Classification Of Totally Umbilical CR-Statistical Submanifolds In Holomorphic Statistical Manifolds With Constant Holomorphic Curvature

Michel Nguiffo Boyom[1], Aliya Naaz Siddiqui[2], Wan Ainun Mior Othman[3] and Mohammad Hasan Shahid[4]

[1] IMAG, Alexander Grothendieck Research Institute, Universit*é* of Montpellier, France
nguiffo.boyom@gmail.com,
[2] Department of Mathematics, Jamia Millia Islamia University, New Delhi - 110025, India
aliyanaazsiddiqui9@gmail.com,
[3] Institute of Mathematical Sciences, University of Malaya, Malaysia
wanainun@um.edu.my,
[4] Department of Mathematics, Jamia Millia Islamia University, New Delhi - 110025, India
hasan_jmi@yahoo.com

Abstract. In 1985, Amari [1] introduced an interesting manifold, i.e., statistical manifold in the context of information geometry. The geometry of such manifolds includes the notion of dual connections, called conjugate connections in affine geometry, it is closely related to affine geometry. A statistical structure is a generalization of a Hessian one, it connects Hessian geometry.

In the present paper, we study CR-statistical submanifolds in holomorphic statistical manifolds. Some results on totally umbilical CR-statistical submanifolds with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ in holomorphic statistical manifolds with constant holomorphic curvature are obtained.

Keywords: CR-statistical submanifolds, Holomorphic statistical manifolds, Totally umbilical submanifolds.

1 Introduction

In 1978, A. Bejancu [3] introduced the notion of a CR-submanifold of Kaehler manifolds with complex structure $\mathcal{J}$. CR submanifolds arise as a natural generalization of both holomorphic and totally real submanifolds in complex geometry. Since then such submanifolds have been investigated extensively by many geometers and many interesting results were obtained. On the other hand, statistical manifolds are abstract generalizations of statistical models. Even if a statistical manifold is treated as a purely geometric object, however, the motivation for the definitions is inspired from statistical models. Geometry of statistical manifolds lies at the confluence of some research areas such as information geometry, affine differential geometry, and Hessian geometry. Beyond expectations, statistical manifolds are familiar to geometers and many interesting results were obtained [11, 10, 2, 14]. In 2004, Kurose [9] defined their complex version, i.e., holomorphic statistical manifolds. It is natural for geometers to try to build the submanifold theory and the complex manifold theory of statistical manifolds. Recently, Furuhata and Hasegawa [7] studied CR-statistical submanifold theory in holomorphic statistical manifolds. Motivated by their work, we wish to give some more results on CR-statistical submanifolds of holomorphic statistical manifolds, which are new objects originating from information geometry.

Our work is structured as follows : *Section* 2 is devoted to preliminaries. *Section* 3 deals with some basic results in holomorphic statistical manifolds. In *Section* 4, we give complete classification of totally umbilical CR-statistical submanifolds with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ in holomorphic statistical manifolds with constant holomorphic curvature.

2 Preliminaries

This section is fully devoted to a brief review of several fundamental notions, formulas and some definitions which are required later.

Definition 1. *[7] A statistical manifold is a Riemannian manifold $\left(\overline{M}, g\right)$ of dimension $(n+k)$, endowed with a pair of torsion-free affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ satisfying*

$$Zg(X,Y) = g(\overline{\nabla}_Z X, Y) + g(X, \overline{\nabla}^*_Z Y)$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. It is denoted by $\left(\overline{M}, g, \overline{\nabla}, \overline{\nabla}^*\right)$. The connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called dual connections on $\overline{M}$ and it is easily shown that $(\overline{\nabla}^*)^* = \overline{\nabla}$. If $\left(\overline{\nabla}, g\right)$ is a statistical structure on $\overline{M}$, then $\left(\overline{\nabla}^*, g\right)$ is also a statistical structure.

Let $\left(\overline{M}, g\right)$ be a Riemannian manifold and M a submanifold of $\overline{M}$. If $\left(M, \nabla, g\right)$ is a statistical manifold, then we call $\left(M, \nabla, g\right)$ a statistical submanifold of $\left(\overline{M}, g\right)$, where ∇ is an affine connection on M and the Riemannian metric for M and $\overline{M}$ is denoted by the same symbol g. Let $\overline{\nabla}$ be an affine connection on $\overline{M}$. If $\left(\overline{M}, g, \overline{\nabla}\right)$ is a statistical manifold and M a submanifold of $\overline{M}$, then $\left(M, \nabla, g\right)$ is also a statistical manifold with the induced connection ∇ and induced metric g.

