Generalized Wintegen type inequality for Lagrangian submanifolds in holomorphic Statistical space forms

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Abstract. Statistical manifolds are abstract generalizations of statistical models introduced by Amari [1] in 1985. Such manifolds have been studied in terms of information geometry which includes the notion of dual connections, called conjugate connection in affine geometry. Recently, Furuhata [3] defined and studied the properties of holomorphic statistical space forms.

In this paper, we obtain the generalized Wintgen type inequality for Lagrangian submanifolds in holomorphic statistical space forms. We also obtain condition under which the submanifold becomes minimal or H is some scalar multiple of H^*.

Keywords: Wintgen inequality, Lagrangian submanifold, holomorphic statistical space forms

1 Introduction

The history of statistical manifold was started from investigations of geometric structures on sets of certain probability distributions. In fact, statistical manifolds introduced, in 1985, by Amari [1] have been studied in terms of information geometry and such manifolds include the notion of dual connections, called conjugate connection in affine geometry, closely related to affine differential geometry and which has application in various fields of science and engineering such as string theory, robot control, digital signal processing etc. The geometry of submanifolds of statistical manifolds is still a young geometry, therefore it attracts our attention.

Moreover, the Wintgen inequality is a sharp geometric inequality for surface in 4-dimensional Euclidean space involving Gauss curvature (intrinsic invariant), normal curvature and square mean curvature (extrinsic invariant). The generalized Wintgen inequality was conjectured by De Smet, Dillen, Verstraelen and Vrancken in 1999 for the submanifolds in real space forms also known as DDVV conjecture.

In present article, we will prove the generalized Wintgen type inequalities for Lagrangian submanifolds in statistical holomorphic space forms some of its applications.

2 Statistical manifolds and submanifolds

A statistical manifold is a Riemannian manifold (M, g) endowed with a pair of torsion-free affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ satisfying

$$Zg(X,Y) = g(\overline{\nabla}_Z X, Y) + g(X, \overline{\nabla}^*_Z Y), \tag{1}$$

for $X, Y, Z \in \Gamma(TM)$. It is denoted by $(M, g, \overline{\nabla}, \overline{\nabla}^*)$. The connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called dual connections and it is easily shown that $(\overline{\nabla}^*)^* = \overline{\nabla}$. The pair $(\overline{\nabla}, g)$ is said to be a statistical structure. If $(\overline{\nabla}, g)$ is a statistical structure on $\overline{M}$, then $(\overline{\nabla}^*, g)$ is also statistical structure on $\overline{M}$. Denote by $\overline{R}$ and $\overline{R}^*$ the curvature tensor fields of $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively. Then the curvature tensor fields $\overline{R}$ and $\overline{R}^*$ satisfies

$$g(\overline{R}^*(X,Y)Z, W) = -g(Z, \overline{R}(X,Y)W). \tag{2}$$

Let $\overline{M}$ be a $2m$-dimensional manifold and let M be a n-dimensional submanifolds of $\overline{M}$. Then, the corresponding Gauss formulas according to [5] are:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3}$$

$$\overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y) \tag{4}$$

where h and h^* are symmetric and bilinear, called imbedding curvature tensor of M in $\overline{M}$ for $\overline{\nabla}$ and the imbedding curvature tensor of M in $\overline{M}$ for $\overline{\nabla}^*$, respectively. Let us denote the normal bundle of M by $\Gamma(TM^\perp)$. Since h and h^* are bilinear, we have the linear transformations A_ξ and A^*_ξ defined by

$$g(A_\xi X, Y) = g(h(X,Y), \xi), \tag{5}$$

$$g(A^*_\xi X, Y) = g(h^*(X,Y), \xi), \tag{6}$$

for any $\xi \in \Gamma(TM^\perp)$ and $X, Y \in \Gamma(TM)$. The corresponding Weingarten formulas [5] are:

$$\overline{\nabla}_X \xi = -A^*_\xi X + \nabla^\perp_X \xi, \tag{7}$$

$$\overline{\nabla}^*_X\xi = -A_\xi X + \nabla^{*\perp}_X\xi, \tag{8}$$

for any $\xi \in \Gamma(TM^\perp)$ and $X \in \Gamma(TM)$. The connections $\nabla^\perp_X$ and $\nabla^{*\perp}_X$ given in the above equations are Riemannian dual connections with respect to the induced metric on $\Gamma(TM^\perp)$.

