}(\mathscr{X}^{\mathbf{T}})$ is the primitive idempotent associated with x (i.e., the fundamental class of x).

Then $\mathscr{D} = \pi^*(\mathscr{C})$ is a $\mathbf{T}$-stable effective Cartier divisor, and we denote by $[\mathscr{D}]$ the class in $\mathrm{K}_{\mathbf{T}}(\mathscr{X})$ of its associated line bundle (which is $\mathbf{T}$-equivariant). We denote by $\mathrm{ch}_{\mathscr{X}}^1([\mathscr{D}]) \in \mathrm{H}_{\mathbf{T}}^2(\mathscr{X})$ its first $\mathbf{T}$-equivariant Chern class. We want to compute $i_{\mathscr{X}}^*(\mathrm{ch}_{\mathbf{T}}^1(\mathscr{D}))$.

First, let I be the ideal of $\mathbb{C}[\mathscr{Y}]$ associated with $\mathscr{C}$. As it is $\mathbf{T}$-stable, we can find a family of $\mathbf{T}$-homogeneous generators $(a_i)_{1 \leqslant i \leqslant k}$ of I. We denote by $\lambda_i \in X(\mathbf{T})$ the $\mathbf{T}$-weight of a_i. The choice of this family of generators induces a $\mathbf{T}$-equivariant closed immersion $\mathscr{X} \hookrightarrow \mathscr{Y} \times \mathbf{P}^{k-1}(\mathbb{C})$. We denote by $\mathscr{X}_i$ the affine chart corresponding to "$a_i \neq 0$". If $x \in \mathscr{X}^{\mathbf{T}}$, we denote by $\mathbf{i}(x) \in \{1, 2, \dots, k\}$ an element such that $x \in \mathscr{X}_{\mathbf{i}(x)}$. Then

$$i_{\mathscr{X}}^*(\mathrm{ch}_{\mathbf{T}}^1(\mathscr{D})) = -\hbar \sum_{x \in \mathscr{D}^{\mathbf{T}}} (d\lambda_{\mathbf{i}(x)})\mathbf{e}_x. \tag{2.6}$$

Indeed, we just need to compute the local equation of $\mathscr{D}$ around $x \in \mathscr{X}^{\mathbf{T}}$, and this can be done in $\mathscr{X}_{\mathbf{i}(x)}$. But $\mathscr{D} \cap \mathscr{X}_i$ is principal for all i, defined by

$$\mathscr{D} \cap \mathscr{X}_i = \{(y, \xi) \in \mathscr{Y} \times \mathbf{P}^{k-1}(\mathbb{C}) \mid (y, \xi) \in \mathscr{X}_i \text{ and } a_i(y) = 0\}.$$

So $\mathscr{D} \cap \mathscr{X}_i$ is defined by a $\mathbf{T}$-homogeneous equation of degree $\lambda_i \in X(\mathbf{T})$, and so (2.6) follows. ■

3. Localization and Calogero-Moser spaces

In this section, we apply the previous discussions to $\mathscr{X} = \mathscr{Z}_c$ and $\mathbf{T} = \mathbb{C}^\times$. Denote by $q : \mathbb{C}^\times \to \mathbb{C}^\times$ the identity map. Then $X(\mathbb{C}^\times) = q^{\mathbb{Z}}$, and we have

$$\mathrm{K}_{\mathbb{C}^\times}(\mathbf{pt}) = \mathbb{Z}[q, q^{-1}], \qquad \mathrm{H}_{\mathbb{C}^\times}^{2\bullet}(\mathbf{pt}) = \mathbb{C}[\hbar],$$

with $\hbar = c_q$, following the notation of Section 2.A. So $\hat{\mathrm{H}}_{\mathbb{C}^\times}^{2\bullet}(\mathbf{pt}) = \mathbb{C}[[\hbar]]$ and the Chern map in this case is given by

$$\mathrm{ch}_{\mathbf{pt}} : \mathbb{Z}[q, q^{-1}] \hookrightarrow \mathbb{C}[[\hbar]], \qquad q \mapsto \exp(\hbar).$$

Note also that a finite $\mathbb{C}^\times$-module is nothing but a finite dimensional $\mathbb{Z}$-graded vector space $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $\mathbb{C}^\times$ acts on M_i by the character q^i. The identification $\mathrm{K}_{\mathbb{C}^\times}(\mathbf{pt}) = \mathbb{Z}[q, q^{-1}]$ sends the class of M to its *graded dimension* (or *Hilbert series*):

$$\dim^{\mathrm{gr}}(M) = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{C}}(M_i) q^i \in \mathbb{N}[q, q^{-1}].$$

3.A. Fixed points. — For $\chi \in \mathrm{Irr}(W)$, we denote by $\omega_\chi : \mathrm{Z}(\mathbb{C}W) \to \mathbb{C}$ the associated morphism of algebras, i.e. $\omega_\chi(z) = \chi(z)/\chi(1)$ is the scalar through which z acts on the irreducible $\mathbb{C}W$-module with the character χ. We denote by e_χ^W the unique primitive idempotent of $\mathrm{Z}(\mathbb{C}W)$ such that $\omega_\chi(e_\chi^W) = 1$. If $\mathscr{E}$ is a subset of $\mathrm{Irr}(W)$, then we set $e_\mathscr{E}^W = \sum_{\chi\in\mathscr{E}} e_\chi^W$.