In the geometry of Riemannian submanifolds [15], the fundamental equations are the Gauss and Weingarten formulas and the equations of Gauss, Codazzi and Ricci. In our case, for any $X, Y \in \Gamma(TM)$ and $\mathcal{V} \in \Gamma(T^{\perp}M)$, Gauss and Weingarten formulas are, respectively, defined by [7]

$$\left.\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + \sigma(X,Y), \qquad & \overline{\nabla}^*_X Y &= \nabla^*_X Y + \sigma^*(X,Y), \\ \overline{\nabla}_X \mathcal{V} &= -\mathcal{A}_{\mathcal{V}}(X) + D_X \mathcal{V}, \qquad & \overline{\nabla}^*_X \mathcal{V} &= -\mathcal{A}^*_{\mathcal{V}}(X) + D^*_X \mathcal{V}, \end{aligned}\right\} \tag{1}$$

where $\overline{\nabla}$ and $\overline{\nabla}^*$ (respectively, ∇ and ∇^*) are the dual connections on $\overline{M}$ (respectively, on M), σ and σ^* are symmetric and bilinear, called the imbedding curvature tensor of M in $\overline{M}$ for $\overline{\nabla}$ and the imbedding curvature tensor of M in $\overline{M}$ for $\overline{\nabla}^*$, respectively. Since σ and σ^* are bilinear, we have the linear transformations $\mathcal{A}_\mathcal{V}$ and $\mathcal{A}^*_\mathcal{V}$, defined by [7]

$$\left.\begin{aligned} g(\sigma(X,Y),\mathcal{V}) &= g(\mathcal{A}^*_\mathcal{V}(X),Y),\\ g(\sigma^*(X,Y),\mathcal{V}) &= g(\mathcal{A}_\mathcal{V}(X),Y) \end{aligned}\right\} \tag{2}$$

for any $X,Y \in \Gamma(TM)$ and $\mathcal{V} \in \Gamma(T^\perp M)$.

Let $\overline{R}$ and R be the curvature tensor fields of $\overline{\nabla}$ and ∇, respectively. The corresponding Gauss, Codazzi and Ricci equations, respectively, are given by [7]

$$\left.\begin{aligned} \overline{R}(X,Y,Z,W) =& R(X,Y,Z,W) + g(\sigma(X,Z),\sigma^*(Y,W))\\ &- g(\sigma^*(X,W),\sigma(Y,Z)),\\ \overline{R}(X,Y,Z,\mathcal{N}) =& g((\overline{\nabla}_X\sigma)(Y,Z),\mathcal{N}) - g((\overline{\nabla}_Y\sigma)(X,Z),\mathcal{N}),\\ \overline{R}(X,Y,\mathcal{N},Z) =& g((\overline{\nabla}_Y\mathcal{A})_\mathcal{N}X,Z) - g((\overline{\nabla}_X\mathcal{A})_\mathcal{N}Y,Z),\\ \overline{R}(X,Y,\mathcal{N},\mathcal{V}) =& R^\perp(X,Y,\mathcal{N},\mathcal{V}) + g(\sigma(Y,\mathcal{A}_\mathcal{N}X,\mathcal{V}) - g(\sigma(X,\mathcal{A}_\mathcal{N}Y,\mathcal{V}) \end{aligned}\right\} \tag{3}$$

for any $X,Y,Z,W \in \Gamma(TM)$ and $\mathcal{N},\mathcal{V} \in \Gamma(T^\perp M)$.

Similarly, $\overline{R}$ and R are the curvature tensor fields of $\overline{\nabla}$ and ∇, respectively and duals of all equations in (3) can be obtained for the connections $\overline{\nabla}^*$ and ∇^* [7].

