TORIC VARIETIES ASSOCIATED TO ROOT SYSTEMS

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Abstract. Given a reductive group G and a parabolic subgroup $P \subset G$, with maximal torus T, we consider the closure X of a generic T-orbit (in the sense of Dabrowski's work), and determine when X is a Gorenstein-Fano variety. We establish a correspondence between the family of fans associated to a closure of a generic orbit and the family *fans associated to a root system*; these fans are characterized as those stable by the symmetries with respect to a facet. This correspondence is not bijective, but allows to determine which complete fans associated to a root system correspond to a Gorenstein-Fano variety. Lattice-regular convex polytopes arise as the polytopes associated to a sub-family of these fans — the *lattice-regular complete* fans.

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1. Introduction

Let R be a root system and Λ_P the associated weight lattice. In [VoKl85], V.E. Voskresenskiĭ and A.A. Klyachko considered a family of strictly convex complete fans in $\Lambda_{\mathbb{R}} = \mathbb{R}\otimes_{\mathbb{Z}}\Lambda$, constructed by "gluing together" selected adjacent Weyl chambers, in such a way that the associated toric varieties are smooth; they classified all the Fano varieties of this family. Later, R. Dabrowski considered in [Dab96] the geometry of the closure of a "generic T–orbit in G/P", where G is a reductive group over the complex numbers, $T \subset P \subset G$ a maximal torus and a parabolic subgroup associated to an anti-dominant weight λ respectively. The combinatorial data associated to these complete toric varieties is given, as in Voskresenskiĭ and Klyachko's work, by a fan in $(\Lambda_P)_{\mathbb{R}}$, such that a cone of maximal dimension is the union of some translates of the Weyl chambers.

On the other hand, given a lattice Λ, O. Karpenkov classified in [Kar06] the *lattice–regular convex polytopes* in $\Lambda_{\mathbb{R}}$ — that is, the convex polytopes generated by elements of Λ, that are regular with respect to the group of affine transformations preserving Λ. Later, in [MR09] the first author and N. Ressayre shown how to canonically associate a root system to any such regular polytope.

In this paper we establish the notion of a *complete toric variety (or fan) associated to a root system*: a complete fan Σ is associated to a root system if the symmetries with respect to any facet — that is a co-dimension one cone of the fan — is an automorphism of Σ. It turns out that in this case the primitive elements of the normal to the support hyperplane of the facets of Σ configure a root system for the ambient space. Moreover, there exists a correspondence between this family and the family of closure of generic T-orbits in G/P — *generic closures* from now on. This correspondence is not bijective, but allows us to completely determine which complete toric varieties associated to a root system are $\mathbb{Q}$–Gorenstein-Fano — this is done establishing which generic closures are $\mathbb{Q}$–Gorenstein-Fano. The key ingredient for this characterization is a description of the combinatorics of the polytope generated by the set of primitive elements of the fan, in terms of the root system and the associated fundamental weights. Lattice-regular polytopes are in duality with a sub-family of the family of fans associated to root systems; namely, the fans Σ that are that a regular for the action of the automorphisms group — the *lattice-regular complete fans*, see Definition 4.12.

Explicit — and rather long — calculations allow to give a complete description of the mentioned families. The computing for exceptional type are made by using the following software: Sage [St] and the version of Gap3 [Sch97] maintained by Jean Michel — that allow us to use the package Chevie (see [GHLMP96] and [Mic2015]). In this regard, we warmly thank Cédric Bonnafé for his short, but effective introduction to Gap3.

We describe now the content of this paper.

In Section 2 we collect some well known basic facts on toric varieties and their associated fans, as well as some (also very well known) key results on root systems and their associated weight lattices.

In Section 3 we first recall Dabrowski's description of the fan associated to the closure of a generic orbit, and establish some key facts on the combinatorics of these fans (see lemmas 3.5 and 3.7 and their corollaries). These results allow us to characterize the $\mathbb{Q}$–Gorenstein-Fano generic closures in Proposition 3.14: a generic closure associated to an anti-dominant weight λ is $\mathbb{Q}$–Gorenstein-Fano if and only if the convex hull of $\mathrm{Prim}(\sigma_\lambda)$ — the primitive vectors of the cone $\sigma_\lambda = \bigcup_{W_\lambda} w\mathcal{C}$, where W is the Weyl group of G and $\mathcal{C}$ the anti-dominant Weyl chamber —, is a $(n-1)$-dimensional polytope, such that the normal of its support hyperplane is interior to the cone generated by $\{\omega_i : i \in I_\lambda\}$ (I_λ is the support of λ, see Definition 2.9).

In Section 4 we define toric varieties associated to a root system (see definitions 4.1 and 4.3, and Proposition 4.7) and establish correspondence between this family and the family of the closures of a generic T-orbit in G/P (Proposition 4.4 and Theorem 4.9) — this correspondence is not a bijection, see Remark 4.10. Lattice-regular polytopes arise as a subfamily of the fans associated to root systems that are Fano: the *lattice-regular complete fans* are those fans associated to a root system such that their associated polytope is regular (see Proposition 4.14 and Corollary 4.15).

We include as an Appendix (Section 6) the explicit calculations of the Fano generic closures. We use the previous results in order to completely classify Fano closures of generic orbits. In view of the results of Section 3, given an anti-dominant weight λ, one needs to construct $\mathrm{Prim}(\sigma_\lambda)$, the set of primitive generators of σ_λ as a sets of orbits of the form $W_\lambda \cdot (-\omega_i)$, for some fundamentals weights ω_i — these weights depend on the given λ. Then one must check if $\big\langle \mathrm{Prim}(\sigma_\lambda)\big\rangle_{\mathrm{aff}}$ is an hyperplane, and that n_λ, the interior normal associated to the the hyperplane, belongs to the interior of the cone $\mathbb{R}^+\langle \omega_i : i \in I_\lambda\rangle$.

The cases A_n, B_n, C_n and D_n are dealt by doing generic calculations; we intensively use the results of Section 3 of generic orbits, in particular corollaries 3.6 and 3.9. In order to deal with the exceptional cases F_4, E_6, E_7 and E_8 we use Gap3 functionalities in order to calculate a generating set of the cone σ_λ, and then Sage's "toric varieties" package in order to calculate the set of primitive elements of σ_λ. Once this is done, we calculate n_λ when $\dim\big\langle \mathrm{Prim}(\sigma_\lambda)\big\rangle_{\mathrm{aff}} = n-1$.

A table resuming all the geometric properties is also given if Section 5.

All the varieties we consider are defined over an algebraically closed field $\Bbbk$ of characteristic 0.

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2. Preliminaries

2.1. **Toric varieties.**

Definition 2.1. Let Λ be a lattice and $\Lambda_\mathbb{R} = \Lambda \otimes_\mathbb{Z} \mathbb{R}$ the *associated ambient space.* A *fan* Σ *in* $\Lambda_\mathbb{R}$ is a finite collection of rational polyhedral, strictly convex cones $\Sigma = \{\sigma_i : i \in I\}$, such that for every $i, j \in I$, $\sigma_i \cap \sigma_j \in \Sigma$ is a common face of σ_i and σ_j, and any face $\tau \subset \sigma_i$ belongs to Σ. The fan Σ is *complete* if $\bigcup_{i\in I} \sigma_i = \Lambda_\mathbb{R}$.

We denote by $\Sigma(r)$ the collection of r–dimensional cones in Σ.

A element a of a monoid S is primitive, if for all $b, c \in S$ such that $a = b+c$ then $b = 0$ or $c = 0$. The set of primitive elements of S will be denoted $\mathrm{Prim}(S)$. The set of *primitive elements of the fan* Σ is defined as $\mathrm{Prim}_\Lambda(\Sigma) = \bigcup_{\sigma\in\Sigma(1)} \mathrm{Prim}(\sigma\cap\Lambda)$. In other words, a lattice element $v \in \Lambda$ is primitive for Σ if v is primitive and generates a one dimensional cone of Σ. If $\sigma\in\Sigma(r)$, then $\mathrm{Prim}_\Lambda(\sigma) = \mathrm{Prim}_\Lambda(\Sigma)\cap\sigma$ is the set of *primitive elements of* σ; clearly $\sigma = \mathbb{R}^+\big\langle \mathrm{Prim}_\Lambda(\sigma)\big\rangle$ and $\mathrm{Prim}_\Lambda(\Sigma) = \cup_{\sigma\in\Sigma}\mathrm{Prim}_\Lambda(\sigma)$.

When no confusion is possible, the subscript Λ will be omitted.

We define the *baricenter* b_σ of a cone $\sigma\in\Sigma$ as $b_\sigma = \sum_{\nu\in\mathrm{Prim}(\sigma)}\nu$; clearly $b_\sigma \in (\sigma)^\circ$, the *interior* of σ.

Remark 2.2. Let T be an algebraic torus and $\mathcal{X}(T)$ its *character group*, and let $\Lambda = \mathcal{X}(T)^\vee$. It is well known that the family of fans in $\Lambda_\mathbb{R} = \mathbb{R}\otimes_\mathbb{Z}\Lambda$ is in bijection with the family of T-toric varieties. Under this correspondence, complete fans correspond to complete toric varieties. If Σ is a fan, we denote X_Σ the associated toric variety. It is well known that the T–stable Weil divisors are in bijection with the $\mathbb{Z}$–linear combinations $\sum_{\sigma\in\Sigma(1)} a_\sigma D_\sigma$, where $a_\sigma\in\mathbb{Z}$ and D_σ is the T-stable divisor associated to the cone $\sigma\in\Sigma(1)$. We refer to [CLS11] for further properties of this correspondence.

We recall now some well known properties that we will use in what follows.

Definition 2.3. Let Σ be a complete fan in $\Lambda_\mathbb{R}$. A *support function* is a function $\varphi : M\to\mathbb{R}$ that is linear in each cone $\sigma\in\Sigma$. A support function φ is *integral* (resp. *rational*) if $\varphi(\Lambda)\subset\mathbb{Z}$ (resp. $\mathbb{Q}$).

The following Lemma is a well-known result on toric varieties and their divisors (for a proof, see for example [CLS11, theorems 4.2.12 and 6.2.1]).

Lemma 2.4. *Let T be a n-dimensional torus with associated one-parameter subgroup lattice Λ and consider the following sets of equivalence classes*

(i) set of pairs (complete toric variety, ample T-stable $\mathbb{Q}$–Cartier divisor)

$$\mathcal{TD} = \left\{ (X,D) : \begin{array}{l} X \textit{ is a complete toric variety,} \\ D \textit{ is a } \mathbb{Q}\textit{–Cartier divisor} \end{array} \right\} / \sim,$$

where $(X_1,D_1)\sim(X_2,D_2)$ if and only if there exists an isomorphism of toric varieties $\psi : X_1\to X_2$, such that $D_1\sim\psi^*(D_2)$ as divisors. Recall that a Weil divisor D is $\mathbb{Q}$–Cartier if some positive integer multiple is Cartier.

(ii) set of pairs (complete fan, strictly convex rational support function)

$$\mathcal{FD} = \left\{ (\Sigma,\varphi) : \begin{array}{l} \Sigma \textit{ a complete fan,} \\ \varphi \textit{ a strictly convex rational support function} \end{array} \right\} / \sim,$$

where $(\Sigma_1,\varphi_1)\sim(\Sigma_2,\varphi_2)$ if and only if there exists an integral isomorphism $\rho : \Lambda_\mathbb{R}\to\Lambda_\mathbb{R}$, such that $\Sigma_2 = \rho(\Sigma_1)$ and $\varphi_1 = \rho^*(\varphi_2)$.