Now, consider the algebra homomorphism

$$\Omega_\chi^c = \omega_\chi \circ \boldsymbol{\Omega}^c : \mathbf{Z}_c \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}.$$

Its kernel is a maximal ideal of $\mathbf{Z}_c$, denote by z_χ the corresponding closed point in $\mathscr{Z}_c$. It follows from [3, Lemma 10.2.3 and (14.2.2)] that $z_\chi \in \mathscr{Z}_c^{\mathbb{C}^\times}$ and that the map

(3.1)
$$z : \begin{array}{ccc} \mathrm{Irr}(W) & \longrightarrow & \mathscr{Z}^{\mathbb{C}^\times} \\ \chi & \longmapsto & z_\chi \end{array}$$

is surjective. The fibers of this map are called the *Calogero-Moser c-families*. They were first consider by Gordon [6] and Gordon-Martino [8]: see also for instance [3, §9.2]. Let $\mathrm{CM}_c(W)$ be the set of Calogero-Moser c-families. For $\mathscr{E} \in \mathrm{CM}_c(W)$, we denote by $z_\mathscr{E} \in \mathscr{Z}^{\mathbb{C}^\times}$ its image under the map z. On the other hand, by [3, (16.1.2)] we have

(3.2)
$$\mathrm{Im}(\boldsymbol{\Omega}^c) = \bigoplus_{\mathscr{E}\in\mathrm{CM}_c(W)} \mathbb{C}e_\mathscr{E}^W.$$

Hence we get an isomorphism of $\mathbb{C}$-algebras

$$\mathrm{H}^{2\bullet}(\mathscr{Z}_c^{\mathbb{C}^\times}) \simeq \mathrm{Im}(\boldsymbol{\Omega}^c), \qquad e_{z_\mathscr{E}} \mapsto e_\mathscr{E}^W,$$

which extends to an isomorphism of $\mathbb{C}[\hbar]$-algebras $\mathrm{H}^{2\bullet}(\mathscr{Z}_c^{\mathbb{C}^\times}) \simeq \mathbb{C}[\hbar] \otimes \mathrm{Im}(\boldsymbol{\Omega}^c)$. For simplification, we set $i_c = i_{\mathscr{Z}_c} : \mathscr{Z}_c^{\mathbb{C}^\times} \hookrightarrow \mathscr{Z}$ and, under the above identification, we view the pullback map i_c^* as a morphism of algebras

$$i_c^* : \mathrm{H}_\mathbf{T}^{2\bullet}(\mathscr{Z}_c) \longrightarrow \mathbb{C}[\hbar] \otimes \mathrm{Im}(\boldsymbol{\Omega}^c).$$

So, by Proposition 2.4, Conjecture 1.5 is implied by the following one:

***Conjecture* 3.3.** — *With the above notation, we have:*

(1) $\mathrm{H}^{2i+1}(\mathscr{Z}_c) = 0$ *for all* i.
(2) $\mathrm{Im}(i_c^*) = \mathrm{Rees}_\mathscr{F}(\mathrm{Im}(\boldsymbol{\Omega}^c))$.

***Remark* 3.4.** — Set

$$\mathscr{F}_i^{\mathrm{H}}(\mathrm{Im}(\boldsymbol{\Omega}^c)) = \{z \in \mathrm{Im}(\boldsymbol{\Omega}^c) \mid \hbar^i z \in \mathrm{Im}(i_c^*)\}.$$

Then, by construction and Proposition 2.4, $\mathscr{F}_\bullet^{\mathrm{H}}(\mathrm{Im}(\boldsymbol{\Omega}^c))$ is the filtration of $\mathrm{Im}(\boldsymbol{\Omega}^c)$ satisfying

$$\mathrm{H}_{\mathbb{C}^\times}^{2\bullet}(\mathscr{Z}_c) \simeq \mathrm{Im}(i_c^*) = \mathrm{Rees}_{\mathscr{F}^{\mathrm{H}}}(\mathrm{Im}(\boldsymbol{\Omega}^c))$$

and
$$\mathrm{H}^{2\bullet}(\mathscr{Z}_c) \simeq \mathrm{gr}_{\mathscr{F}^{\mathrm{H}}}(\mathrm{Im}(\boldsymbol{\Omega}^c)).$$

So showing Conjecture 3.3 is then equivalent to showing that the filtrations $\mathscr{F}_\bullet(\mathrm{Im}(\boldsymbol{\Omega}^c))$ and $\mathscr{F}_\bullet^{\mathrm{H}}(\mathrm{Im}(\boldsymbol{\Omega}^c))$ coincide.

Note also that, since $\mathscr{Z}_c$ is an affine variety of dimension $2n$, we have $\mathrm{H}^i(\mathscr{Z}_c) = 0$ for $i > 2n$, so this shows that

(♭)
$$\mathscr{F}_n(\mathrm{Im}(\boldsymbol{\Omega}^c)) = \mathrm{Im}(\boldsymbol{\Omega}^c) = \mathscr{F}_n^{\mathrm{H}}(\mathrm{Im}(\boldsymbol{\Omega}^c)).$$

On the other hand, since $\mathscr{Z}_c$ is connected, we have

$$(\sharp) \qquad \mathscr{F}_0(\mathrm{Im}(\boldsymbol{\Omega}^c)) = \mathscr{F}_0^{\mathrm{H}}(\mathrm{Im}(\boldsymbol{\Omega}^c)) = \mathbb{C}.$$

These two particular cases will be used below. ■

The following result, based on the previous remark, can be viewed as a reduction step for the proof of Conjecture 3.3:

Proposition 3.5. — *Assume that* $\mathrm{H}^{2i+1}(\mathscr{Z}_c) = 0$ *and*

$$\dim_{\mathbb{C}}(\mathrm{H}^{2i}(\mathscr{Z}_c)) = \dim_{\mathbb{C}}(\mathscr{F}_i(\mathrm{Im}(\boldsymbol{\Omega}^c))/\mathscr{F}_{i-1}(\mathrm{Im}(\boldsymbol{\Omega}^c)))$$

for all i*. Then Conjecture 3.3 holds if and only if* $\mathrm{Rees}_{\mathscr{F}}(\mathrm{Im}(\boldsymbol{\Omega}^c)) \subset \mathrm{Im}(i_c^*)$.