The corresponding Gauss, Codazzi and Ricci equations are given by the following results.

Proposition 1 ([5]) *Let $\overline{\nabla}$ be a dual connection on $\overline{M}$ and ∇ the induced connection on M. Let $\overline{R}$ and R be the Riemannian curvature tensors of $\overline{\nabla}$ and ∇, respectively. Then,*

$$\begin{aligned} g(\overline{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + g(h(X,Z),h^*(Y,W)) \\ &\quad - g(h^*(X,W),h(Y,Z)), \end{aligned} \tag{9}$$

$$\begin{aligned} (\overline{R}(X,Y)Z)^\perp &= \nabla^\perp_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) \\ &\quad - \{\nabla^\perp_Y h(X,Z) - h(\nabla_Y X,Z) - h(X,\nabla_Y Z)\}, \end{aligned} \tag{10}$$

$$g(R^\perp(X,y)\xi,\eta) = g(\overline{R}(X,y)\xi,\eta) + g([A^*_\xi,A_\eta]X,Y), \tag{11}$$

where $R^\perp$ is the Riemannian curvature tensor on $TM^\perp$, $\xi,\eta \in \Gamma(TM^\perp)$ and $[A^*_\xi,A_\eta] = A^*_\xi A_\eta - A_\eta A^*_\xi$.

Similarly, for the dual connection $\overline{\nabla}^*$ on $\overline{M}$, we have

Proposition 2 ([5]) *Let $\overline{\nabla}^*$ be a dual connection on $\overline{M}$ and ∇^* the induced connection on M. Let $\overline{R}^*$ and R^* be the Riemannian curvature tensors of $\overline{\nabla}^*$ and ∇^*, respectively. Then,*

$$\begin{aligned} g(\overline{R}^*(X,Y)Z,W) &= g(R^*(X,Y)Z,W) + g(h^*(X,Z),h(Y,W)) \\ &\quad - g(h(X,W),h^*(Y,Z)), \end{aligned} \tag{12}$$

$$\begin{aligned} (\overline{R}^*(X,Y)Z)^\perp &= \nabla^{*\perp}_X h^*(Y,Z) - h^*(\nabla^*_X Y,Z) - h^*(Y,\nabla^*_X Z) \\ &\quad - \{\nabla^{*\perp}_Y h^*(X,Z) - h^*(\nabla^*_Y X,Z) - h^*(X,\nabla^*_Y Z)\}, \end{aligned} \tag{13}$$

$$g(R^{*\perp}(X,y)\xi,\eta) = g(\overline{R}^*(X,y)\xi,\eta) + g([A_\xi,A^*_\eta]X,Y), \tag{14}$$

where $R^{*\perp}$ is the Riemannian curvature tensor for $\nabla^{\perp *}$ on $TM^\perp$, $\xi,\eta \in \Gamma(TM^\perp)$ and $[A_\xi,A^*_\eta] = A_\xi A^*_\eta - A^*_\eta A_\xi$.

Definition 1 ([3]). *A $2m$-dimensional statistical manifold M is said to be a holomorphic statistical manifold if it admits an endomorphism over the tangent bundle $\Gamma(M)$ and a metric g and a fundamental form ω given by $\omega(X,Y) = g(X,JY)$ such that*

$$J^2 = -Id; \qquad \overline{\nabla}\omega = 0, \tag{15}$$

for any vector fields $X, Y \in \Gamma(M)$. Since ω is skew-symmetric, we have $g(X,JY) = -g(JX,Y)$.

Definition 2 ([3]). *A holomorphic statistical manifold M is said to be of constant holomorphic curvature $c \in R$ if the following curvature equation holds :*

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(X,JZ)JY \\ &-g(Y,JZ)JX + 2g(X,JY)JZ\}.\end{aligned} \tag{16}$$

According to the behavior of the tangent space under the action of J, submanifolds in a Hermitian manifold is divided into two fundamental classes namely: *Invariant submanifold* and *totally real submanifold*.