(iii) set of full dimensional convex rational lattice polytopes — that is, full dimensional polytopes P in $M^\vee = \mathbb{R}\otimes_\mathbb{Z}\Lambda^\vee$ such that an integer multiple aP, $a>0$, is a lattice polytope.

$$\mathcal{P} = \Big\{ P\subset\Lambda_\mathbb{R}^\vee : P \textit{ full dimensional convex rational lattice polytope}\Big\}/\sim,$$

where $P_1\sim P_2$ if and only if there exists an integral isomorphism $g:\Lambda^\vee_\mathbb{R}\to\Lambda^\vee_\mathbb{R}$ such that $P_2 = g(P_1)$.

Then the assignments $(\Sigma,\varphi)\mapsto(X_\Sigma,D_\varphi)$ *and* $(\Sigma,\varphi)\mapsto P_\varphi$ *induce bijections* $F:\mathcal{FD}\to\mathcal{TD}$ *and* $G:\mathcal{FD}\to\mathcal{P}$.

Under these correspondences Cartier divisors correspond to strictly convex integral support functions respectively lattice polytopes. □

Definition 2.5. Let Σ be a fan. The *set of flags of cones of* Σ is the set

$$\mathcal{F}_\Sigma=\big\{0=\tau_0\subsetneq\tau_1\subsetneq\cdots\subsetneq\tau_n:\tau_i\in\Sigma(i)\big\}.$$

Definition 2.6. A complete fan Σ is $\mathbb{Q}$–*Gorenstein-Fano* (resp. *Gorenstein-Fano*, resp. *Fano*) if the associated complete toric variety X_Σ is $\mathbb{Q}$–Gorenstein-Fano (resp. Gorenstein-Fano, resp. Fano); that is, the anti-canonical divisor $-K_{X_\Sigma}$ is an ample $\mathbb{Q}$–Cartier divisor (resp. an ample Cartier divisor, resp. X_Σ is a smooth Gorenstein-Fano variety).

Recall that if X_Σ is the toric variety associated to the complete fan Σ, then $-K_{X_\Sigma}=\sum D_{\sigma\in\Sigma(1)}D_\sigma$. By using this equality and [CLS11, Lemma 6.1.13], we deduce the following equivalences.

Lemma 2.7. *Let* Σ *be a complete fan in* $\mathbb{R}\otimes\Lambda\cong\mathbb{R}^n$. *Then the following assertions are equivalent:*

(i) Σ *is* $\mathbb{Q}$–*Gorenstein-Fano;*

(ii) the elements of $\big\{\operatorname{Conv}\big(\operatorname{Prim}(\sigma)\big):\Sigma(s)\,,\ s=1,\dots,n\big\}$ *are the proper faces of the polytope* $\operatorname{Conv}\big(\operatorname{Prim}(\Sigma)\big)$*;*

(iii) for every cone $\sigma\in\Sigma(n)$*, the polytope* $F=\operatorname{Conv}\big(\operatorname{Prim}(\sigma)\big)$ *is* $(n-1)$*-dimensional, and if* $\langle\operatorname{Prim}(\sigma)\rangle_{\mathrm{aff}}=n_\sigma^\perp$*, with* $\langle n_\sigma,v\rangle=-1$ *for* $v\in\operatorname{Prim}(\sigma)$*, then* $\langle n,w\rangle>-1$ *for every* $w\in\operatorname{Prim}(\Sigma)\setminus\operatorname{Prim}(\sigma)$*. In this case,* $\varphi_K(v)=\langle n_\sigma,v\rangle$ *if* $v\in\sigma$ *is the support function associated to the anti-canonical divisor* $-K_{X_\Sigma}$*.*

(iv) for every cone $\sigma\in\Sigma(n)$*, the polytope* $F=\operatorname{Conv}\big(\operatorname{Prim}(\sigma)\big)$ *is* $(n-1)$*-dimensional, and if* $\langle\operatorname{Prim}(\sigma)\rangle_{\mathrm{aff}}=n_\sigma^\perp$*, then* $-n_\sigma\in(\sigma)^\circ$*.*

In this case, Σ *is Gorenstein-Fano if and only if* $n_\lambda\in\Lambda$*.* □

2.2. **Root systems.**

In what follows, G is a semi-simple algebraic group and $T\subset B\subset G$ are a maximal torus and a Borel subgroup respectively. Denote by R,W the associated root system and Weyl group respectively. Let $\alpha_1,\dots,\alpha_n$ be the simple roots and $\omega_1,\dots,\omega_n$ be the fundamental weights. The *type* of G is the type of the associated root system R.

We denote by Λ_R the root lattice and by Λ_P the weight lattice. Let $N=\mathbb{R}\otimes_{\mathbb{Z}}\Lambda_P$ and $M=\mathbb{R}\otimes_{\mathbb{Z}}\Lambda_R$. Then the duality given by $\langle\omega_i,\alpha_j^\vee\rangle=\delta_{i,j}$, where $\alpha^\vee$ is the co-root associated to α_i, induces an identification of M and N. Under such identification $\Lambda_R\subset\Lambda_P$, and hyperplanes $H_i=\alpha_i^\perp$ are generated by $\{\omega_1,\dots,\omega_n\}\setminus\omega_i$. The corresponding subdivision of M is given by the Weyl chambers associated to the root system, which are simplicial rational cones. In particular, W acts by isometries, transitively and freely on the Weyl chambers. We denote by $\mathcal{D}$ the dominant Weyl chamber; $\mathcal{C}=-\mathcal{D}$ is the *anti-dominant (Weyl) chamber*.

Definition 2.8. If $\lambda\in\mathcal{D}$ is a dominant weight, let $P\subset G$ be the *parabolic subgroup associated to* λ, that is P is a parabolic subgroup containing B^- (the Borel subgroup opposite to B), such that the Weyl group of P is $W_P=W_\lambda$. Then λ can be extend to P. We denote by $V(\lambda)$ the *Weyl G–module*

$$V(\lambda)=H^0(G/P,\mathcal{L}^\lambda)=\big\{f\in \Bbbk[G] : f(xy)=\lambda^{-1}(y)f(x)\ \forall x\in G, y\in P\big\}.$$

Definition 2.9. If $\lambda=\sum_{i=1}^n a_i\omega_i\in\Lambda_P$ is a weight, we define the *support* of λ as the set $I_\lambda=\{i : a_i\neq 0\}$.

If $\mathcal{S}$ is the Dynkin diagram associated to R, let $\langle I_\lambda\rangle\subset\mathcal{S}$ generated by I_λ, that is, $\mathcal{S}$ is the full sub-graph with set of vertices I_λ. We say λ is *connected* if $\langle I_\lambda\rangle$ is an irreducible Dynkin diagram.

Remark 2.10. Let $\lambda=\sum_{i=1}^n a_i\omega_i\in\mathcal{D}$ be a dominant weight, and let $W_\lambda\subset W$ be the isotropy group of λ under the action of W on the weight lattice. Then

$$W_\lambda=W_{\sum_{i\in I_\lambda}\omega_i}=W_{\sum_{i\in I_\lambda}}-\omega_i=\big\langle s_{\alpha_i} : i\in I_\lambda^c\big\rangle=\prod_{i=1}^{\ell_\lambda}\langle\mathcal{S}_i\rangle,$$

where $\mathcal{S}_i$, $i=1,\dots,\ell_\lambda$, are the irreducible components of $\big\langle I_\lambda^c\big\rangle$. In particular, W_λ depends only on I_λ.

If $j\in I_\lambda^c$, we denote $i(j)\in\{1,\dots,\ell_\lambda\}$ the index such that $j\in\mathcal{S}_{i(j)}$; then $(W_\lambda)_{\omega_j}=\big\langle s_{\alpha_i} : i\in\mathcal{S}_{i(j)}\setminus\{j\}\big\rangle\times\prod_{i\neq i(j)}\mathcal{S}_i$.

In particular, if $a\neq 0$, then $W_\lambda\cdot(a\omega_j)=\big\langle s_{\alpha_t} : t\in S_{i(j)}\big\rangle\cdot(a\omega_j)=W_{\mathcal{S}_{i(j)}}\cdot(a\omega_j)$, where if $A\subset S$ is a sub-graph, then W_A denotes the Weyl group associated to the root system of the Levi subgroup associated to A. It follows that

$$\#W_\lambda\cdot a\omega_i=\frac{\#W_{\mathcal{S}_i}}{\#W_{\big\langle s_{\alpha_i}:i\in\mathcal{S}_{i(j)}\setminus\{j\}\big\rangle}}.$$

Observe that the same results hold if $\lambda\in C$ is an anti-dominant weight.

3. Generic orbits of G/P

From now on, we assume that G is a semi-simple algebraic group, and $T\subset B\subset G$ a maximal torus and a Borel subgroup respectively.

3.1. General results.

Definition 3.1 (see [Dab96, §1])**.** Let $\lambda\in\mathcal{C}\setminus\{0\}$ be a non trivial anti-dominant weight, and $P\supset B$ be the parabolic subgroup associated to $-\lambda$. Let $\Delta_P=\{\alpha_i : s_{\alpha_i}\in W_\lambda=W_{-\lambda}\}$, and consider $S^P\subset\Lambda_R$, the sub-lattice generated by the positive roots that are not sums of simple roots in Δ_P .

Let $\Pi_\lambda=\big\{\mu\in\Lambda_P : V(-\lambda)_\mu\neq 0\big\}$ the set of T-weights of $V(-\lambda)$ and $\mathcal{A}_\lambda$ be the list of the T-weights counted with multiplicity. A *set of Plücker coordinates* $\{f_\mu : \mu\in\mathcal{A}_\lambda\}$ is a choice of a basis of T-semi-invariants functions $f_\mu\in V(-\lambda)_\mu$.

If $x=uP\in G/P$, we consider

$$\Pi_\lambda(x):=\big\{\mu\in\Pi_\lambda : f_\mu(x)\neq 0 \text{ for some } f_\mu \text{ in the Plücker basis}\big\}.$$

It is easy to see that $\Pi_\lambda(x)$ does not depends on the choice of the Plücker coordinates. Moreover, $\lambda-w\Pi_\lambda(x)\subset S^P\subset\Lambda_R$, for every $w\in W$.

We say that the T-orbit $T\cdot x$ is *generic* if:

(i) $W\cdot\lambda\subset\Pi_\lambda(x)$

(ii) The set $\lambda - w\Pi_\lambda(x)$ generates S^P as a lattice.

We recall in the next theorem some of the properties of generic orbits shown on [Dab96], that we need for the rest of this work.

Theorem 3.2. *If $x \in G/P$ is such that all its Plücker coordinates does not vanish, then $T \cdot x$ is a generic orbit. In particular, generic orbits exist.*

Let $\sigma_\lambda = \bigcup_{w\in W_\lambda} w\mathcal{C}$. Then σ_λ is a convex convex rational cone. If $T\cdot x$ is a generic orbit, then T-orbit closure $\overline{T\cdot x} \subset G/P$ is a toric variety with associated fan Σ_λ, with cones of maximal dimension given by

$$\Sigma_\lambda(n) = \{w\sigma_\lambda : w \in W^\lambda\},$$

where $W^\lambda \subset W$ is a set-theoretical section of W/W_λ.

Moreover, $\Sigma_\lambda(n)$ is the fan associated to the polytope $-\mathcal{P}_\lambda = -\operatorname{Conv}(\Pi_\lambda)$ — that is, the fan obtained by considering the cones with vertex 0, generated by the strict faces of $-\mathcal{P}_\lambda$. □

Remark 3.3. (1) If λ is a regular anti-dominant weight, then $\sigma_\lambda = \mathcal{C} = -\mathcal{D}$.