Proof. — Assume that $\mathrm{H}^{2i+1}(\mathscr{Z}_c) = 0$ and

$$\dim_{\mathbb{C}}(\mathrm{H}^{2i}(\mathscr{Z}_c)) = \dim_{\mathbb{C}}(\mathscr{F}_i(\mathrm{Im}(\boldsymbol{\Omega}^c))/\mathscr{F}_{i-1}(\mathrm{Im}(\boldsymbol{\Omega}^c)))$$

for all i. We keep the notation of Remark 3.4. It then follows from this remark and the hypothesis that

$$\dim_{\mathbb{C}}(\mathscr{F}_i(\mathrm{Im}(\boldsymbol{\Omega}^c))/\mathscr{F}_{i-1}(\mathrm{Im}(\boldsymbol{\Omega}^c))) = \dim_{\mathbb{C}}(\mathrm{H}^{2i}(\mathscr{Z}_c)) = \dim(\mathscr{F}_i^{\mathrm{H}}(\mathrm{Im}(\boldsymbol{\Omega}^c))/\mathscr{F}_{i-1}^{\mathrm{H}}(\mathrm{Im}(\boldsymbol{\Omega}^c)))$$

for all i. So, by induction, we get that $\dim_{\mathbb{C}}(\mathscr{F}_i(\mathrm{Im}(\boldsymbol{\Omega}^c))) = \dim(\mathscr{F}_i^{\mathrm{H}}(\mathrm{Im}(\boldsymbol{\Omega}^c)))$ for all i (by the equality ($\sharp$) of Remark 3.4). This shows that $\mathrm{Rees}_{\mathscr{F}}(\mathrm{Im}(\boldsymbol{\Omega}^c)) = \mathrm{Im}(i_c^*)$ if and only if $\mathrm{Rees}_{\mathscr{F}}(\mathrm{Im}(\boldsymbol{\Omega}^c)) \subset \mathrm{Im}(i_c^*)$, as desired. □

Example 3.6. — Assume in this example, and only in this example, that $\dim_{\mathbb{C}}(V) = 1$. It is proved in [**3**, Theorem 18.5.8] that in this case $\mathrm{H}^{2i+1}(\mathscr{Z}_c) = 0$ and

$$\dim_{\mathbb{C}}(\mathrm{H}^{2i}(\mathscr{Z}_c)) = \dim_{\mathbb{C}}(\mathscr{F}_i(\mathrm{Im}(\boldsymbol{\Omega}^c))/\mathscr{F}_{i-1}(\mathrm{Im}(\boldsymbol{\Omega}^c)))$$

for all i. Since $\mathscr{Z}_c$ is affine of dimension 2, we have $\mathrm{H}^i(\mathscr{Z}_c) = 0$ if $i \notin \{0,2\}$. So it follows from the equalities ($\flat$) and ($\sharp$) that Conjecture 3.3 holds in this case (this proves Proposition 1.6). ■

3.B. Chern map. — If $\mathscr{E}$ is a Calogero-Moser family, we denote by $\mathfrak{m}_{\mathscr{E}} \subset \mathbf{Z}_c$ the ideal of functions vanishing at $z_{\mathscr{E}} \in \mathscr{Z}_c^{\mathbb{C}^\times}$. We also set

$$\mathrm{Im}(\boldsymbol{\Omega}^c)_{\mathbb{Z}} = \bigoplus_{\mathscr{E} \in \mathrm{CM}_c(W)} \mathbb{Z} e_{\mathscr{E}}^W.$$

We make the natural identification $\mathrm{K}_{\mathbb{C}^\times}(\mathscr{Z}_c^{\mathbb{C}^\times}) = \mathbb{Z}[q,q^{-1}] \otimes_{\mathbb{Z}} \mathrm{Im}(\boldsymbol{\Omega}^c)_{\mathbb{Z}}$. Through these identifications, the Chern map $\mathrm{ch}_{\mathscr{Z}_c^{\mathbb{C}^\times}}$ just becomes the natural inclusion $\mathbb{Z}[q,q^{-1}] \otimes_{\mathbb{Z}} \mathrm{Im}(\boldsymbol{\Omega}^c)_{\mathbb{Z}} \hookrightarrow \mathbb{C}[[\hbar]] \otimes \mathrm{Im}(\boldsymbol{\Omega}^c)$. Moreover, if P is a $\mathbb{Z}$-graded finitely generated projective $\mathbf{Z}_c$-module, then the commutativity of the diagram (2.3) just says that

$$i_c^*(\mathrm{ch}_{\mathscr{Z}_c}([P])) = \sum_{\mathscr{E} \in \mathrm{CM}_c(W)} \dim^{\mathrm{gr}}(P/\mathfrak{m}_{\mathscr{E}} P) e_{\mathscr{E}}^W \subset \mathbb{C}[[\hbar]] \otimes \mathrm{Im}(\boldsymbol{\Omega}^c). \tag{3.7}$$

4. Proof of Theorem A

Hypothesis and notation. *We assume in this section, and only in this section, that $\mathscr{Z}_c$ is smooth.*

If $M = \bigoplus_{i\in\mathbb{Z}} M_i$ and $N = \bigoplus_{i\in\mathbb{Z}} N_i$ are two finite-dimensional graded $\mathbb{C}W$-modules, we set

$$\langle M,N\rangle_W^{\mathrm{gr}} = \sum_{i,j\in\mathbb{Z}} \langle M_i,N_j\rangle_W \; q^{i+j},$$

where $\langle E,F\rangle_W = \dim \mathrm{Hom}_{\mathbb{C}W}(E,F)$ for any finite dimensional $\mathbb{C}W$-modules E and F. We extend this notation to the case where M or N is a graded virtual character (i.e. an element of $\mathbb{Z}[q,q^{-1}]\otimes_{\mathbb{Z}}\mathbb{Z}\mathrm{Irr}(W)$). Finally, we denote by $\mathbb{C}[V^*]^{\mathrm{co}(W)}$ the *coinvariant algebra*, i.e. the quotient of the algebra $\mathbb{C}[V^*]$ by the ideal generated by the elements $f\in\mathbb{C}[V^*]^W$ such that $f(0)=0$. Then $\mathbb{C}[V^*]^{\mathrm{co}(W)}$ is a graded $\mathbb{C}W$-module, which is isomorphic to the regular representation $\mathbb{C}W$ when one forgets the grading (we define similarly $\mathbb{C}[V]^{\mathrm{co}(W)}$).