Definition 2. *[7, 8] Let M be a submanifold of a statistical manifold $\overline{M}$. Then M is said to be a*

(A) *totally geodesic with respect to $\overline{\nabla}$ if $\sigma = 0$.*
(A*) *totally geodesic with respect to $\overline{\nabla}^*$ if $\sigma^* = 0$.*
(B) *totally tangentially umbilical with respect to $\overline{\nabla}$ if $\sigma(X,Y) = g(X,Y)\mathcal{H}$ for any $X,Y \in \Gamma(TM)$. Here $\mathcal{H}$ is the mean curvature vector of M in $\overline{M}$ for $\overline{\nabla}$.*
(B*) *totally tangentially umbilical with respect to $\overline{\nabla}^*$ if $\sigma^*(X,Y) = g(X,Y)\mathcal{H}^*$ for any $X,Y \in \Gamma(TM)$. Here $\mathcal{H}^*$ is the mean curvature vector of M in $\overline{M}$ for $\overline{\nabla}^*$.*
(C) *totally normally umbilical with respect to $\overline{\nabla}$ if $\mathcal{A}_\mathcal{N}X = g(\mathcal{H},\mathcal{N})X$ for any $X \in \Gamma(TM)$ and $\mathcal{N} \in \Gamma(T^\perp M)$.*
(C*) *totally normally umbilical with respect to $\overline{\nabla}^*$ if $\mathcal{A}^*_\mathcal{N}X = g(\mathcal{H}^*,\mathcal{N})X$ for any $X \in \Gamma(TM)$ and $\mathcal{N} \in \Gamma(T^\perp M)$.*

Definition 3. *[7] Let $\left(\overline{M},\mathcal{J},g\right)$ be a Kähler manifold and $\overline{\nabla}$ be an affine connection on $\overline{M}$. Then $\left(\overline{M},\overline{\nabla},g,\mathcal{J}\right)$ is said to be a holomorphic statistical manifold if $\left(\overline{M},\overline{\nabla},g\right)$ is a statistical manifold and a 2−form ω on $\overline{M}$, given by*

$$\omega(X,Y) = g(X,\mathcal{J}Y)$$

for any $X,Y \in \Gamma(T\overline{M})$, is a $\overline{\nabla}$−parallel, i.e., $\overline{\nabla}\omega = 0$.

Let $(\overline{M}, \mathcal{J}, g)$ be a holomorphic statistical manifold. Then [7]

$$\overline{\nabla}_X(\mathcal{J}Y) = \mathcal{J}\overline{\nabla}^*_X Y \tag{4}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}^*$ is the dual connection of $\overline{\nabla}$ with respect to g.

Remark 1. A holomorphic statistical manifold $(\overline{M}, \overline{\nabla}, g, \mathcal{J})$ is nothing but a special Kähler manifold if $\overline{\nabla}$ is flat.

For any vector field $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, respectively, we put [15]

$$\mathcal{J}X = GX + LX \quad and \quad \mathcal{J}V = CV + BV, \tag{5}$$

where $GX = tan(\mathcal{J}X)$, $LX = nor(\mathcal{J}X)$, $CV = tan(\mathcal{J}V)$ and $BV = nor(\mathcal{J}V)$. It is easy to see that [15]

$$g(GX, Y) = -g(X, GY), \;\; g(BV, \mathcal{N}) = -g(V, B\mathcal{N}) \;\; and \;\; g(LX, \mathcal{N}) = -g(X, C\mathcal{N}) \tag{6}$$

for any $X, Y \in \Gamma(TM)$ and $V, \mathcal{N} \in \Gamma(T^\perp M)$.

Definition 4. *[7] A holomorphic statistical manifold $\overline{M}$ of constant holomorphic curvature $k \in \mathbb{R}$ is said to be a holomorphic statistical space form $\overline{M}(k)$ if the following curvature equation holds*

$$\begin{aligned} \overline{R}(X, Y)Z = \frac{k}{4}\big\{ & g(Y, Z)X - g(X, Z)Y + g(\mathcal{J}Y, Z)\mathcal{J}X \\ & - g(\mathcal{J}X, Z)\mathcal{J}Y + 2g(X, \mathcal{J}Y)\mathcal{J}Z\big\}. \end{aligned} \tag{7}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$.

The statistical version of definition of CR-submanifold as follows :

Definition 5. *[7] A statistical submanifold M is called a CR-statistical submanifold in a holomorphic statistical manifold $\overline{M}$ of dimension $2m \geq 4$ if M is CR-submanifold in $\overline{M}$, i.e., there exists a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subseteq T_x M$ on M satisfying the following conditions :*

(A) *$\mathcal{D}$ is holomorphic, i.e., $\mathcal{J}\mathcal{D}_x = \mathcal{D}_x \subset T_x M$ for each $x \in M$, and*

(B) *the complementary orthogonal distribution $\mathcal{D}^\perp : x \to \mathcal{D}^\perp_x \subseteq T_x M$ is totally real, i.e., $\mathcal{J}\mathcal{D}^\perp_x \subset T^\perp_x M$ for each $x \in M$.*

Remark 2. [7] If $\mathcal{D} \neq 0$ and $\mathcal{D}^\perp \neq 0$, then M is said to be *proper*.