(2) Let G be a simple group — equivalently, the associated root system is irreducible. If $\lambda \in \mathcal{C}$ is an anti-dominant weight, then $\mathcal{P}_\lambda$ is non-degenerate (i.e. of maximal dimension), and thus Σ_λ is a strictly convex fan.

(3) If G has associated root system $R = \prod_{i=1}^r R_i$, with R_i an irreducible root system, then $\Lambda_P = \prod_i \Lambda_{P_i}$, where P_i is the weight lattice associated to R_i, $\mathcal{C} = \prod_i \mathcal{C}_{R_i}$ and and $W_R \cong \prod_i W_{R_1}$.

It is clear that if $\lambda = \sum_i \lambda_i$, $\lambda_i \in \mathcal{C}_{R_i}$, is an anti-dominant weight, then $\Sigma_\lambda = \prod_i \Sigma_{\lambda_i}$. It follows that Σ_λ is a strictly convex fan if and only if $\lambda_i \neq 0$ for all $i = 1, \dots, r$.

On the other hand, if $\lambda_i = 0$ then every cone in Σ_λ contains the subspace $\langle \Lambda_{P_i}\rangle_\mathbb{R}$. It follows that that $\overline{T\cdot x}$ is a complete T/T_x-toric variety, with associated fan the projection of Σ_λ over $N/\langle \prod_{\lambda_i=0}\Lambda_{P_i}P_\lambda\rangle_\mathbb{R}$.

Therefore, we can always assume that the polytope $\mathcal{P}_\lambda$ is non–degenerate and Σ_λ a strictly convex fan. Moreover, Σ_λ is $\mathbb{Q}$–Gorenstein-Fano (resp. Gorenstein-Fano, resp. Fano) if and only if Σ_{λ_i} is so for all $i = 1, \dots, r$. Hence, we can restrict our calculations to the simple case.

Proposition 3.4. *Let $\lambda \in \mathcal{C}$ be a non trivial anti-dominant weight. Then*

(1) *The weight λ is in the interior of σ_λ.*
(2) *Let $\gamma_\lambda = \mathbb{R}^+\langle -\omega_i : i \in I_\lambda\rangle$ be the biggest face $\gamma \subset \mathcal{C}$ such that $\lambda \in (\gamma)^\circ$. Then*
$$\gamma_\lambda = \bigcap_{w\in W_\lambda} w\mathcal{C} = (\sigma_\lambda)^{W_\lambda}$$
(3) *The fan Σ_λ is stable under the action of W, and $W_\lambda = W_{\sigma_\lambda}$. In particular, $\#\Sigma(n) = \#(W/W_\lambda) = \#W/\#W_\lambda$.*

Proof. (1) In order to prove that $\lambda \in (\sigma_\lambda)^\circ$ observe that, in the notations of Theorem 3.2, $-\lambda$ is a vertex of $-\mathcal{P}_\lambda$, and hence it corresponds to an interior point of σ_λ under the duality between M and N.

(2) If $w \in W_\lambda$, then $w \cdot \gamma_\lambda$ is the maximal face of $w\mathcal{C}$ containing $w \cdot \lambda = \lambda$. It follows that $w \cdot \gamma_\lambda = \gamma_\lambda$, and $\gamma_\lambda \subset \bigcap_{w \in W_\lambda} w\mathcal{C}$. On the other hand, since the decomposition of the ambient space in Weyl chambers induces a fan — in the notations of Theorem 3.2, the fan $\Sigma_{-\sum \omega_i}$ —, it follows that $\bigcap_{w \in W_\lambda} w\mathcal{C}$ is a face of $\mathcal{C}$ containing λ. Thus, $\bigcap_{w \in W_\lambda} w\mathcal{C} \subset \gamma_\lambda$. It is clear that $\gamma_\lambda \subset (\sigma_\lambda)^{W_\lambda}$, and that $(\sigma_\lambda)^{W_\lambda} \subset \bigcap_{w \in W_\lambda} w\mathcal{C}$.
(3) By construction, the Weyl group W acts transitively on $\Sigma(n)$; therefore Σ_λ is stable under the action of W. In order to prove the rest of the assertions, we can assume that $\lambda = -\sum_{i \in I_\lambda} \omega_i$. It is clear then that $W_\lambda \subset W_{\sigma_\lambda}$. Since $\gamma_\lambda = \mathbb{R}^+\langle -\omega_i : i \in I_\lambda \rangle$ is it follows Let $w' \in W_\sigma$. Then $w'(w\mathcal{C}) \subset \sigma$ for any $w \in W_\lambda$. It follows that $w'\gamma_\lambda = \gamma_\lambda$, and thus $w'\{-\omega_i : i \in I_\lambda\} = \{-\omega_i : i \in I_\lambda\}$. Hence $w'\lambda = \lambda$.

It is clear that $\{w \in W : \lambda \in w\mathcal{C}\} \subset \bigcup_{\{w \in W : \lambda \in \mathcal{C}_w\}}$. Let $w \in W$ be such that $\lambda \in w\mathcal{C}$. Then $\lambda \in w \cdot C \cap (\sigma_\lambda)^\circ$. Since W acts transitively on the cones of $\Sigma_\lambda(n)$, it follows that $w \cdot \mathcal{C} \subset \sigma_\lambda$. □

Lemma 3.5. *Let $\lambda, \mu \in \mathcal{C}$ be anti-dominant weights. Then the following are equivalent: (i) $\sigma_\mu \subset \sigma_\lambda$; (ii) $\mu \in (\sigma_\lambda)^\circ$; (iii) $I_\lambda \subset I_\mu$; (iv) $W_\mu \subset W_\lambda$.*

Proof. It is well known that (iii) is equivalent to (iv), see Remark 2.10.

Since σ_μ is a cone of maximal dimension and that $\mu \in (\sigma_\mu)^\circ$, it follows that (i) implies (ii). On the other hand, if $\mu \in (\sigma_\lambda)^\circ$ and $w \in W_\mu$, it follows from the transitivity of the W-action (as in the proof of Proposition 3.4) that $w \cdot \mathcal{C} \subset \sigma_\lambda$.

It is clear that (iv) implies (i). Assume now that $\sigma_\mu \subset \sigma_\lambda$. If $w \in W_\mu$, then $w\mathcal{C} \subset \sigma_\lambda$ and hence $w \in W_\lambda$ — we are using here that W acts freely on the set of Weyl chambers. □

Corollary 3.6. *Let $\lambda \in \mathcal{C}$ be an anti-dominant weight and consider a face $\tau = \mathbb{R}^+\big\langle -\omega_i : i \in J \subset \{1, \dots, n\}\big\rangle \subset \mathcal{C}$. Then τ is contained in a proper face of σ_λ if and only if $J \not\supset I_\lambda$; that is, if and only if $\lambda \notin \tau$.*

Proof. By Lemma 3.5, $\tau \cap (\sigma_\lambda)^\circ \neq \emptyset$ if and only if there exists an anti-dominant weight $\mu \in \tau$ such that $I_\mu \supset I_\lambda$, that is if and only if $J \supset I_\lambda$. □

Lemma 3.7. *Let $\lambda \in \mathcal{C}$ be an anti-dominant weight and consider a face $\gamma \in \sigma_\lambda(r)$. Then there exist faces $\gamma_1, \dots, \gamma_{s(\gamma)} \subset \mathcal{C}(r)$ and elements $\omega_1, \dots, \omega_{s(\gamma)} \in W_\lambda$, $w_i \neq w_j$ if $i \neq j$, such that $\gamma = \bigcup_{i=1}^{s(\gamma)} w_i \cdot \gamma_i$.*

Let $\gamma' \in \sigma_\lambda(r)$ be another r–dimensional face, such that there exists $1 \leq j \leq s(\gamma)$ and $w' \in W_\lambda$ with $\gamma' = w'\gamma_j \bigcup_{i=1}^{s(\gamma')-1} w'_i \cdot \gamma'_i$, $w'_i \in W_\lambda$ and $\gamma'_i \in \mathcal{C}(r)$. Then $s(\gamma) = s(\gamma')$ and $\{\gamma_1, \dots, \gamma_{s(\gamma)}\} = \{\gamma'_1, \dots, \gamma'_{s(\gamma)-1}, \gamma_j\}$.

Proof. It is clear that if $\gamma \in \sigma_\lambda(r)$ is a r-face, then there exists at most only one face $\tau \in \mathcal{C}(r)$ such that $\tau \subset \gamma$. Indeed, is $\tau' \in \mathcal{C}(r)$ is another r-face, then $\dim\langle \tau \cup \tau' \rangle > r$. Thus, the first assertion follows immediately from the transitivity of the W-action on the Weyl chambers.

Let $\gamma = \bigcup_{i=1}^{s(\gamma)} w_i \cdot \gamma_i \in \sigma_\lambda(r)$ and assume that $j = 1$. Then $w \cdot \gamma \in \sigma_\lambda(r)$ for all $w \in W_\lambda$, and it follows that $s(\gamma) = s(w\gamma)$ for all $w \in W_\lambda$. In particular,

$$w' w_1^{-1} \cdot \gamma = \bigcup_{i=1}^{s(\gamma)} w' w_1^{-1} w_i \cdot \gamma_i = w'\gamma_1 \cup \bigcup \bigcup_{i=2}^{s(\gamma)} w' w_1^{-1} w_i \cdot \gamma_i \in \sigma_\lambda(r)$$

It follows that $w' w_1^{-1} \cdot \gamma = \gamma'$ and the result easily follows. □

Corollary 3.8. *Let $\lambda \in \mathcal{C}$ be an anti-dominant weight. Then:*

(1) If ω_i is a fundamental weight, then $-\omega_i \in (\sigma_\lambda)^\circ$ if and only if $\lambda = a\omega_i$, $a < 0$.

(2) There exists a subset $J_\lambda \subset \{1, \dots, n\}$, such that $\operatorname{Prim}(\sigma_\lambda) = W_\lambda \cdot J_\lambda$. If $\lambda = -\omega_i$, then $\omega_i \notin J_\lambda$.

(3) A facet $\mathcal{C}_i = \mathbb{Z}^+\langle -\omega_1, \dots, -\omega_{i-1}, -\omega_{i+1}, \dots, -\omega_n \rangle \subset \mathcal{C}$ is contained in a facet of σ_λ if and only if $i \in I_\lambda$. In particular, $\# I_\lambda$ facets of $\mathcal{C}$ are contained in a facet of σ_λ, whereas the remaining $n - \# I_\lambda$ facets of $\mathcal{C}$ contain points in $(\sigma_\lambda)^\circ$, and if $-\lambda = \omega_i$ is a fundamental weight, then

$$\bigcup_{\tau \in \sigma_\lambda(n-1)} \tau = \bigcup_{w \in W_\lambda} w\mathcal{C}_i.$$

Proof. Assertion (1) is an easy consequence of Lemma 3.5. Assertion (2) follows from Lemma 3.7 applied to the case $r = 1$. Assertion (3) is proved combining lemmas 3.5 and 3.7. □

The comprehension of $\operatorname{Prim}(\sigma_\lambda)$ given in Corollary 3.8 can be improved, by describing the affine sub-space that this set generates.