4.A. Localization in K-theory. — Recall from (1.2) that the smoothness of $\mathscr{Z}_c$ implies that $\mathrm{Im}(\boldsymbol{\Omega}^c)=\mathrm{Z}(\mathbb{C}W)$, so that $z:\mathrm{Irr}(W)\to\mathscr{Z}_c^{\mathbb{C}^\times}$ is bijective. Let $e=e_1^W=(1/|W|)\sum_{w\in W} w$. The smoothness of $\mathscr{Z}_c$ also implies that the functor

$$\begin{array}{ccc} \mathbf{H}_c\text{-mod} & \longrightarrow & \mathbf{Z}_c\text{-mod} \\ M & \longmapsto & eM = e\mathbf{H}_c\otimes_{\mathbf{H}_c} M \end{array}$$

is an equivalence of categories [**4**, Theorem 1.7]. If E is a finite dimensional $\mathbb{Z}$-*graded* $\mathbb{C}W$-module, the $\mathbf{H}_c$-module $\mathbf{H}_c\otimes_{\mathbb{C}W} E$ is finitely generated, $\mathbb{Z}$-graded and projective. Therefore, $e\mathbf{H}_c\otimes_{\mathbb{C}W} E$ is a finitely generated graded projective $\mathbf{Z}_c$-module, which can be viewed as a $\mathbb{C}^\times$-equivariant vector bundle on $\mathscr{Z}_c$. For simplification, we set

$$\mathrm{ch}_c(E) = i_c^*(\mathrm{ch}_{\mathscr{Z}_c}(e\mathbf{H}_c\otimes_{\mathbb{C}W} E))\in\mathbb{C}[[\hbar]]\otimes\mathrm{Im}(\boldsymbol{\Omega}^c).$$

Proposition 4.1. — *Assume that $\mathscr{Z}_c$ is smooth. Let E be a finite dimensional graded $\mathbb{C}W$-module. Then*

$$\mathrm{ch}_c(E) = \sum_{\chi\in\mathrm{Irr}(W)} q^{b_\chi}\,\frac{\langle \chi,\mathbb{C}[V^*]^{\mathrm{co}(W)}\otimes E\rangle_W^{\mathrm{gr}}}{\langle\chi,\mathbb{C}[V^*]^{\mathrm{co}(W)}\rangle_W^{\mathrm{gr}}}\, e_\chi^W,$$

where b_χ is the valuation of $\langle\chi,\mathbb{C}[V^*]^{\mathrm{co}(W)}\rangle_G^{\mathrm{gr}}$.

Proof. — As the formula is additive, we may, and we will, assume that E is an irreducible $\mathbb{C}W$-module, concentrated in degree 0, and we denote by ψ its character.

Now, let $\chi\in\mathrm{Irr}(W)$: we denote by $\mathfrak{m}_\chi$ the maximal ideal of $\mathbf{Z}_c$ corresponding to the fixed point z_χ. We set $\mathfrak{p}=\mathfrak{m}_\chi\cap\mathbf{P}$: it does not depend on χ (it is the maximal ideal of $\mathbf{P}=\mathbb{C}[V/W\times V^*/W]$ of functions which vanishes at $(0,0)$; see for instance [**3**, (14.2.2)]). By (3.7),

$$\mathrm{ch}_c(E) = \sum_{\chi\in\mathrm{Irr}(W)} \dim^{\mathrm{gr}}((e\mathbf{H}_c\otimes_{\mathbb{C}W} E)/\mathfrak{m}_\chi(e\mathbf{H}_c\otimes_{\mathbb{C}W} E))e_\chi^W.$$

Now, let $\bar{\mathbf{Z}}_c=\mathbf{Z}_c/\mathfrak{p}\mathbf{Z}_c$ and $\bar{\mathbf{H}}_c=\mathbf{H}_c/\mathfrak{p}\mathbf{H}_c$. We set $\bar{\mathfrak{m}}_\chi=\mathfrak{m}_\chi/\mathfrak{p}\mathbf{Z}_c$. Then $\bar{\mathbf{H}}_c$ is a finite dimensional $\mathbb{C}$-algebra (called the *restricted rational Cherednik algebra*) and again the bimodule

$e\bar{\mathbf{H}}_c$ induces a Morita equivalence between $\bar{\mathbf{H}}_c$ and $\bar{\mathbf{Z}}_c$. This implies that $e\bar{\mathbf{H}}_c/\bar{\mathfrak{m}}_\chi\bar{\mathbf{H}}_c = (\bar{\mathbf{Z}}_c/\bar{\mathfrak{m}}_\chi)\otimes_{\bar{\mathbf{Z}}_c} e\bar{\mathbf{H}}_c$ is a simple right $\bar{\mathbf{H}}_c$-module (that will be denoted by $\mathscr{L}_c(\chi)$).

Since $\mathbb{C}W$ is semisimple, the $\mathbb{C}W$-module E is flat, and so

$$\begin{aligned}(e\mathbf{H}_c\otimes_{\mathbb{C}W}E)/\mathfrak{m}_\chi(e\mathbf{H}_c\otimes_{\mathbb{C}W}E) &\simeq (e\mathbf{H}_c/\mathfrak{m}_\chi(e\mathbf{H}_c))\otimes_{\mathbb{C}W}E\\ &\simeq (e\bar{\mathbf{H}}_c/\bar{\mathfrak{m}}_\chi(e\bar{\mathbf{H}}_c))\otimes_{\mathbb{C}W}E.\\ &\simeq \mathscr{L}_c(\chi)\otimes_{\mathbb{C}W}E.\end{aligned}$$

But the graded dimension of $\mathscr{L}_c(\chi)\otimes_{\mathbb{C}W}E$ is known whenever $\mathscr{Z}_c$ is smooth and is given by the expected formula (see [**1**, Lemma 3.3 and its proof]). □

Let $\mathrm{K}_{\mathbb{C}^\times}(\mathbb{C}W)$ denote the Grothendieck group of the category of finite dimensional graded $\mathbb{C}W$-modules. If E is a finite dimensional graded $\mathbb{C}W$-module, we denote by $[E]$ its class in $\mathrm{K}_{\mathbb{C}^\times}(\mathbb{C}W)$. We still denote by $\mathrm{ch}_c : \mathrm{K}_{\mathbb{C}^\times}(\mathbb{C}W)\to\mathbb{C}[[\hbar]]\otimes\mathrm{Z}(\mathbb{C}W)$ the map defined by

$$\mathrm{ch}_c([E])=\mathrm{ch}_c(E).$$

Now, let W' be a parabolic subgroup of W and set $V'=V^{W'}$ and $r=\mathrm{codim}_{\mathbb{C}}(V')$. We identify the dual V'^* of V' with $V^{*W'}$ and note that

$$V=V'\oplus(V'^*)^\perp. \tag{4.2}$$

We denote by $\wedge(V'^*)^\perp$ the element of $\mathrm{K}_{\mathbb{C}^\times}(\mathbb{C}W')$ defined by

$$\wedge(V'^*)^\perp=\sum_{i\geqslant 0}(-1)^i[\wedge^i(V'^*)^\perp].$$

Recall also that there exists n algebraically independent homogeneous polynomials $f_1,\dots,$ f_n in $\mathbb{C}[V]^W$ such that $\mathbb{C}[V]^W=\mathbb{C}[f_1,\dots,f_n]$, and we denote by d_i the degree of f_i.