Definition 3. *A totally real submanifold of maximal dimension is called Lagrangian submanifold.*

Let $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m}\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on M. The mean curvature vector field is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) \tag{17}$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i). \tag{18}$$

We also set

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) \tag{19}$$

and

$$\|h^*\|^2 = \sum_{i,j=1}^{n} g(h^*(e_i, e_j), h^*(e_i, e_j)). \tag{20}$$

3 Generalized Wintgen type inequality

We denote by K and $R^{\perp}$ the sectional curvature function and the normal curvature tensor on M, respectively. Then the normalized scalar curvature ρ is given by [5]

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} K(e_i \wedge e_j), \tag{21}$$

where τ is scalar curvature, and the normalized normal scalar curvature by [2]

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1\leq i<j\leq n} \sum_{1\leq\alpha<\beta\leq 2m} (R^{\perp}(e_i, e_j, \xi_\alpha, \xi_\beta))^2}. \tag{22}$$

Following [6] we put

$$K_N = \frac{1}{4} \sum_{r,s=1}^{2m-n} Trace[A_r^*, A_s]^2 \tag{23}$$

and called it the scalar normal curvature of M. The normalized scalar normal curvature is given by [4] $\rho_N = \frac{2}{n(n-1)}\sqrt{K_N}$.
Obviously

$$\begin{aligned} K_N &= \frac{1}{2} \sum_{1\leq r<s\leq 2m-n} Trace[A_r^*, A_s]^2 \\ &= \sum_{1\leq r<s\leq 2m-n} \sum_{1\leq i<j\leq n} g([A_r^*, A_s]e_i.e_j)^2, \end{aligned} \tag{24}$$

for $i, j \in \{1, \ldots, n\}$ and $r, s \in \{1, \ldots, 2m-n\}$.
In term of the components of the second fundamental form, we can express K_N by the formula [4]

$$K_N = \sum_{1\leq r<s\leq 2m-n} \sum_{1\leq i<j\leq n} \big(\sum_{k=1}^{n} h_{jk}^{*r} h_{ik}^{s} - h_{jk}^{r} h_{ik}^{*s} \big)^2. \tag{25}$$

We prove the following.

Theorem 1 *Let M be a Lagrangian submanifold of a holomorphic statistical space form $\overline{M}$. Then*

$$(\rho^{\perp})^2 \geq \frac{c}{n(n-1)}(\rho - \frac{c}{4}) + \frac{c}{(n-1)^2}\big[g(H^*, H) - \|H\|\|H^*\|\big]. \tag{26}$$

Proof. Let M be a Lagrangian submanifold of a holomorphic statistical space form $\overline{M}$ and $\{e_1, \ldots, e_n\}$ an orthonormal frame on M; then $\{\xi_1 = Je_1, \ldots, \xi_n =$

$Je_n\}$ is the orthonormal frame in the normal bundle $\Gamma(TM^{\perp})$. Putting $X = W = e_i$, $Y = Z = e_j$, $i \neq j$ from (16), we have

$$\begin{aligned}\overline{R}(e_i, e_j, e_j, e_i) = \frac{c}{4}\{&g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\\ &+g(e_i, Je_j)g(Je_j, e_i) - g(e_j, Je_j)g(Je_i, e_i)\\ &+2g(e_i, Je_j)g(Je_j, e_i)\}. \end{aligned} \tag{27}$$

Combining equations (9) and (27), we obtain

$$\begin{aligned}R(e_i, e_j, e_j, e_i) = \frac{c}{4}\{&g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\\ &+g(e_i, Je_j)g(Je_j, e_i) - g(e_j, Je_j)g(Je_i, e_i)\\ &+2g(e_i, Je_j)g(Je_j, e_i)\} + g(h(e_i, e_i), h^*(e_j, e_j))\\ &-g(h^*(e_i, e_j), h(e_i, e_j)). \end{aligned} \tag{28}$$

By taking summation $1 \leq i, j \leq n$ and using (17), (18) in (28), we derive

$$2\tau = n(n-1)\frac{c}{4} + n^2 g(H, H^*) - g(h^*(e_i, e_j), h(e_i, e_j)). \tag{29}$$

Using (21) in (29), we get

$$\rho = \frac{c}{4} + \frac{n}{n-1}g(H, H^*) - \frac{1}{n(n-1)}g(h^*(e_i, e_j), h(e_i, e_j)). \tag{30}$$

Now, using Cauchy-Schwarz inequality,(19) and (20) in the above equation, we find

$$\rho \leq \frac{c}{4} + \frac{n}{n-1}g(H, H^*) - \frac{1}{n(n-1)}\|h^*\|\|h\|, \tag{31}$$

which imply

$$\|h^*\|\|h\| \leq n(n-1)(\frac{c}{4} - \rho) + n^2\|H\|\|H^*\|. \tag{32}$$

Further, equation (11) implies

$$R^{\perp}(e_i, e_j, \xi_r, \xi_s) = \frac{c}{4}\{-(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is})\} + g([A^*_{\xi_r}, A_{\xi_s}]e_i, e_j), \tag{33}$$

for all $i, j \in \{1, \dots, n\}$ and $r, s \in \{1, \dots, n\}$.