Corollary 3.9. *Let $\lambda \in \mathcal{C}$ be an anti-dominant weight, and consider J_λ as in Corollary 3.8. If $-\omega_k \in J_\lambda$, then, with the notations of Remark 2.10,*

$$\big\langle \operatorname{Prim}(\sigma_\lambda) \big\rangle_{\mathrm{aff}} = -\omega_k + \Big\langle \big(\bigcup_{j \in J_\lambda} W_\lambda \cdot (\omega_j) - \omega_j \big) \cup \{ \omega_i - \omega_j : i, j \in J_\lambda \} \Big\rangle_{\mathbb{R}} =$$

$$-\omega_k + \Big\langle \{ \alpha_i : i \notin I_\lambda \} \cup \{ \omega_i - \omega_k : i \in J_\lambda \} \Big\rangle_{\mathbb{R}}$$

Proof. Indeed, if $i, j \in J_\lambda$ and $f, g \in W_\lambda$, then

$$f \cdot (-\omega_i) - g \cdot (-\omega_j) = f \cdot (-\omega_i) - \omega_i + \omega_i - \omega_j + \omega_j - g \cdot (-\omega_j),$$

and the first equality follows. As for the second equality, let $f = s_\ell \cdots s_1 \in W_\lambda$, with $s_i \in \{ s_{\alpha_i} : i \notin I_\lambda \}$. Then $f \cdot (-\omega_i) - \omega_i \in \langle \alpha_i \rangle_{\mathbb{R}}$, and the inclusion $\subset$ follows.

Let $i \notin I_\lambda$; if $s_{\alpha_i}(\nu) = \nu$ for all $\nu \in \operatorname{Prim}(\sigma_\lambda)$, then $s_{\alpha_i} = \mathrm{Id}$; since σ_λ is of maximal dimension, this is a contradiction. It follows that there exists $\nu \in \operatorname{Prim}(\sigma_\lambda)$ such that $s_{\alpha_i}(\nu) \neq \nu$, and therefore $\alpha_i \in \big\langle \operatorname{Prim}(\sigma_\lambda) \big\rangle_{\mathrm{aff}} - \omega_k$. □

Corollary 3.10. *Let $\lambda \in \mathcal{C}$ be an anti-dominant weight. Then $I_{b_{\sigma_\lambda}} = I_\lambda$. In particular, $b_{\sigma_\lambda} = \sum_{\nu \in \operatorname{Prim}(\sigma_\lambda)} \nu \in (\gamma_\lambda)^\circ$.*

Proof. By construction, $b_{\sigma_\lambda} \in (\sigma_\lambda)^\circ$. Thus, by Lemma 3.5, $I_\lambda \subset I_{b_{\sigma_\lambda}}$. On the other hand, it follows from Corollary 3.9 that $W_\lambda \cdot b_{\sigma_\lambda} = b_{\sigma_\lambda}$. Hence, $I_{b_{\sigma_\lambda}} \subset I_\lambda$. □

Remark 3.11. Let G be a semi-simple group and $\lambda,\mu\in\mathcal{C}$ two anti-dominant weights, such that $I_\mu\subset I_\lambda$. If $\tau\in\sigma_\mu(r)$, then $(\tau\cap\sigma_\lambda)\in\sigma_\lambda(s)$, where $s\leq r$. In particular, if $\mathbb{R}^+\langle-\omega_i\rangle\in\sigma_\mu(1)$, then $\mathbb{R}^+\langle-\omega_i\rangle\in\sigma_\lambda(1)$. This well known fact on the geometry of rational cones will be useful for the description of the geometry of the closure of the generic orbits — namely, the description of $\mathrm{Prim}(\sigma_\lambda)\cap\mathcal{C}$.

3.2. **Fano closures of generic orbits.**

Definition 3.12. Let G be a semi-simple group and $\lambda\in\mathcal{C}$ an anti-dominant weight. Then $F_\lambda=\mathrm{Conv}\big(\mathrm{Prim}(\sigma_\lambda)\big)$ is a face of $-P_\lambda=\mathrm{Conv}\big(\mathrm{Prim}(\Sigma_\lambda)\big)$. We denote by n_λ the *interior normal to the face* $F_{\mathcal{C}}$; that is, n_λ is the unique element of $\Lambda_\mathbb{R}$ such that $\langle n_\lambda,\nu\rangle=-1$ for all $\nu\in\mathrm{Prim}(\sigma_\lambda)$.

Since the cones σ_λ, and in particular the generating sets $\mathrm{Prim}(\sigma_\lambda)$, are stable by the W_λ-action, one can give partial information about $\mathrm{Conv}\big(W_\lambda\cdot(-\omega_j)\big)$, for $-\omega_j\in\mathcal{C}\cap\mathrm{Prim}(\sigma_\lambda)$, in terms of λ and the baricenter b_λ.

Lemma 3.13. *Let G be a simple group and $\lambda=-\sum_{i\in I_\lambda}\omega_i\in\mathcal{C}$ an anti-dominant weight. Let $-\omega_j\in\mathrm{Prim}(\sigma_\lambda)$ be an anti-fundamental weight that is a generator of a ray of σ_λ. Then*

$$\big\langle W_\lambda\cdot(-\omega_j)\big\rangle_{\mathrm{aff}}\subset-\omega_j+\big(b_{\sigma_\lambda}^\perp\cap\lambda^\perp\cap n_\lambda^\perp\big).$$

Proof. By Corollary 3.10, $I_\lambda=I_{b_{\sigma_\lambda}}$; therefore, b_{σ_λ} is fixed by W_λ. It follows that if $w\in W_\lambda$, then $\big\langle b_{\sigma_\lambda},w\cdot(-\omega_j)+\omega_j\big\rangle=\big\langle b_{\sigma_\lambda},-\omega_j\big\rangle+\big\langle b_{\sigma_\lambda},\omega_j\big\rangle=0$. It follows that $\big\langle W_\lambda(-\omega_j)\big\rangle_{\mathrm{aff}}\subset-\omega_j+b_{\sigma_\lambda}^\perp$. The same kind of calculations show that $\big\langle W_\lambda(-\omega_j)\big\rangle_{\mathrm{aff}}\subset-\omega_j+\lambda^\perp$. □

Lemma 3.13 gives general but partial information on the polytope $F_{\mathcal{C}}$. However, the previous results allow to give a combinatorial description of Fano generic closures, as follows:

Proposition 3.14. *Let G be a simple group and $\lambda\in\mathcal{C}$ an anti-dominant weight. Then $n_\lambda=\sum_{i\in I_\lambda}a_i\omega_i$, and the fan Σ_λ is $\mathbb{Q}$–Gorenstein-Fano if and only if $n_\lambda\in(\gamma_\lambda)^\circ=\big(\big(\mathbb{R}^+\langle\omega_i:i\in I_\lambda\rangle\big)\big)^\circ$. That is, if and only if $n_\lambda=\sum_{i\in I_\lambda}a_i\omega_i$, with $a_i>0$ for all $i\in I_\lambda$.*

Proof. It follows from Corollary 3.9 that $n_\lambda=\sum_{i\in I_\lambda}a_i\omega_i$. By Lemma 2.7, Σ_λ is $\mathbb{Q}$–Gorenstein-Fano if and only if $-n_\lambda\in(\sigma_\lambda)^\circ$. This implies our result. □

4. Fans associated to root systems

4.1. **Fans associated to root systems.**

Definition 4.1. Let $\Sigma\subset\Lambda_\mathbb{R}=\mathbb{R}\otimes_\mathbb{Z}\Lambda$ be a complete fan. The *automorphisms group* of Σ is defined as

$$\mathrm{Aut}(\Sigma)=\Big\{f\in\mathrm{GL}\big(\Lambda\big):f(\sigma)\in\Sigma(r)\ \forall\sigma\in\Sigma(r)\Big\}$$

Remark 4.2. Since any $f\in\mathrm{Aut}(\Sigma)\subset\mathrm{GL}\big(\Lambda\big)$ acts by permutations in $\Sigma(1)$, it follows that $\mathrm{Aut}(\Sigma)$ acts by permutations on $\mathrm{Prim}(\Sigma)$. In particular, $\mathrm{Aut}(\Sigma)$ is a finite group. From now on, we fix an internal product on $\Lambda_\mathbb{R}$ in such a way that any automorphism of Σ is an isometry of $\Lambda_\mathbb{R}$.

Definition 4.3. Let Σ be a complete fan in $\Lambda_{\mathbb{R}}$, with $\dim \Lambda_{\mathbb{R}} = n$. We say that Σ is *associated to a root system* if for every facet $\sigma \in \Sigma(n-1)$, there exists a reflection s_σ that fixes the hyperplane $H_\sigma = \langle \sigma \rangle_{\mathbb{R}}$, that is also an element of $\operatorname{Aut}(\Sigma)$.

Since any automorphism is an isometry for the chosen internal product, it follows that if Σ is associated to a root system and $\sigma \in \Sigma(n-1)$ then the orthogonal reflection fixing H_σ is the unique reflexion fixing that hyperplane belonging to $\operatorname{Aut}(\Sigma)$. Let $\alpha_\sigma \in \Lambda$ be such that $\pm\alpha_\sigma$ are the unique primitive elements of $H_\sigma^\perp$. Let $\Phi(\Sigma) = \big\{ \pm\alpha_\sigma : \sigma \in \Sigma(n-1) \big\}$. be the set of such primitive elements. In Proposition 4.7, we will prove that $\Phi(\Sigma)$ is a root system; this fact justifies the above definition.

Proposition 4.4. *Let G be a simple group and $\lambda \in \mathcal{C}$ an anti-dominant weight. Then Σ_λ is a fan associated to a root system.*

Proof. If $\tau \in \Sigma(n-1)$ is a facet then, by Lemma 3.7, there exists $w \in W$ and a facet $\sigma \in \mathcal{C}(n-1)$ such that $\langle \tau \rangle_{\mathbb{R}} = w \cdot \langle \sigma \rangle_{\mathbb{R}}$. Now, by construction, there exists a simple root α_i such that $\langle \sigma \rangle_{\mathbb{R}} = \alpha_i^\perp$. It follows that $w s_{\alpha_i} w^{-1} \in W$ is the reflection associated to τ. Since $W \subset \operatorname{Aut}(\Sigma_\lambda)$ (see Proposition 3.4), the result follows. □

Remark 4.5. The converse of Proposition 4.4 we will be proved in Theorem 4.9. The reader should note however that the associated root system $\Phi(\Sigma_\lambda)$ is not necessarily the root system associated to G, see Example 4.6 and Remark 4.10 below.

Example 4.6 (G_2-type groups)**.** Let G be a simple group of type G_2. It is easy to see that $\Sigma_{-\omega_1}$ and $\Sigma_{-\omega_2}$ have A_2 as associated root system. Thus, the root system associated to a generic orbit is not necessarily the one associated to the original group G.

Proposition 4.7. *Let Σ a complete fan in $\Lambda_{\mathbb{R}}$ associated to a root system. Then*

(1) The set $\Phi(\Sigma)$ is a root system of $\Lambda_{\mathbb{R}}$.

(2) The weight and root lattices of $\Phi(\Sigma)$, denoted respectively by Λ_P and Λ_R, satisfy:

$$\Lambda_R \subset \Lambda \subset \Lambda_P.$$

(3) If W and $\operatorname{Aut}(\Phi(\Sigma))$ denote the Weyl and the automorphisms group of $\Phi(\Sigma)$ respectively, then

$$W \subset \operatorname{Aut}(\Sigma) \subset \operatorname{Aut}\big(\Phi(\Sigma)\big).$$

Proof. (1) It is clear that $\Phi(\Sigma)$ is finite, does not contain zero, spans $\Lambda_{\mathbb{R}}$ and that $\mathbb{Z} \cdot \alpha \cap \Phi(\Sigma) = \{\pm\alpha\}$ for every $\alpha \in \Phi(\Sigma)$. Indeed, $\Sigma(n)$ is composed of strictly convex rational cones.

By construction, $\alpha \in \Phi(\Sigma)$ if and only if $-\alpha \in \Phi(\Sigma)$, and $s_\alpha(\alpha) = -\alpha$.