Corollary 4.3. — *Assume that $\mathscr{Z}_c$ is smooth. Let E' be a finite dimensional graded $\mathbb{C}W'$-module. Then*

$$\mathrm{ch}_c\big(\mathrm{Ind}_{W'}^W(\wedge(V'^*)^\perp\otimes[E'])\big)=\frac{(1-q^{d_1})\cdots(1-q^{d_n})}{(1-q)^{n-r}}\sum_{\chi\in\mathrm{Irr}(W)}q^{b_\chi}\frac{\langle\chi,\mathrm{Ind}_{W'}^W E'\rangle_W^{\mathrm{gr}}}{\langle\chi,\mathbb{C}[V^*]^{\mathrm{co}(W)}\rangle_W^{\mathrm{gr}}}e_\chi^W.$$

Proof. — The group W' acts trivially on V' so it acts trivially on $\wedge^i V'$ for all i. Therefore,

$$(1-q)^{n-r}\,\mathrm{ch}_c\big(\mathrm{Ind}_{W'}^W(\wedge(V'^*)^\perp\otimes[E'])\big)=\mathrm{ch}_c\big(\mathrm{ind}_{W'}^W\big(\wedge V'\otimes\wedge(V'^*)^\perp\otimes[E']\big)\big).$$

But $\wedge V'\otimes\wedge(V'^*)^\perp=\mathrm{Res}_{W'}^W(\wedge V)$ by (4.2), so, by Frobenius formula,

$$(1-q)^{n-r}\,\mathrm{ch}_c\big(\mathrm{Ind}_{W'}^W(\wedge(V'^*)^\perp\otimes[E'])\big)=\mathrm{ch}_c\big(\wedge V\otimes\mathrm{Ind}_{W'}^W E'\big).$$

So it follows from Proposition 4.1 that

$$\begin{aligned}&(1-q)^{n-r}\,\mathrm{ch}_c\big(\mathrm{Ind}_{W'}^W(\wedge(V'^*)^\perp\otimes[E'])\big)\\ &\qquad=\sum_{\chi\in\mathrm{Irr}(W)}\frac{q^{b_\chi}}{\langle\chi,\mathbb{C}[V^*]^{\mathrm{co}(W)}\rangle_W^{\mathrm{gr}}}\langle\chi,\mathbb{C}[V^*]^{\mathrm{co}(W)}\otimes(\wedge V)\otimes\mathrm{Ind}_{W'}^W(E')\rangle_W^{\mathrm{gr}}\,e_\chi^W.\end{aligned}$$

But, if $w \in W$, the Molien's formula implies that the graded trace of w on $\mathbb{C}[V^*]^{\mathrm{co}(W)}$ is equal to

$$\frac{(1-q^{d_1})\cdots(1-q^{d_n})}{\det(1-wq)},$$

while its graded trace on $\wedge V$ is equal to $\det(1-wq)$. So the class of $\mathbb{C}[V^*]^{\mathrm{co}(W)}\otimes\wedge V$ is equal to $(1-q^{d_1})\cdots(1-q^{d_n})$ times the class of the trivial module, and the corollary follows. □

Proof of Theorem A. — Assume that $\mathscr{Z}_c$ is smooth. This implies in particular that Conjecture 1.5(1) and (2) hold (Etingof-Ginzburg Theorem 1.3), and so the hypotheses of Proposition 3.5 are satisfied. Note also that $\mathrm{Im}(\mathbf{\Omega}^c) = \mathrm{Z}(\mathbb{C}W)$ by (1.2). It is then sufficient to prove that $\hbar^i \mathscr{F}_i(\mathrm{Z}(\mathbb{C}W)) \subset \mathrm{Im}(i_c^*)$ for all i.

Let us introduce some notation. If G is a finite group and H is a subgroup, we define a $\mathbb{C}$-linear map $\mathrm{Tr}_H^G : \mathrm{Z}(\mathbb{C}H) \longrightarrow \mathrm{Z}(\mathbb{C}G)$ by

$$\mathrm{Tr}_H^G(z) = \sum_{g\in[G/H]} {}^g z = \frac{1}{|H|}\sum_{g\in G} {}^g z$$

(here, $[G/H]$ denotes a set of representatives of elements of G/H and ${}^g z = gzg^{-1}$). It is easily checked that

$$\mathrm{Tr}_H^G(e_\eta^H) = \frac{\eta(1)}{|H|}\sum_{\gamma\in\mathrm{Irr}(G)} \frac{|G|}{\gamma(1)}\langle\gamma, \mathrm{Ind}_H^G(\eta)\rangle_G\, e_\gamma^G \tag{4.4}$$

for all $\eta \in \mathrm{Irr}(H)$. Also, if $h \in H$ and $\Sigma_H(h) \in \mathrm{Z}(\mathbb{C}H)$ denotes the sum of the conjugates of h in H, then

$$\mathrm{Tr}_H^G(\Sigma_H(h)) = \frac{|C_G(h)|}{|C_H(h)|}\Sigma_G(h). \tag{4.5}$$

Let $\mathscr{P}_r(W)$ denote the set of parabolic subgroups W' of W such that $\mathrm{codim}_{\mathbb{C}}(V^{W'}) = r$. It follows from (4.5) that

$$\mathscr{F}_r(\mathrm{Z}(\mathbb{C}W)) = \sum_{W'\in\mathscr{P}_r(W)} \mathrm{Tr}_{W'}^W(\mathrm{Z}(\mathbb{C}W')). \tag{4.6}$$

Therefore, by Proposition 3.5, it is sufficient to prove that

(★) $\hbar^r\, \mathrm{Tr}_{W'}^W(e_{\chi'}^{W'}) \in \mathrm{Im}(i_c^*)$ *for all* $W' \in \mathscr{P}_r(W)$ *and all* $\chi' \in \mathrm{Irr}(W')$.