Remark 3. [7] CR-statistical submanifolds are characterized by the condition $LG = 0$.

Definition 6. *[7] A statistical submanifold M of a holomorphic statistical manifold $\overline{M}$ is called holomorphic ($L = 0$ and $C = 0$) if the almost complex structure $\mathcal{J}$ of $\overline{M}$ carries each tangent space of M into itself whereas it is said to be totally real ($G = 0$) if the almost complex structure $\mathcal{J}$ of $\overline{M}$ carries each tangent space of M into its corresponding normal space.*

For a CR-statistical submanifold M we shall denote by μ the orthogonal complementary subbundle of $\mathcal{J}\mathcal{D}^\perp$ in $T^\perp M$, we have [7]

$$T^\perp M = \mathcal{J}\mathcal{D}^\perp \oplus \mu \tag{8}$$

3 Some basic results in holomorphic statistical manifolds

In this section, we propose some basic results which are based on $D_X LY = L\nabla^*_X Y$:

Theorem 1. *Let M be a statistical submanifold of a holomorphic statistical manifold $\overline{M}$. Then $D_X LY = L\nabla^*_X Y$ holds if and only if $\mathcal{A}^*_{\mathcal{N}} GY = -\mathcal{A}_{B\mathcal{N}} Y$ for any $X, Y \in \Gamma(TM)$ and $\mathcal{N} \in \Gamma(T^\perp M)$.*

Proof. From Lemma 5(7.40) of [7], we have

$$\sigma(X, GY) + D_X LY = L\nabla^*_X Y + B\sigma^*(X, Y).$$

In the light of (6) and (2), we get

$$g(\mathcal{A}^*_N GY, X) = -g(\mathcal{A}_{B\mathcal{N}} Y, X)$$

for any $X, Y \in \Gamma(TM)$ and $\mathcal{N} \in \Gamma(T^\perp M)$. This proves our theorem.

The following theorem shall be required to prove some results in the next section :

Theorem 2. *Let M be a statistical submanifold of a holomorphic statistical manifold $\overline{M}$. If $D_X LY = L\nabla^*_X Y$ holds, then the curvature tensor R^* and the normal curvature tensor $R^\perp$ satisfy $LR^*(X, Y)Z = R^\perp(X, Y)LZ$ for any $X, Y, Z \in \Gamma(TM)$.*

Proof. Since, we have assumed that $D_X(LY) = L\nabla^*_X Y$ for any $X, Y \in \Gamma(TM)$. Therefore, we derive the following :

$$\begin{aligned} LR^*(X, Y)Z &= L\big(\nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z\big) \\ &= D_X D_Y LZ - D_Y D_X LZ - D_{[X,Y]} LZ \\ &= R^\perp(X, Y)LZ \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$. This proves our assertion.

4 Classification of totally umbilical CR-statistical submanifolds

B. Y. Chen [5] studied totally umbilical submanifolds in the case of spaces of constant curvature. Also, Chen and Ogiue [6] considered such immersions in complex space forms. Blair and Vanhecke [4] considered in Sasakian space forms. In 2002, Kurose [8] studied totally tangentially umbilical and totally normally umbilical in statistical manifolds. In this section, we study a special class of CR-statistical submanifolds which is totally umbilical CR-statistical submanifolds with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ in holomorphic statistical manifolds.

Theorem 3. *Let M be a CR-statistical submanifold in a holomorphic statistical manifold $\overline{M}$. If M is totally umbilical with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ such that $\mathcal{J}\mathcal{H}^* \in \Gamma(\mu)$, then we have either*

(A) *M is a totally geodesic with respect to $\overline{\nabla}^*$, or*
(B) *dim $\mathcal{D} \geq 2$.*

Proof. For any $X, Y \in \Gamma(\mathcal{D})$, we have

$$\overline{\nabla}_X \mathcal{J}Y = \overline{\nabla}_X GY.$$

By our assumption, last relation takes the following form :

$$\mathcal{J}\nabla^*_X Y + g(Y, X)\mathcal{J}\mathcal{H}^* = \nabla_X GY + g(X, GY)\mathcal{H}. \tag{9}$$

Taking inner product with $\mathcal{J}\mathcal{H}^*$ on both sides of (9), we obtain

$$g(Y, X)||\mathcal{H}^*||^2 = g(X, GY)g(\mathcal{J}\mathcal{H}^*, \mathcal{H}). \tag{10}$$