If $\beta \in \Phi(\Sigma)$, then $\beta^\perp = \langle \sigma_\beta \rangle$ for some $\sigma_\beta \in \Sigma(n-1)$. Since $s_\alpha \in \operatorname{Aut}(\Sigma)$, it follows that $s_\alpha(\sigma_\beta) \in \Sigma(n-1)$, with $s_\alpha(\beta)^\perp = \langle s_\alpha(\sigma_\beta) \rangle_{\mathbb{R}}$. Therefore, $s_\alpha\big(\Phi(\Sigma)\big) = \Phi(\Sigma)$.

It is clear that $s_\alpha(\beta) - \beta$ is an element of Λ proportional to α. Since α has been chosen primitive, it follows that $s_\alpha(\beta) - \beta$ is an entire multiple of α. This ends the proof of Assertion (1).

In order to prove Assertion (2) it suffices to prove that $\Lambda \subset \Lambda_P$, the other inclusion being obvious. Let $\alpha \in \Phi(\Sigma)$ and $\lambda \in \Lambda$. Then $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \in \Lambda$, where $\alpha^\vee$ is the co-root associated to α. Since α is primitive on Λ, we deduce that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$.

Finally, the inclusions $W \subset \operatorname{Aut}(\Sigma) \subset \operatorname{Aut}\big(\Phi(\Sigma)\big)$ are a direct consequence of the proof of Assertion (1). □

Definition 4.8. The root system $\Phi(\Sigma)$ is called the *root system associated to* Σ.

In Proposition 4.4 we showed that any fan associated to a generic orbit is associated to a root system. The following theorem shows that the converse is also true.

Theorem 4.9. *Let Σ be a complete fan in $\Lambda_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} \Lambda \cong \mathbb{R}^n$, associated to a root system. Then*

(1) If $\sigma \in \Sigma(n)$, then σ is an union of Weyl chambers. If σ is such that $\mathcal{C} \subset \sigma$, then there exists an anti-dominant weight $\lambda = \sum_{i \in I_\lambda} \omega_i$ such that $\sigma = \sigma_\lambda$ and $\Sigma = \Sigma_\lambda$.

In particular, the Weyl group W acts transitively on the set of cones of maximal dimension $\Sigma(n)$, and $\#\Sigma(n) = \#(W/W_\lambda) = \#W/\#W_\lambda$.

(2) If $\lambda = \sum_{i \in I_\lambda} \omega_i$, then $\operatorname{Aut}\big(\Phi(\Sigma)\big)_{\sigma_\lambda} = \operatorname{Aut}\big(\Phi(\Sigma)\big)_\lambda$ and

$$\operatorname{Aut}(\Sigma) = W \cdot \big(\operatorname{Aut}\big(\Phi(\Sigma)\big)_{\sigma_\lambda} \cap \operatorname{Aut}(\Sigma)\big) = W \cdot \big(\operatorname{Aut}\big(\Phi(\Sigma)\big)_\lambda \cap \operatorname{Aut}(\Sigma)\big).$$

Proof. (1) Let $\sigma \in \Sigma(n)$; by definition of $\Phi(\Sigma)$, there exist $\ell \geq n$ and $\beta_i \in \Phi(\Sigma)$, $i = 1, \dots, \ell$, such that

$$\sigma = \bigcap_{s=1}^{\ell} \Big\{ v \in \Lambda_\mathbb{R} : \langle v, \beta_i \rangle \geq 0 \Big\}.$$

Let $\mathcal{E}$ be a Weyl chamber such that $(\mathcal{E})^\circ \cap (\sigma)^\circ \neq \emptyset$, and consider a choice of positive roots $\Phi(\Sigma)^+$ such that $\mathcal{E} = \bigcap_{\alpha \in \Phi(\Sigma)^+} \Big\{ v \in \Lambda_\mathbb{R} : \langle v, \alpha \rangle \geq 0 \Big\}$. If $\beta_i \in \Phi(\Sigma)^-$ for some i, then $v \in (\mathcal{E})^\circ \cap (\sigma)^\circ$ is such that $\langle v, \beta_i \rangle \geq 0$ and $\langle v, -\beta_i \rangle > 0$; this is a contradiction. It follows that $\mathcal{E} \subset \sigma$.

Assume now that $\mathcal{C} \subset \sigma$ and let $\lambda = \sum_{\nu \in \operatorname{Prim}(\sigma)} \nu$. We affirm that $\lambda \in \mathcal{C}$ and that $\sigma = \sigma_\lambda$; then Assertion (1) follows from Theorem 3.2.

We first prove that $W_\lambda = W_\sigma$. If $w \in W_\lambda$ then $\lambda = w \cdot \lambda \in (\sigma)^\circ \cap (w \cdot \sigma)^\circ$, and it follows from Proposition 4.7 that $w \cdot \sigma = \sigma$. On the other hand, if $w \in W_\sigma$ then $w \cdot \operatorname{Prim}(\sigma) = \operatorname{Prim}(\sigma)$; therefore, $w \cdot \lambda = w \cdot \sum_{\nu \in \operatorname{Prim}(\sigma)} \nu = \sum_{\nu \in \operatorname{Prim}(\sigma)} \nu = \lambda$.

Let $w' \in W$ be such that $w' \cdot \lambda \in \mathcal{C}$. Then $w' \cdot \lambda \in (w' \cdot \sigma)^\circ \cap \mathcal{C} \subset (w' \cdot \sigma)^\circ \cap \sigma$. Hence, $w' \cdot \sigma = \sigma$, and it follows that $w' \in W_\sigma = W_\lambda$. Therefore, $\lambda \in \mathcal{C}$.

Next, observe that we have proved in particular that $\sigma_\lambda \subset \sigma$. If $\sigma_\lambda \subsetneq \sigma$, then there exists a weight $\mu \in (\sigma \setminus \sigma_\lambda) \cap \Lambda_P$. Let $w' \in W$ be such that $w' \cdot \mu \in \mathcal{C}$. Applying the same reasoning as before, since W acts on $\Sigma(n)$, it follows that $w' \cdot \sigma = \sigma$ and thus $w' \in W_\sigma = W_\lambda$ is such that ${w'}^{-1} \cdot \mathcal{C} \not\subset \sigma_\lambda = \bigcup_{w \in W_\lambda} w \cdot \mathcal{C}$; this is a contradiction.

(2) If $f \in \operatorname{Aut}\big(\Phi(\Sigma)\big)$, then f is an isometry (f respects the angles and lengths of the simple roots) such that $f(\Lambda_R) = \Lambda_R$ and therefore $f(\Lambda_P) = \Lambda_P$. Moreover, f stabilizes the set of Weyl chambers.

if $f \in \operatorname{Aut}\big(\Phi(\Sigma)\big)_\lambda$ and $\mathcal{E}$ is a Weyl chamber contained in σ_λ, then $\lambda = f(\lambda) \in f(\mathcal{E})$. It follows that $f(\mathcal{E}) \subset \sigma_\lambda$; that is, $f \in \operatorname{Aut}\big(\Phi(\Sigma)\big)_{\sigma_\lambda}$.

If $g \in \operatorname{Aut}\big(\Phi(\Sigma)\big)_{\sigma_\lambda}$, consider as usual $\gamma_\lambda = \mathbb{R}^+\big\langle \omega_i : i \in I_\lambda \big\rangle = \bigcap_{w \in W_\lambda} w \cdot \mathcal{C}$. Since $g(\sigma_\lambda) = \sigma_\lambda$, it follows that $g\big(\{w \cdot \mathcal{C} : w \in W_\lambda\}\big) = \{w \cdot \mathcal{C} : w \in W_\lambda\}$. Thus, $g(\gamma_\lambda) = \gamma_\lambda$. Since g is an automorphism, it follows that $g\big(\operatorname{Prim}_{\Lambda_P}(\sigma_\lambda)\big) = \operatorname{Prim}_{\Lambda_P}(\sigma_\lambda)$, and we deduce that $g\big(\sum_{i \in I_\lambda} \omega_i\big) = \sum_{i \in I_\lambda} \omega_i$. Therefore, $g \in \operatorname{Aut}\big(\Phi(\Sigma)\big)_\lambda$.

Recall that, by Proposition 4.7, $W \subset \operatorname{Aut}(\Sigma) \subset \operatorname{Aut}\big(\Phi(\Sigma)\big)$. Moreover, $W \subset \operatorname{Aut}\big(\Phi(\Sigma)\big)$ is a normal subgroup that acts transitively on $\Sigma(n)$. Thus, $\operatorname{Aut}(\Sigma) \subset W \cdot \operatorname{Aut}\big(\Phi(\Sigma)\big)_{\sigma_\lambda}$.

Let $f \in \operatorname{Aut}(\Sigma)$; then $f(\mathcal{C}) = w \cdot \mathcal{C}$ for some $w \in W$. Therefore, $w^{-1} f \in \operatorname{Aut}\big(\Phi(\Sigma)\big)_{\sigma_\lambda}$. Hence, $\operatorname{Aut}(\Sigma) = W \cdot \big(\operatorname{Aut}\big(\Phi(\Sigma)\big)_\lambda \cap \operatorname{Aut}(\Sigma)\big)$ and the result follows. □

Remark 4.10. Observe that even if Theorem 4.9 provides a converse for Proposition 4.4, these results do *not* establish a bijection between closure of generic orbits in G/P and fans associated to root systems. Indeed, as noted in Remark 4.5, the root system associated to a closure of a generic orbit in G/P is not necessarily the root system associated to G.

In particular, Example 4.6, shows that if we consider G of type G_2 and $\lambda = -\omega_1$, then the fan $\Sigma_{-\omega_1}$ is isomorphic to the fan associated to the pair (G a group of type A_2, $\lambda = -\sum \omega_i$) — the subdivision in Weyl chambers for A_2.

On the other hand; $\Sigma_{-\omega_2}$ is also associated to the root system A_2, but $\Sigma_{-\omega_1} \ncong \Sigma_{-\omega_2}$ as fans. Indeed, $\operatorname{Prim}(\sigma_{-\omega_2})$ is a basis of the lattice Λ — in other words $X_{\Sigma_{-\omega_2}}$ is a smooth toric variety, whereas $\operatorname{Prim}(\sigma_{-\omega_1})$ is *not* a basis of the lattice Λ, since $-\omega_1 \notin \langle -\omega_1, -\omega_2 \rangle_{\mathbb{Z}}$.

Definition 4.11. Let G be a simple group and $\lambda \in \mathcal{C}$ a non trivial anti-dominant weight. The *minimal pair associated to* Σ_λ is the a pair (R', μ), where R' is the root system associated to the fan Σ_λ (as in Proposition 4.7) and μ is the anti-dominant weight given by Theorem 4.9 (for the fan Σ_λ in $(\Lambda_P)_{\mathbb{R}}$); we will describe R' by its type. It should be noted that the lattice of weights associated to the root system R of G, is a sub–lattice of $\Lambda_{P'}$, the lattice of weights associated to R'.

Note that the minimal pair is uniquely determined by (G, λ).

4.2. **Lattice-regular complete fans.**

Definition 4.12. Let Σ be a complete fan in $\Lambda_{\mathbb{R}}$. We say that Σ is *lattice-regular complete* if $\operatorname{Aut}(\Sigma)$ acts transitively on the flags of cones of Σ. A *lattice-regular toric variety* is a complete toric variety which associated fan is lattice-regular.

Example 4.13. Let G be a simple group of type A_3. Then it is easy to see that $\Sigma_{-\omega_i}$ is lattice-regular; see also Theorem 4.16.

Proposition 4.14. *Let Σ be a lattice-regular complete fan in $\Lambda_{\mathbb{R}}$. Then Σ is associated to a root system.*

Proof. Consider an inner product on $\Lambda_{\mathbb{R}}$ such that $\operatorname{Aut}(\Sigma)$ is composed of isometries. We need to prove that if $\tau \in \Sigma(n-1)$, then s_τ, the reflection associated to τ, is an element of $\operatorname{Aut}(\Sigma)$.