But it turns out that the coefficient of $\hbar^r$ in

$$\frac{\chi'(1)}{|W'|}\mathrm{ch}_c\big(\mathrm{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes \chi')\big)$$

is equal, according to Corollary 4.3, to

$$\frac{\chi'(1)}{|W'|}\sum_{\chi\in\mathrm{Irr}(W)} \frac{|W|\,\langle\chi, \mathrm{Ind}_{W'}^W \chi'\rangle_W}{\chi(1)}\, e_\chi^W$$

(since $d_1\cdots d_n = |W|$). Note that by definition $\frac{\chi'(1)}{|W'|}\mathrm{ch}_c\big(\mathrm{Ind}_{W'}^W(\wedge(V'^*)^\perp \otimes \chi')\big)$ belongs to $\mathbb{C}[[\hbar]]\otimes_{\mathbb{C}[\hbar]}\mathrm{Im}(i_c^*)$, so each of its homogeneous components belong to $\mathrm{Im}(i_c^*)$. By (4.4), this proves that (★) holds, and the proof of Theorem A is complete. □

5. Proof of Theorem B

Hypothesis and notation. *We assume in this section, and only in this section, that the symplectic quotient singularity* $\mathscr{Z}_0 = (V \times V^*)/W$ *admits a symplectic resolution* $\mathscr{X} \to \mathscr{Z}_0$.

Recall [**11**, Theorem 1.3(ii)] that the $\mathbb{C}^\times$-action on $\mathscr{Z}_0$ lifts uniquely to $\mathscr{X}$. As it is explained in Remark 1.4, the existence of a symplectic resolution of $\mathscr{Z}_0$ implies that all the irreducible components of W are of type $G(d,1,n)$ or G_4. Since the proof of Theorem B can be easily reduced to the irreducible case, we will separate the proof in two cases.

Proof of Theorem B for $W = G(d,1,n)$. — Assume here that $W = G(d,1,n)$. Let $\mathbf{S}^1$ be the group of complex numbers of modulus 1. In this case, it follows from [**7**] that $\mathscr{X}$ is diffeomorphic to some smooth $\mathscr{Z}_c$, and that the diffeomorphism might be chosen to be $\mathbf{S}^1$-equivariant. As the $\mathbf{S}^1$-equivariant cohomology is canonically isomorphic to the $\mathbb{C}^\times$-equivariant cohomology, this proves Theorem B in this case. □

Proof of Theorem B for $W = G_4$. — Assume here that $W = G_4$. It is possible (probable?) that again $\mathscr{X}$ is $\mathbf{S}^1$-diffeomorphic to some smooth $\mathscr{Z}_c$, but we are unable to prove it (it is only known that they are diffeomorphic). So we will prove Theorem B in this case by brute force computations.

We fix a primitive third root of unity ζ and we assume that $V = \mathbb{C}^2$ and that $W = \langle s, t \rangle$, where

$$s = \begin{pmatrix} \zeta & 0 \\ \zeta^2 & 1 \end{pmatrix} \qquad \text{and} \qquad t = \begin{pmatrix} 1 & -\zeta^2 \\ 0 & \zeta \end{pmatrix}.$$

By the work of Bellamy [**2**], there are only two symplectic resolutions of $\mathscr{Z}_0 = (V \times V^*)/W$. They have both been constructed by Lehn and Sorger [**12**]: one can be obtained from the other by exchanging the role of V and V^*, so we will only prove Theorem B for one of them. Let us describe it.

Let $H = V^s$ and let $\mathscr{H}$ denote the image of $H \times V^*$ in $\mathscr{Z}_0$, with its reduced structure of closed subvariety. We denote by $\beta : \mathscr{Y} \to \mathscr{Z}_0$ the blowing-up of $\mathscr{Z}_0$ along $\mathscr{H}$ and we denote by $\alpha : \mathscr{X} \to \mathscr{Y}$ the blowing-up of $\mathscr{Y}$ along its reduced singular locus $\mathscr{S}$. Then [**12**]

$$\pi = \beta \circ \alpha : \mathscr{X} \longrightarrow \mathscr{Z}_0$$

is a symplectic resolution.

We now give more details, which all can be found in [**12**, §1]. First, $\mathbb{C}[\mathscr{Z}_0]$ is generated by 8 homogeneous elements $(z_i)_{1 \leqslant i \leqslant 8}$ whose degrees are given by the following table:

(5.1)

z	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
$\deg(z)$	0	4	−4	2	−2	−6	6	0

The defining ideal of $\mathscr{H}$ in $\mathbb{C}[\mathscr{Z}_0]$ is generated by 6 homogeneous elements $(b_j)_{1 \leqslant j \leqslant 6}$ whose degrees are given by the following table:

(5.2)

b	b_1	b_2	b_3	b_4	b_5	b_6
$\deg(b)$	2	6	0	12	8	4

This defines a $\mathbb{C}^\times$-equivariant closed immersion $\mathscr{Y} \hookrightarrow \mathscr{Z}_0 \times \mathbf{P}^5(\mathbb{C})$. We denote by $\mathscr{Y}_i$ the affine chart defined by "$b_i \neq 0$". The equations of the zero fiber $\beta^{-1}(0)$ given in [**12**, §1] show that $\mathscr{Y}^{\mathbb{C}^\times} = \{p_2, p_3, p_4, p_6\}$, where p_i is the unique element of $\mathscr{Y}_i^{\mathbb{C}^\times}$. We use the notation of Example 2.5. By (2.6) and (5.2), we have

$$(\clubsuit) \qquad i_{\mathscr{Y}}^*(\mathrm{ch}^1_{\mathscr{Y}}([\beta^*\mathscr{H}])) = -\hbar(6\mathbf{e}_{p_2} + 12\mathbf{e}_{p_4} + 4\mathbf{e}_{p_6}).$$

Now, $\mathscr{S}$ is contained in $\mathscr{Y}_2 \cup \mathscr{Y}_3$, so α is an isomorphism in a neighborhood of p_4 and p_6. So let $q_4 = \alpha^{-1}(p_4)$ and $q_6 = \alpha^{-1}(p_6)$. These are elements of $\mathscr{X}^{\mathbb{C}^\times}$.