Interchanging the role of X and Y in above equation, we get

$$g(X, Y)||\mathcal{H}^*||^2 = g(Y, GX)g(\mathcal{J}\mathcal{H}^*, \mathcal{H}). \tag{11}$$

Combining both equations (10) and (11), we find that

$$g(X, Y)||\mathcal{H}^*||^2 = 0. \tag{12}$$

From (12), we conclude that $\mathcal{H}^* = 0$ or $g(X, Y)$ for any $X, Y \in \Gamma(\mathcal{D})$ which shows that M is totally geodesic with respect to $\overline{\nabla}^*$ or dim $\mathcal{D} \geq 2$, respectively. This completes the proof of the theorem.

Theorem 4. *Let M be a CR-statistical submanifold in a holomorphic statistical manifold $\overline{M}$. If M is totally umbilical with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, then, for any $X \in \Gamma(\mathcal{D}^\perp)$, we have*

(A) *$\mathcal{D}_X\mathcal{H} \in \mathcal{J}\mathcal{D}^\perp$, or*
(B) *$\mathcal{D}_X\mathcal{H} = 0$,*

(A$*$) *$\mathcal{D}^*_X\mathcal{H}^* \in \mathcal{J}\mathcal{D}^\perp$, or*
(B*) *$\mathcal{D}^*_X\mathcal{H}^* = 0$.*

Proof. For any $X \in \Gamma(\mathcal{D}^\perp)$ and $Y, Z \in \Gamma(\mathcal{D})$, we have the following [7] :

$$\mathcal{J}\overline{R}^*(X, Y)Z = \overline{R}(X, Y)\mathcal{J}Z$$
$$G\overline{R}^*(X, Y)Z + L\overline{R}^*(X, Y)Z = g(Y, GZ)D_X\mathcal{H}, \tag{13}$$

where we have used Codazzi equation for a totally tangentially umbilical submanifold. Taking inner product on both sides of (13) with $\mathcal{N} \in \Gamma(\mu)$ and putting $Y = GZ$, we arrive at

$$||Z||^2 g(D_X\mathcal{H}, \mathcal{N}) = 0. \tag{14}$$

Similarly, we can easily obtain dual of (14), i.e.,

$$||Z||^2 g(D^*_X \mathcal{H}^*, \mathcal{N}) = 0, \tag{15}$$

where we have used Codazzi equation for a totally normally umbilical submanifold. From (14), we conclude that $D_X\mathcal{H} \in \mathcal{J}\mathcal{D}^\perp$ or $D_X\mathcal{H} = 0$. And (15) gives $D^*_X\mathcal{H}^* \in \mathcal{J}\mathcal{D}^\perp$ or $D^*_X\mathcal{H}^* = 0$. This completes the proof of the theorem.

For CR-statistical submanifolds in holomorphic statistical space forms, we have the followings :

Theorem 5. *Let M be a CR-statistical submanifold in a holomorphic statistical space form $\overline{M}(k)$. If M is totally umbilical with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ such that $\mathcal{D}_X\mathcal{H} = 0$ and $\mathcal{D}^*_X\mathcal{H}^* = 0$ for any $X \in \Gamma(\mathcal{D}^\perp)$, then*

(A) *$k = 0$, or*
(B) *dim $\mathcal{D} \geq 2$, or*
(C) *H and H^* are perpendicular to $J\mathcal{D}^\perp$, or*
(D) *M is totally geodesic with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$.*

Proof. From Proposition 3(7.33) of [7] and (7), we have

$$\begin{aligned}(\overline{\nabla}_X\sigma)(Y,Z) - (\overline{\nabla}_Y\sigma)(X,Z) + (\overline{\nabla}^*_X\sigma^*)(Y,Z) - (\overline{\nabla}^*_Y\sigma^*)(X,Z)\\ = 2\left\{\frac{k}{4}\big[g(GY,Z)LX - g(GX,Z)LY + 2g(X,GY)LZ\big]\right\}\end{aligned} \tag{16}$$

for any $X, Y, Z \in \Gamma(TM)$. Now we evaluate (16) for $Z = GW$, $W, Y \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}^\perp)$, we get

$$\begin{aligned}(\overline{\nabla}_X\sigma)(Y,GW) - (\overline{\nabla}_Y\sigma)(X,GW) + (\overline{\nabla}^*_X\sigma^*)(Y,GW) - (\overline{\nabla}^*_Y\sigma^*)(X,GW)\\ = \frac{k}{4}\Big[g(GY,GW)LX\Big].\end{aligned}$$

By virtue of Codazzi equation for a totally umbilical with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, we arrive at $\frac{k}{4}\Big[g(Y,W)LX\Big] = 0$. If we take inner product with H (respectively, H^*) on both sides of above relation, then we conclude that $k = 0$ or dim $\mathcal{D} \geq 2$ or both H and H^* are perpendicular to $J\mathcal{D}^\perp$ or M is totally geodesic with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$. This completes our proof.