Let $\sigma_1, \sigma_2 \in \Sigma(n)$ be such that $\tau = \sigma_1 \cap \sigma_2$, and consider the two flags of the form F_1) $\{0\} \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_{n-1} = \tau \subsetneq \sigma_1$ and F_2) $\{0\} \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_{n-1} = \tau \subsetneq \sigma_2$.

Then there exists $f \in \operatorname{Aut}(\Sigma)$ such that $f(F_1) = F_2$. We affirm that $f\big|_{\langle\tau\rangle_{\mathbb{R}}} = \operatorname{Id}\big|_{\langle\tau\rangle_{\mathbb{R}}}$, and thus, since f is an isometry, f is the reflection by $\langle\tau\rangle_{\mathbb{R}}$.

We will prove by induction that if $f \in \operatorname{Aut}(\Sigma)$ is such that f stabilizes a partial flag $\{0\} \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_r$, $\tau_i = \Sigma(i)$, $i = 1, \dots, r \leq n$, then $f\big|_{\langle\tau_r\rangle_{\mathbb{R}}} = \operatorname{Id}\big|_{\langle\tau_r\rangle_{\mathbb{R}}}$; the assertion will then follow.

Let ν in $\operatorname{Prim}(\Sigma)$ be such that $f\big(\mathbb{R}^+\langle\nu\rangle\big) = \mathbb{R}^+\langle\nu\rangle$. Since ν is primitive, it follows that $f(\nu) = \nu$. Assume now that we have proved the assertion for any partial flag of the form $\{0\} \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_r$, $\tau_i \in \Sigma(i)$, and let $\{0\} \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_{r+1}$, $\sigma_i \in \Sigma(i)$, be a partial flag stabilized by f. Then $f\big|_{\langle\sigma_r\rangle_{\mathbb{R}}} = \operatorname{Id}\big|_{\langle\sigma_r\rangle_{\mathbb{R}}}$. If we prove that $f(\nu) = \nu$ for some $v\nu \in \operatorname{Prim}(\sigma_{r+1}) \setminus \operatorname{Prim}(\sigma_r)$ we are done. Since σ_{r-1} is a $(r-1)$-dimensional face of σ_{r+1}, it follows that there exists an unique r-dimensional face τ of σ_{r+1} such that $\sigma_{r-1} = \sigma_r \cap \tau$. Since $f(\tau)$ is a r-dimensional face of $f(\sigma_{r+1}) = \sigma_{r+1}$ and that $f(\sigma_r) = \sigma_r$, it follows that $f(\tau) = \tau$. Then f stabilizes the partial flag $\{0\} \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_{r-1}, \tau$ and the result follows by induction. □

As an easy consequence of Proposition4.14, we have the following result.

Corollary 4.15. *Let Σ be a lattice-regular complete fan and let $\operatorname{Prim}(\Sigma)$ the set of generating primitive vectors. Then $P(\Sigma)$, the convex hull of $\operatorname{Prim}(\Sigma)$ is a regular polytope, with $\operatorname{Aut}\big(P(\Sigma)\big) \cong \operatorname{Aut}(\Sigma)$.*

Conversely, let Q be a centered lattice-regular convex polytope, and let Σ be the complete fan obtained by considering the cones from 0 to any face F of Q. Then Σ is a lattice-regular complete fan.

Proof. Indeed, it follows from Proposition 4.14 that $P(\Sigma) = \big\{u \in N : \varphi(u) \geq -a\big\}$ is a convex polytope, with integer vertex. Let $f : N \to N$ be such that $f \in \operatorname{GL}\big(\mathcal{X}(T)\big)$. Then, by construction, $f \in \operatorname{Aut}\big(P(\Sigma)\big)$ if and only if $f \in \operatorname{Aut}(\Sigma)$.

The converse is proved in the same way. □

Theorem 4.16. *Let $F : \mathcal{FD} \to \mathcal{TD}$ and $G : \mathcal{FD} \to \mathcal{P}$ be the canonical bijections given in Lemma 2.4. Then G induces a bijection between the (class of the) pairs (Σ, φ_Σ), where Σ is a lattice-regular complete fan and φ_Σ is as in Definition 2.3, and the (class of the) lattice-regular polytopes. In particular, $G \circ F^{-1}$ induces a bijection between the lattice-regular toric varieties and the lattice-regular polytopes, where we associate to each lattice-regular variety the first multiple of the anti-canonical divisor that is a Cartier divisor.*

Moreover, if P the reduced, centered, lattice-regular polytope corresponding to (Σ, φ_Σ), then the duality $\operatorname{GL}(M) \to \operatorname{GL}(N)$, $f \mapsto f^*$, $f^*(u)(v) = u\big(f(v)\big)$ induces an isomorphism between the abstract groups $\operatorname{Aut}(\Sigma)$ and $\operatorname{Aut}(P)$.

Proof. Let Σ be a lattice-regular complete fan. Then the convex hull $\langle\operatorname{Prim}(\Sigma)\rangle_{conv} = \big\{v \in N : \varphi_\Sigma(v) \geq -a\big\}$ is a regular polytope. By construction, $P = G(\Sigma, \varphi_\Sigma) = \operatorname{Prim}(\Sigma)^\vee$, where the notation $\cdot^\vee$ stands for the polar dual of a polytope. It easily follows that P is a regular polytope (see for example [MR09, §3]). Furthermore, observe that by construction, and in view of Corollary 4.15, $f \in \operatorname{Aut}(\Sigma)$ if and only if $f^* \in \operatorname{Aut}(P)$.

Let $P \subset N$ be a centered, reduced, regular polytope and consider $(\Sigma, \varphi) = G^{-1}(P)$. If $\operatorname{Prim}(\Sigma) = \{v_1, \dots, v_\ell\}$, then there exist $a_1, \dots, a_\ell > 0$ such that

$$P = \{u \in N : \langle v_i, u\rangle \geq a_i\}.$$

In particular, $\operatorname{Prim}(\Sigma) = \{v_1, \dots, v_\ell\}$, and $\{v_{i_1}, \dots, v_{i_s}\}$ are the primitive generators of a cone $\sigma \in \Sigma(r)$, if and only if $P \cap \bigcap_{j=1}^r \{v_i(u) = -a_i\}$ is a $(n-r)$–dimensional face of P.

If $f \in \operatorname{Aut}(P)$, then it is clear that $f^* \in \operatorname{Aut}(\Sigma)$. The regularity of P implies then that Σ is a lattice-regular complete fan.

We affirm that $a_1 = \cdots = a_\ell = a$. It follows that

$$P^\vee = \bigcup_{i=1}^{\ell} \{v \in \Lambda\mathbb{R} : \varphi(v) \geq -a\},$$

and hence a is the smallest positive integer number such that $-aK$ is a Cartier divisor, with support function φ, and P as associated polytope.

In order to prove the assertion, let $i \neq j \in \{1, \dots, \ell\}$ and consider the corresponding facets $F_i = P \cap \{v_i(u) = -a_i\}$ and $F_j := \{v_j(u) = -a_j\}$. Then there exists $f \in \operatorname{Aut}(P)$ such that $f(F_i) = F_j$. It follows that

$$P = f(P) = \{u \in N : \langle v_i, f^{-1}(u)\rangle \geq a_i\} = \{u \in N : \langle f^*(v_i), u\rangle \geq a_i\}.$$

Since $f \in \operatorname{Aut}(\Sigma)$, f permutes all the facets of P, and $f^* \in \operatorname{Aut}(\Sigma)$ permutes the primitive generators accordingly. It follows that $f^*(v_i) = v_j$, and thus $a_i = a_j$. □

In view of the results proved in [MR09] for lattice-regular polytopes, Proposition 4.19 below and its corollaries can be obtained as an easy consequence of Theorem 4.16. We include direct proofs of these results, in order to explicitly show how the strategies developed in [MR09] can be applied in this context.

Definition 4.17. Let Σ be a fan and $\sigma \in \Sigma(r)$. The *star of base* σ, denoted by $\mathcal{S}(\sigma)$, is the fan composed by all the cones $\tau \in \Sigma$ such that $\sigma \subset \tau$.

Remark 4.18. It is well known that if $\mathcal{O}_\sigma = T \cdot x \subset X_\Sigma$ is an $(n-r)$-dimensional orbit corresponding to σ, and $\pi : M \to M/\langle\sigma\rangle_\mathbb{R}$, then $\pi(\mathcal{S}(\sigma))$ is the fan associated to the toric T/T_x-variety $X = \overline{\mathcal{O}_\sigma}$. Under this projection, $\sigma_Y(s) = \pi(\mathcal{S}(\sigma)(r+s))$.

Proposition 4.19. *Let Σ be a lattice-regular complete fan and consider $\sigma \in \Sigma(1)$ and $\mathcal{S}(\sigma)$, the star of base σ (see Definition 4.17). Then $\pi_\sigma(\mathcal{S}(\sigma)) \subset \sigma^\perp$ is a lattice-regular complete fan. Moreover, $\Phi(\pi_\sigma(\mathcal{S}(\sigma))) = \Phi(\Sigma) \cap \sigma^\perp$; this is a Levi subsystem of $\Phi(\Sigma)$.*

Proof. It follows from Remark 4.18 that $\pi_\sigma(\mathcal{S}(\sigma))$ is a complete fan. Let $f \in \operatorname{Aut}(\Sigma)$ be such that $f|_\sigma = \operatorname{Id}|_\sigma$. Since $\operatorname{Aut}(\Sigma) \subset \operatorname{O}(\Lambda_\mathbb{R}) \cap \operatorname{GL}(\Lambda)$, it follows that $f(\Lambda \cap \sigma^\perp) = \Lambda \cap \sigma^\perp$ and $f \circ \pi_\sigma = \pi_\sigma \circ f$. In particular $f(\sigma^\perp) = \sigma^\perp$ and $f(\pi_\sigma(\tau)) = \pi_\sigma(f(\tau))$. In other words, $f|_{\sigma^\perp} \in \operatorname{Aut}(\pi_\sigma(\mathcal{S}(\sigma)))$. Moreover, since $\operatorname{Aut}(\Sigma)$ acts transitively on the flags containing σ, it follows that $\pi_\sigma(\mathcal{S}(\sigma))$ is a lattice-regular complete fan.