Then we have

$$\begin{aligned}(\tau^{\perp})^2 &= (R^{\perp}(e_i, e_j, \xi_r, \xi_s))^2\\ &= (\frac{c}{4}\{(\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is})\} - g([A^*_{\xi_r}, A_{\xi_s}]e_i, e_j))^2\\ &= \frac{c^2}{16}\frac{n(n-1)}{2} + K_N - \frac{c}{4}g(h(e_i, e_j), h^*(e_i, e_j))\\ &\quad+\frac{c}{4}g(h^*(e_i, e_i), h(e_j, e_j)) \end{aligned} \tag{34}$$

Above equation can be re-written as

$$\begin{aligned}(\rho^{\perp})^2 &= \frac{c^2}{8n(n-1)} + \rho_N^2 - \frac{c}{n^2(n-1)^2} g(h(e_i, e_j), h^*(e_i, e_j)) \\ &\quad + \frac{c}{n^2(n-1)^2} g(h^*(e_i, e_i), h(e_j, e_j)) \\ &\geq \frac{c^2}{8n(n-1)} + \rho_N^2 - \frac{c}{n^2(n-1)^2} \|h\| \|h^*\| \\ &\quad + \frac{c}{n^2(n-1)^2} g(h^*(e_i, e_i), h(e_j, e_j)). \qquad (35)\end{aligned}$$

Now, from (32) and (35), we have

$$\begin{aligned}(\rho^{\perp})^2 &\geq \frac{c^2}{8n(n-1)} + \rho_N^2 - \frac{c}{n^2(n-1)^2} \big[n(n-1)(\frac{c}{4} - \rho) \\ &\quad + n^2 \|H\| \|H^*\|\big] + \frac{c}{n^2(n-1)^2} g(h^*(e_i, e_i), h(e_j, e_j)) \\ &\geq \frac{c^2}{8n(n-1)} + \rho_N^2 + \frac{c}{n(n-1)}(\rho - \frac{c}{4}) \\ &\quad + \frac{c}{(n-1)^2} \big[g(H, H^*) - \|H\| \|H^*\|\big] \\ &\geq \frac{c}{n(n-1)}(\rho - \frac{c}{4}) + \frac{c}{(n-1)^2} \big[g(H, H^*) - \|H\| \|H^*\|\big]\end{aligned}$$

An immediate consequence of the Theorem 1 yields the following.

Corollary 1. *Let M be a Lagrangian submanifold of negatively curved holomorphic space form with flat normal bundle. If $\rho = \frac{c}{4}$, then M is either minimal or H is some scalar multiple of H^*.*

Further, we observe

Proposition 3 *Let M be a Lagrangian submanifold of a holomorphic statistical space form $\overline{M}$. If θ be the angle between H and H^*, then*

$$(\rho^{\perp})^2 \geq \frac{c}{n(n-1)}(\rho - \frac{c}{4}) + \frac{c}{(n-1)^2} \big[\|H\| \|H^*\| (\cos\theta - 1)\big].$$

Corollary 2. *Let M be a Lagrangian submanifold of a holomorphic statistical space form $\overline{M}$. If H and H^* are parallel, then*

$$(\rho^{\perp})^2 \geq \frac{c}{n(n-1)}(\rho - \frac{c}{4}).$$

Corollary 3. *Let M be a Lagrangian submanifold of a holomorphic statistical space form $\overline{M}$. If H and H^* are perpendicular, then*

$$(\rho^{\perp})^2 \geq \frac{c}{n(n-1)}(\rho - \frac{c}{4}) - \frac{c}{(n-1)^2} \|H\| \|H^*\|.$$

Remark 1. The above results are verified for Lagrangian submanifold in complex space form, which is ordinary case of Lagrangian submanifold in holomorphic statistical space form when H and H^* coincides.

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