Theorem 6. *Let M be a proper CR-statistical submanifold in a holomorphic statistical space form $\overline{M}(k)$. If M is totally umbilical with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ such that $D_X LY = L\nabla^*_X Y$ for any $Y \in \Gamma(\mathcal{D}^\perp)$ and $X \in \Gamma(\mathcal{D})$, then we have*

(A) *$k = 0$, or*
(B) *dim $\mathcal{D}^\perp \geq 2$.*

Proof. From Theorem 2, 4^{th} equation of (3) and its dual, we can easily get $\frac{k}{4}\Big[g(Y,Z)GX\Big]=0$ for any $Y,Z\in\Gamma(\mathcal{D}^{\perp})$ and $X\in\Gamma(\mathcal{D})$. Thus, we get $k=0$ or dim $\mathcal{D}^{\perp}\geq 2$. This completes our proof.

An immediate consequence of Theorem 6 as follows :

Corollary 1. *Let M be a proper CR-statistical submanifold in a holomorphic statistical space form $\overline{M}(k)$. If M is totally geodesic with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ such that $D_X LY = L\nabla^*_X Y$ for any $Y\in\Gamma(\mathcal{D}^{\perp})$ and $X\in\Gamma(\mathcal{D})$, then we have*

(A) $k=0$, *or*
(B) *dim* $\mathcal{D}^{\perp}\geq 2$.

Acknowledgment. The authors are grateful to the referee for his/her valuable comments and suggestions.

References

1. Amari, S.: Differential Geometric Methods in Statistics. Lecture Notes in Statistics, Springer, New York. 28 (1985)
2. Aydin, M. E., Mihai, A., Mihai, I.: Some Inequalities on Submanifolds in Statistical Manifolds of Constant Curvature. Filomat. 29(3) (2015).
3. Bejancu A.: CR submanifolds of a Kaehler manifold I. Proc. Am. MAth. Soc. 69, 135-142 (1978)
4. Blair, D. E., Vanhecke, L.: Umbilical submanifolds of Sasakian space forms. J. Differential Geometry. 13(2), 273-278 (1978)
5. Chen, B. Y.: Geometry of Submanifolds. Marcel Dekker, New York (1973)
6. Chen, B. Y., Ogiue K.: On totally real submanifolds. Trans. Amer. Math. Soc. 193, 257-266 (1974)
7. Furuhata, H., Hasegawa, I.: Submanifold Theory in Holomorphic Statistical Manifolds. In: Dragomir, S., Shahid, M. H., Falleh R. Al-Solamy (eds.) Geometry of Cauchy-Riemann Submanifolds, pp 179–215. Springer (2016)
8. Kurose, T.: Conformal-projective geometry of statistical manifolds. Interdisciplinary Information Sciences. 8(1), 89-100 (2002)
9. Kurose, T.: Geometry of statistical manifolds. Mathematics in the 21st Century, Nihon-hyouron-sha. 34-43 (2004) (in Japanese)
10. Nguiffo Boyom, M.: Foliations-Webs-Hessian Geometry-Information Geometry-Entropy and Cohomology. Entropy. 18, 433 (2016). doi:10.3390/e18120433
11. Nguiffo Boyom, M., Jamali, M., Shahid, M. H.: Mulitply CR-Warped product statistical submanifolds of a holomorphic statistical space form. Geometric Science of Information. 257-268 (2015)
12. Nomizu, K.: Generalized central spheres and the notion of spheres in Riemannian geometry. Tohoku Math. J. 25, 129-137 (1973)
13. Nomizu, K., Yano, K.: On circles and spheres in Riemannian geometry. Math. Ann. 210(2) 163-170, (1974)
14. Siddiqui, A. N., Al-Solamy, F. R., and Shahid, M. H.: On CR-statistical submanifolds of holomorphic statistical manifolds. Communicated
15. Yano, K., Kon, M.: Structures on manifolds. Worlds Scientific, Singapore (1984)