By Remark 4.18 again, it follows that $\Phi(\pi_\sigma(\mathcal{S}(\sigma))) \subset \Phi(\Sigma) \cap \sigma^\perp$. Let $\alpha \in \Phi(\Sigma) \cap \sigma^\perp$, we need to prove that there exists $\tau \in \mathcal{S}(\sigma)(n-1)$ such that $\alpha \in \tau^\perp$. Let H_α be the hyperplane fixed by the reflection s_α. Then $\sigma \subset H_\alpha$, and $H_\alpha \cap \sigma^\perp$ is an hyperplane of $\sigma^\perp$. In particular, $\tau\big(\mathcal{S}(\sigma)\big) = \mathcal{S}(\sigma)$. Since $\pi_\sigma\big(\mathcal{S}(\sigma)\big)$ is a complete strictly convex polyhedral fan, it follows that there exists $\gamma \in \pi_\sigma\big(\mathcal{S}(\sigma)\big)(n-2)$ such that if β is a root associated to γ, then $\beta \not\subset H_\alpha \cap \sigma^\perp$. If β is co-linear to α, we are done. Assume that β is not co-linear to α and let $\tau \in \mathcal{S}(\sigma)(n-1)$ be such that $\pi_\sigma(\tau) = \gamma$ – the cone τ exists by construction. Then β is a root associated to τ, and it follows that $\Phi(\Sigma) \cap \operatorname{Vect}(\alpha, \beta)$ is a root system of type A_2. Changing eventually α by $-\alpha$ we may assume that $\alpha + \beta$ is a root. Since $\sigma \subset (\alpha^\perp \cap \beta^\perp)$, it follows that $s_{\alpha+\beta}(\tau) \in \mathcal{S}(\sigma)$. It suffices now to observe that $s_{\alpha+\beta}(\beta)$ is a root associated to $\pi_\sigma\big(s_{\alpha+\beta}(\tau)\big)$, and that this root is parallel to α. □

Corollary 4.20. *Let X be a lattice-regular toric variety, and $\mathcal{O} \subset X$ an orbit. Then $\overline{\mathcal{O}}$ is a lattice-regular toric variety.*

Proof. We prove this result by nœtherian induction. Let Σ be the complete lattice-regular complete fan associated X. Recall that if $\mathcal{O}$ is an orbit of co-dimension 1, then $\overline{\mathcal{O}}$ is a toric variety. Moreover, if $\sigma \in \Sigma(1)$ is the cone associated to $\mathcal{O}$, then, with the notations of Proposition 4.19, it follows that $\pi_\sigma\big(\mathcal{S}(\sigma)\big)$ is the fan associated to $\mathcal{O}$, see for example [CLS11, §3.2] for more details. The result is thus a straightforward consequence of Proposition 4.19. □

5. Description of Fano toric varieties associated to root systems

The following table describes all the pairs (root system type, anti-dominant weight) such that the closure of the associated generic orbit is Fano. The last column indicates the minimal pair associated to the generic orbit. We refer to Section 6 for the explicit calculations that led to this description.

The first column describes the generic orbit by stating the simple group G and the corresponding anti-dominant weight λ. Since W_G acts transitively on $\Sigma(n)$ and that $(W_G)_\lambda$ acts transitively on $\operatorname{Prim}(\sigma_\lambda)$, it suffices to explicitly describe $\operatorname{Prim}(\sigma_\lambda) \cap \{-\omega_1, \dots, -\omega_n\}$ in order to completely describe Σ_λ (see Corollary 3.8); this is done in the second column. The third column describes the geometry of Σ_λ. The fourth column shows the minimal pair (P, μ) determined by (G, λ); we also identify the lattice Λ in terms of Λ_R and Λ_P (see Proposition 4.7). Finally, the fifth column collects the information on the action of $\operatorname{Aut}(\Sigma_\lambda)$ on the (partial flags): there it is indicated if the fan Σ_λ is lattice-regular or not.

Observe that *a fortiori*, we obtain the following characterization of Fano closures of generic orbits.

Corollary 5.1. *Let $\lambda = \sum_{i \in I_\lambda} -\omega_i$ be a an anti-dominant weight. Then Σ_λ is Fano if and only if $n_\lambda = -\lambda$.*

$(\text{type}(G), \lambda)$	$\text{Prim}(\sigma_\lambda) \cap \{-\omega_1, \ldots, -\omega_n\}$	Geometry	minimal pair	lattice-regular
$(A_n, -\omega_1)$, $n \geq 1$	$\{-\omega_n\}$	Smooth, Fano	same pair, $\Lambda = \Lambda_P$	
$(A_n, -\omega_n)$	$\{-\omega_1\}$	Smooth, Fano	same pair, $\Lambda = \Lambda_P$	
$(A_n, -\omega_1 - \omega_n)$	$\{-\omega_1, \ldots, -\omega_n\}$	Smooth, Fano	same pair, $\Lambda = \Lambda_P$	
$(A_{2s-1}, -\omega_s)$, $s \geq 2$	$\{-\omega_1, -\omega_n\}$	$\mathbb{Q}$–Gorenstein Fano	same pair $\Lambda = \Lambda_P$	
$(A_{2s}, -\omega_s - \omega_{s+1})$	$\{-\omega_1, -\omega_n\}$	Smooth, Fano	same pair, $\Lambda = \Lambda_P$	
$(B_2, -\omega_1)$	$\{-\omega_2\}$	Smooth, Fano	$\big(A_1 \times A_1, (-\omega_1, -\omega_1)\big)$, $\Lambda = \Lambda_{P'}$	yes
$(B_2, -\omega_2)$	$\{-\omega_1\}$	Smooth, Fano	$\big(A_1 \times A_1, (-\omega_1, -\omega_1)\big)$, $\Lambda = \Lambda_{A_1^2}$	yes
$(B_n, -\omega_1)$, $n \geq 3$	$\{-\omega_n\}$	Gorenstein Fano	$(D_n, -\omega_1)$, $\Lambda = \Lambda_{P'}$	
$(B_n, -\omega_2)$, $n \geq 3$	$\{-\omega_1, -\omega_n\}$	Gorenstein Fano	same pair, $\Lambda = \Lambda_P$	
$(B_n, -\omega_n)$, $n \geq 2$	$\{-\omega_1\}$	Gorenstein Fano	$\big(A_1^n, (-\omega_1, \ldots, -\omega_1)\big)$, $\Lambda_{A_1^n} \subsetneq \Lambda = \big\langle \varepsilon_1, \ldots, \varepsilon_n, \frac{1}{2}\sum \varepsilon_i \big\rangle_{\mathbb{Z}} \subsetneq \Lambda_{P'}$	
$(C_2, -\omega_1)$	$\{-\omega_1\}$	Smooth, Fano	$(A_2, -\omega_1)$, $\Lambda = \Lambda_R$	yes
$(C_2, -\omega_2)$	$\{-\omega_1\}$	Smooth, Fano	$(A_2, -\omega_1)$, $\Lambda = \Lambda_P$	yes
$(C_n, -\omega_1)$	$\{-\omega_n\}$	Gorenstein Fano	$(D_n, -\omega_1)$	
$(C_n, -\omega_2)$, $n \geq 3$	$\{-\omega_1, -\omega_n\}$	Gorenstein Fano	same pair, $\Lambda = \Lambda_P$	
$(C_n, -\omega_n)$, $n \geq 3$	$\{-\omega_1\}$	Gorenstein Fano	$\big(A_1^n, (-\omega_1, \ldots, -\omega_1)\big)$ $\Lambda_{A_1^n} \subsetneq \Lambda = \big\langle \varepsilon_1, \ldots, \varepsilon_n, \frac{1}{2}\sum \varepsilon_i \big\rangle_{\mathbb{Z}} \subsetneq \Lambda_{P'}$	
$(D_n, -\omega_1)$, $n \geq 4$	$\{-\omega_{n-1}, -\omega_n\}$	Gorenstein Fano	same pair, $\Lambda = \Lambda_P$	
$(D_n, -\omega_2)$, $n \geq 4$	$\{-\omega_1, -\omega_{n-1}, -\omega_n\}$	Gorenstein Fano	same pair, $\Lambda = \Lambda_P$	
$(D_4, -\omega_2)$	$\{-\omega_1, -\omega_3, -\omega_4\}$	Smooth, Fano	same pair, $\Lambda = \Lambda_P$	
$(D_4, -\omega_3)$	$\{-\omega_1, -\omega_4\}$	Gorenstein Fano	same pair, $\Lambda = \Lambda_P$	
$(D_4, -\omega_4)$	$\{-\omega_1, -\omega_3\}$	Gorenstein Fano	same pair $\Lambda = \Lambda_P$	
$(E_6, -\omega_2)$		Gorenstein Fano	same pair, $\Lambda = \Lambda_P$	
$(F_4, -\omega_1)$		$\mathbb{Q}$–Gorenstein Fano	same pair, $\Lambda = \Lambda_P$	
$(F_4, -\omega_4)$		$\mathbb{Q}$–Gorenstein Fano	same pair, $\Lambda = \Lambda_P$	
$(G_2, -\omega_1)$	$\{-\omega_2\}$	$\mathbb{Q}$–Gorenstein Fano	$(A_2, -\omega_1 - \omega_2)$, $\Lambda = \Lambda_{A_2}$	yes
$(G_2, -\omega_2)$	$\{-\omega_1\}$	Smooth, Fano	$(A_2, -\omega_1 - \omega_2)$, $\Lambda = \Lambda_{P'}$	yes

6. Appendix: Explicit calculations

In this section we present the explicit calculations we made in order to describe which toric varieties associated to a root system are Fano (Section 5). In order to produce this classification, we take two different approaches, according to the type of root system:

Toric varieties associated to root systems of type G_2, F_4, E_6, E_7, E_8: we make explicit calculation with GAP3 and Sage (see [MR17] for the code and explicit results).

Toric varieties associated to root systems of type A_n, B_n, C_n, D_n: in this case we deal with the whole family, as follows:

(i) In order to apply Proposition 3.14, we use Lemma 3.13 in order to describe $\mathrm{Prim}(\sigma_\lambda)$.

(ii) Remark 3.11 allows us to find some elements of $\mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda)$ by looking onto weights μ with support $I_\mu \subset I_\lambda$.

(iii) Recall that if $v \in \sigma_\lambda$, then $\mathbb{R}^+ v \in \Sigma_\lambda(1)$ if and only if $v = \sum_{i\in I} a_i v_i$, with $v_i \in \mathrm{Prim}(\sigma_\lambda)$, $a_i > 0$ implies $\#I = 1$. Since $-\omega_i \in \mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda)$ if and only if $W_\lambda(-\omega_i) \subset \mathrm{Prim}(\sigma_\lambda)$, it follows that $-\omega_i$ is not primitive if and only if $-\omega_i \in \mathbb{R}^+ \big\langle \cup_{j\neq i} W_\lambda(-\omega_j) \big\rangle$.

(iv) Concerning the minimal pair (R', μ) associated to Σ_λ, note that if $\tau \subset \mathcal{C}$ is a maximal face such that $\langle \tau \rangle_{\mathbb{R}} = \alpha^\perp$ is a supporting hyperplane of σ_λ, then $W \cdot \alpha \subset R$ (see Lemma 3.7) . Since W acts transitively in the roots of same length, it follows that R contains all the roots of length $\|\alpha\|$. Hence, by Corollary 3.8, $R = W \cdot \{\alpha_i : i \in I_\lambda\}$, the set of roots having length equal to $\|\alpha_i\|$ for some $i \in I_\lambda$. In particular, if the root system R of G is simply laced, then the minimal pair associated to Σ_λ is (R, λ), with lattice $\Lambda = \Lambda_P$.

We begin by presenting explicit and complete calculations for root systems R of rank $1, 2$ and for R of type A_3, F_4. Then we deal with the A_n case in a rather complete way. The types B_n, C_n, D_n and A_n present a very similar behavior "far away from α_n", so we focus mainly in the relationship of λ and $-\omega_n$.

Notations 6.1. We follow the following conventions:

• A sub-list of elements indexed by i is *always* presented in ascending order. For instance, if $a > b$, then $\{u_1, v_a, \dots v_b, w_1, w_2\} = \{u_1, w_1, w_2\}$. This not-so orthodox convention allows to simplify the description of the sets $\mathrm{Prim}(\sigma_\lambda)$ and the sets of generators of $\big\langle \mathrm{Prim}(\sigma_\lambda) \big\rangle_{\mathrm{aff}}$.

• When dealing with root systems and their associated weights, we follow the notations of Bourbaki (see [Bou68, pp 250–276], or its English translation [B68en]).

• In figures 1–8, for each pair (R, λ) of root system R and anti-dominant weight λ, we draw the chamber $-w_0\sigma_\lambda$ in gray, and the convex hull of the weights $W \cdot \mathrm{Prim}(\sigma_\lambda)$ in thick lines.