On the other hand, $\mathscr{Y}_2$ is a transversal A_1-singularity, so the defining ideal of $\mathscr{S} \cap \mathscr{Y}_2$ in $\mathbb{C}[\mathscr{Y}_2]$ is generated by three homogeneous elements a_+, $a_\circ$ and a_- (of degree 6, 0 and -6, by [**12**, §1]), and it is easily checked that $\alpha^{-1}(p_2)^{\mathbb{C}^\times} = \{q_2^+, q_2^-\}$ where $q_2^\pm$ is the unique $\mathbb{C}^\times$-fixed element in the affine chart defined by "$a_\pm \neq 0$".

Also, $\mathscr{Y}_3$ is isomorphic to $(\mathfrak{h} \times \mathfrak{h}^*)/\mathfrak{S}_3$, where $\mathfrak{h}$ is the diagonal Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{C})$ and $\mathfrak{S}_3$ is the symmetric group on 3 letters, viewed as the Weyl group of $\mathfrak{sl}_3(\mathbb{C})$. Let $\mathbf{Hilb}_3(\mathbb{C}^2)$ denote the Hilbert scheme of 3 points in $\mathbb{C}^2$, and let $\mathbf{Hilb}_3^0(\mathbb{C}^2)$ denote the (reduced) closed subscheme defined as the Hilbert scheme of three points in $\mathbb{C}^2$ whose sum is equal to $(0,0)$. By [**10**, Proposition 2.6],

$$(5.3) \qquad \mathscr{X}_3 \simeq \mathbf{Hilb}_3^0(\mathbb{C}^2).$$

It just might be noticed that the isomorphism $\mathscr{Y}_3 \simeq (\mathfrak{h} \times \mathfrak{h}^*)/\mathfrak{S}_3$ becomes $\mathbb{C}^\times$-equivariant if one "doubles the degrees" in $(\mathfrak{h} \times \mathfrak{h}^*)/\mathfrak{S}_3$, that is, if $\mathbb{C}^\times$ acts on $\mathfrak{h}$ (respectively $\mathfrak{h}^*$) with weight 2 (respectively -2). Recall that $\mathbb{C}^\times$-fixed points in $\mathbf{Hilb}_3^0(\mathbb{C}^2)$ are parametrized by partitions of 3: we denote by q_3^+, q_3° and q_3^- the fixed points in $\mathscr{X}_3$ corresponding respectively to the partitions (3), $(2,1)$ and $(1,1,1)$ of 3, so that

$$(5.4) \qquad \mathscr{X}_3^{\mathbb{C}^\times} = \{q_3^+, q_3^\circ, q_3^-\}.$$

Finally, we have

$$(5.5) \qquad \mathscr{X}^{\mathbb{C}^\times} = \{q_2^+, q_2^-, q_3^+, q_3^\circ, q_3^-, q_4, q_6\}.$$

Also, $\pi^*(\mathscr{H}) = \alpha^*(\beta^*(\mathscr{H}))$ is an effective Cartier divisor of $\mathscr{X}$, and it follows from ($\clubsuit$) and the commutativity of the diagram 2.3 that

$$(\diamondsuit) \qquad i_{\mathscr{X}}^*(\mathrm{ch}^1_{\mathscr{X}}([\pi^*\mathscr{H}])) = -\hbar(6\mathbf{e}_{q_2^+} + 6\mathbf{e}_{q_2^-} + 12\mathbf{e}_{q_4} + 4\mathbf{e}_{q_6}).$$

We now wish to compute $i_{\mathscr{X}}^*(\mathrm{ch}^1_{\mathscr{X}}([\alpha^*\mathscr{S}]))$. As the singular locus $\mathscr{S}$ is contained in $\mathscr{Y}_2 \cup \mathscr{Y}_3$, and contains p_2 and p_3, there exists n_2^+, n_2^-, n_3^+, n_3° and n_3^- in $\mathbb{Z}$ such that

$$i_{\mathscr{X}}^*(\mathrm{ch}^1_{\mathscr{X}}([\alpha^*\mathscr{S}])) = -\hbar(n_2^+\mathbf{e}_{q_2^+} + n_2^-\mathbf{e}_{q_2^-} + n_3^+\mathbf{e}_{q_3^+} + n_3^\circ\mathbf{e}_{q_3^\circ} + n_3^-\mathbf{e}_{q_3^-}).$$

Since a_+ and a_- have degree 6 and -6, it follows from (2.6) that $n_2^+ = 6$ and $n_2^- = -6$.

As we can exchange the roles of $\mathfrak{h}$ and $\mathfrak{h}^*$ in the description of $\mathscr{Y}_3$ and its singular locus (because $\mathfrak{h} \simeq \mathfrak{h}^*$ as an $\mathfrak{S}_3$-module), this shows that $n_3^- = -n_3^+$ and $n_3^\circ = -n_3^\circ$. So $n_3^\circ = 0$ and it remains to compute n_3^+. So let U_+ be the open subset of $\mathbf{Hilb}_3^0(\mathbb{C}^2)$ consisting of ideals J of codimension 3 of $\mathbb{C}[x,y]$ such that the classes of 1, x and x^2 form a basis of $\mathbb{C}[x,y]/J$. Then we have an isomorphism $J_+ : \mathbb{C}^4 \xrightarrow{\sim} U_+$ given by

$$(a,b,c,d) \longmapsto J_+(a,b,c,d) = \langle x^3 + ax + b, y - cx^2 - dx - \frac{2}{3}ac\rangle.$$

The form of the generators of the ideal $J_+(a,b,c,d)$ is here to ensure that $J_+(a,b,c,d) \in \mathbf{Hilb}_3^0(\mathbb{C}^2)$. The fixed point q_3^+ is the unique one in U_+. Through this identification, the action of $\mathbb{C}^\times$ on $\mathbb{C}^4$ is given by

$$\xi \cdot (a,b,c,d) = (\xi^4 a, \xi^6 b, \xi^{-6} c, \xi^{-4} d)$$

(remember that we must "double the degrees" of the usual action). The equation of $\alpha^*(\mathscr{S})$ on this affine chart $\simeq \mathbb{C}^4$ can then be computed explicitly and is given by $4a^3 + 27b^2 = 0$, so is of degree 12. This shows that $n_3^+ = 12$. Finally,

$$(\heartsuit) \qquad i_{\mathscr{X}}^*(\mathrm{ch}^1_{\mathscr{X}}([\alpha^*\mathscr{S}])) = -\hbar(6\mathbf{e}_{q_2^+} - 6\mathbf{e}_{q_2^-} + 12\mathbf{e}_{q_3^+} - 12\mathbf{e}_{q_3^-}).$$

Let us now conclude. First, recall from [**5**, Theorem 1.2] that

$$(5.6) \qquad \begin{cases} \mathrm{H}^{2i+1}(\mathscr{X}) = 0 & \text{for all } i, \\ \mathrm{H}^{2\bullet}(\mathscr{X}) \simeq \mathrm{gr}_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W)). \end{cases}$$

By Proposition 2.4, we get

$$(\spadesuit) \qquad \dim_{\mathbb{C}} \mathrm{H}^{2i}_{\mathbb{C}^\times}(\mathscr{X}) = \begin{cases} 1 & \text{if } i = 0, \\ 3 & \text{if } i = 1, \\ 7 & \text{if } i \geqslant 2. \end{cases}$$

Now, W has seven irreducible characters 1, ε, ε^2, χ, $\chi\varepsilon$, $\chi\varepsilon^2$ and θ, where χ is the unique irreducible character of degree 2 with rational values and θ is the unique one of degree 3. We denote by

$$\Psi : \mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{X}^{\mathbb{C}^\times}) \xrightarrow{\sim} \mathbb{C}[[\hbar]] \otimes \mathrm{Z}(\mathbb{C}W)$$

the isomorphism of $\mathbb{C}[[\hbar]]$-algebras such that

$$\Psi(\mathbf{e}_{q_4}) = e_1^W, \quad \Psi(\mathbf{e}_{q_6}) = e_\theta^W, \quad \Psi(\mathbf{e}_{q_2^+}) = e_{\chi\varepsilon}^W, \quad \Psi(\mathbf{e}_{q_2^-}) = e_{\chi\varepsilon^2}^W,$$

$$\Psi(\mathbf{e}_{q_3^+}) = e_{\varepsilon^2}^W, \quad \Psi(\mathbf{e}_{q_3^\circ}) = e_\chi^W \quad \text{and} \quad \Psi(\mathbf{e}_{q_3^-}) = e_\varepsilon^W.$$

By Proposition 2.4, $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{X})$ is isomorphic to its image by $\Psi \circ i_{\mathscr{X}}^*$ in $\mathbb{C}[[\hbar]] \otimes \mathrm{Z}(\mathbb{C}W)$. But, by ($\diamondsuit$) and ($\heartsuit$), and after investigation of the character table of W, this image contains

$$\hbar(6\mathbf{e}_{\chi\varepsilon}^W - 6\mathbf{e}_{\chi\varepsilon^2}^W + 12e_1^W + 4e_\theta^W) = \hbar(4 + \Sigma_W(s) + \Sigma_W(s^2))$$

and $$\hbar(12e_{\varepsilon^2}^W - 12e_\varepsilon^W + 6e_{\chi\varepsilon} - 6e_{\chi\varepsilon^2}) = \hbar((1+2\zeta)\Sigma_W(s) + (1+2\zeta^2)\Sigma_W(s^2)).$$

So it contains $\hbar\Sigma_W(s)$ and $\hbar\Sigma_W(s^2)$. Also, by ($\spadesuit$) and Proposition 2.4, it also contains $\hbar^2\mathrm{Z}(\mathbb{C}W)$, so

$$\mathrm{Rees}^\bullet_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W)) \subset \mathrm{Im}(\Psi \circ i_{\mathscr{X}}^*).$$

Using again ($\spadesuit$) and Proposition 2.4, a comparison of dimensions yields that

$$\mathrm{Rees}^\bullet_{\mathscr{F}}(\mathrm{Z}(\mathbb{C}W)) = \mathrm{Im}(\Psi \circ i_{\mathscr{X}}^*),$$

and the proof is complete. □

References

[1] G. BELLAMY, *On singular Calogero-Moser spaces*, Bull. LMS **41**, 315-326.

[2] G. BELLAMY, *Counting resolutions of symplectic quotient singularities*, Compos. Math. **152** (2016), 99-114.

[3] C. BONNAFÉ & R. ROUQUIER, *Cherednik algebras and Calogero-Moser cells*, preprint (2017), `arXiv:1708.09764`.

[4] P. ETINGOF & V. GINZBURG. *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Inventiones Mathematicae **147**, no. 2 (2002), 243-348.

[5] V. GINZBURG & D. KALEDIN, *Poisson deformations of symplectic quotient singularities*, Adv. in Math. **186** (2004), 1-57.

[6] I. GORDON, *Baby Verma modules for rational Cherednik algebras*, Bull. London Math. Soc. **35** (2003), 321-336.

[7] I. GORDON, *Quiver varieties, category $\mathcal{O}$ for rational Cherednik algebras, and Hecke algebras*, Int. Math. Res. Papers **2008** (2008), Art. ID rpn006, 69 pages.

[8] I. GORDON & M. MARTINO, *Calogero-Moser space, restricted rational Cherednik algebras, and two-sided cells*, Math. Res. Lett. **16** (2009), 255-262.

[9] M. GORESKY & R. MACPHERSON, *On the spectrum of the equivariant cohomology ring*, Canadian Journal of Mathematics **62**, no. 2 (2010), 262-283.

[10] M. HAIMAN, *t, q-Catalan numbers and the Hilbert scheme*, Discrete Math. **193** (1998), 201-224. Selected papers in honor of Adriano Garsia (Taormina, 1994).

[11] D. KALEDIN, *On crepant resolutions of symplectic quotient singularities*, Selecta Math. **9** (2003), 529-555.

[12] M. LEHN & C. SORGER, *A symplectic resolution for the binary tetrahedral group*, Geometric methods in representation theory II, 429-435, Sémin. Congr., 24-II, Soc. Math. France, Paris, 2012.

[13] G. BELLAMY, T. SCHEDLER & U. THIEL, *Hyperplane arrangements associated to symplectic quotient singularities*, preprint (2017), `arXiv:1702.04881`.

[14] É. VASSEROT, *Sur l'anneau de cohomologie du schéma de Hilbert de $\mathbb{C}^2$*, C. R. Acad. Sci. **332** (2001), 7-12.

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