When describing the minimal pair (R', μ) associated to a Fano variety Σ_λ, we also describe the lattice Λ in terms of $\Lambda_{R'}$ and $\Lambda_{P'}$ (see Proposition 4.7).

Recall that if G is simply laced, then the minimal pair associated to a generic orbit Σ_λ is (R, λ), with lattice $\Lambda = \Lambda_P$. In view of this remark, we will omit the (trivial) data of the associated minimal pair in the simply laced cases.

6.1. **Explicit calculation for small ranks.**

Example 6.2 (Explicit calculations for A_1)**.**

If G is of type A_1, then $G/B \cong \mathcal{P}^1(\Bbbk)$ is a Fano (trivially lattice-regular) toric variety, with associated fan the subdivision of M by the Weyl chambers: $\Sigma_{\mathcal{P}^1(\Bbbk)} = \big\{0, \mathbb{R}^+\langle w_1\rangle, \mathbb{R}^+\langle -w_1\rangle\big\}$.

Example 6.3 (Explicit calculations for $R = A_1 \times A_1$)**.**

Clearly, Σ_λ is Fano if and only if $\lambda = -(\omega_1 + \omega_2)$; in this case, $\Sigma_{-\omega_1-\omega_2}$ is a smooth, lattice-regular toric variety. See Figure 1.

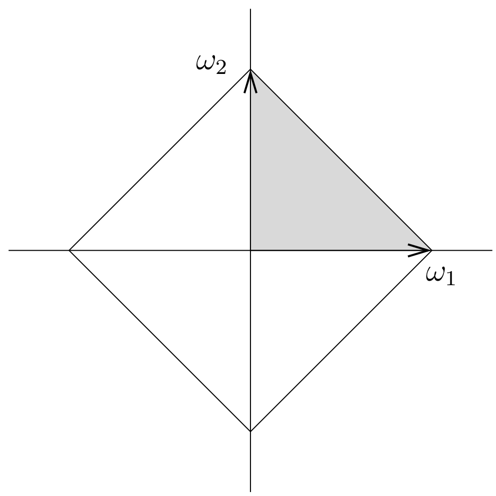

Fig. 1

Example 6.4 (Explicit calculations for $R = A_2$)**.**

As follows from figures 2 and 3, Σ_λ is Fano for $\lambda = -\omega_1, -\omega_2$ or $-(\omega_1 + \omega_2)$. Clearly, $\Sigma_{-\omega_1} \cong \Sigma_{-\omega_2}$ as toric varieties. In in all three cases Σ_λ is a smooth, lattice-regular toric variety.

Example 6.5 (Explicit calculations for $R = B_2 = C_2$)**.**

If $R = B_2$, then Σ_λ is Fano if and only if $\lambda = -\omega_i$. The minimal pair associated to $\Sigma_{-\omega_1}$ is $\big(A_1 \times A_1, (-\omega_1, -\omega_1)\big)$, with lattice $\Lambda = \Lambda_{P'}$. The minimal pair associated to $\Sigma_{-\omega_2}$ is $\big(A_1 \times A_1, (-\omega_1, -\omega_1)\big)$, with lattice $\Lambda = \Lambda_{A_1^2}$. See figure 1. Note that both varieties $\Sigma_{-\omega_i}$ are smooth and and lattice-regular.

Example 6.6 (Explicit calculations for $R = G_2$)**.**

In this case Σ_λ is Fano if and only if $\lambda = -\omega_i$. The toric variety $\Sigma_{-\omega_2}$ is smooth, whereas $\Sigma_{-\omega_1}$ is $\mathbb{Q}$–Gorenstein Fano; both varieties are lattice-regular. The minimal pair associated to $\Sigma_{-\omega_1}$ is $(A_2, -\omega_1 - \omega_2)$, with lattice $\Lambda = \Lambda_{A_2}$. The minimal pair associated to $\Sigma_{-\omega_2}$ is $(A_2, -\omega_1 - \omega_2)$, with lattice $\Lambda = \Lambda_{P'}$. See figures 6 and 7.

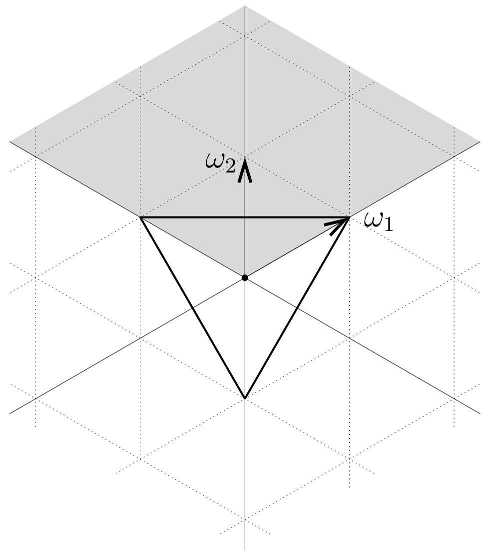

FIG. 2

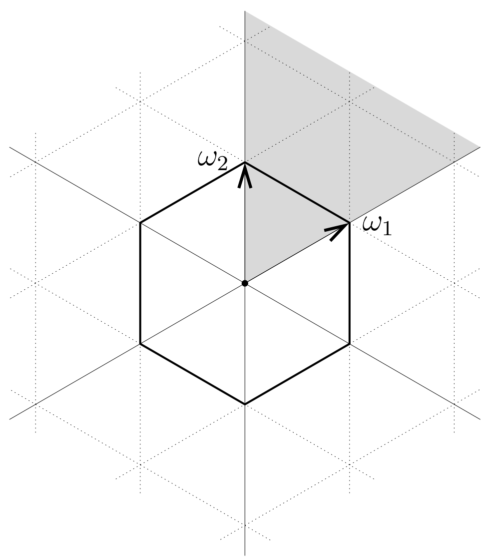

FIG. 3

Example 6.7. Let G be of type G_2, and consider Σ as in the picture 8; clearly, Σ is a lattice-regular complete fan. However, Σ does not correspond to a generic T-orbit of an homogeneous space G/P. Thus, there exist Fano toric varieties whose maximal cones are union of Weyl chambers, but are not of the form Σ_λ for some anti-dominant weight.

Note that Σ is associated to the root system $A_1 \times A_1$ with lattice $\Lambda_R \subsetneq \Lambda = \big\langle (\varepsilon/2, 0), (0, \varepsilon) \big\rangle_{\mathbb{Z}} = \Lambda_{P_{A_1}} \times \Lambda_{A_1} \subsetneq \Lambda_P$ and $(-\omega_1, -\omega_1)$ as associated anti-dominant weight.

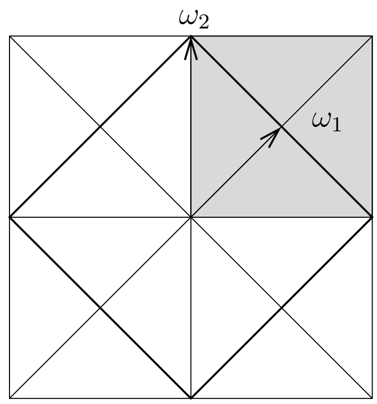

FIG. 4

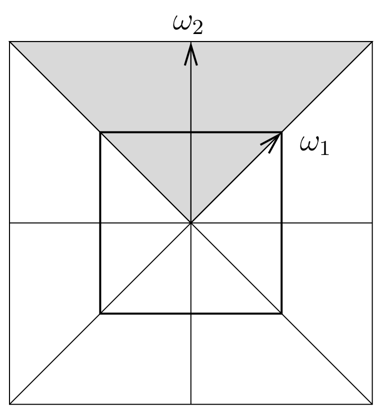

FIG. 5

6.2. **Explicit calculations for A_n, $n \geq 3$.**

CASE $\lambda = -\omega_n$

In this case $W_{-\omega_n}(-\omega_1) = \left\{-\varepsilon_1 + \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i, \ldots, -\varepsilon_n + \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i\right\}$. It is then clear that $-\omega_1$ is the unique anti-fundamental weight that generates a ray. Indeed, $-\omega_i \in \mathbb{R}^+\left\langle W_{-\omega_n}(-\omega_1)\right\rangle$ for $i = 2, \ldots, n$. Thus,

$$\operatorname{Prim}(\sigma_{-\omega_n}) = W_{-\omega_n} \cdot (-\omega_1)$$
$$\left\langle \operatorname{Prim}(\sigma_{-\omega_n})\right\rangle_{\text{aff}} = -\omega_1 + \langle \alpha_1, \ldots, \alpha_{n-1}\rangle_{\mathbb{R}}.$$

In particular, $\dim F_{\mathcal{C}} = n - 1$, and $n_{-\omega_n} = \omega_n$. Since $\left\langle (n+1)\omega_n, -\varepsilon_i + \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i\right\rangle = -1$, and that $\operatorname{Prim}(\sigma_{-\omega_n})$ is a basis for the weight lattice Λ_P, it follows that $\Sigma_{-\omega_n}$ is a smooth Fano toric variety.

CASE $\lambda = -\omega_1$

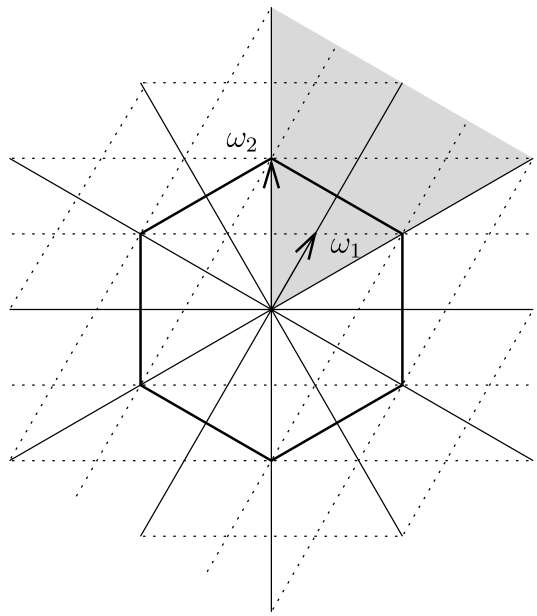

FIG. 6

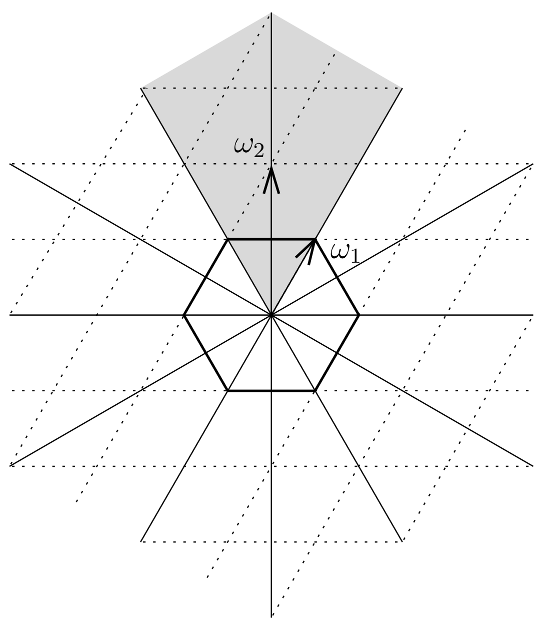

FIG. 7

This case is dual to the previous one, but it is convenient for us to recall that

$$\begin{aligned}\mathrm{Prim}(\sigma_{-\omega_1}) &= W_{-\omega_1}(-\omega_n)\\ &= \Big\{\varepsilon_2 - \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i, \dots, \varepsilon_{n+1} - \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i = -\omega_n\Big\}\\ \big\langle \mathrm{Prim}(\sigma_{-\omega_1})\big\rangle_{\mathrm{aff}} &= -\omega_n + \langle \alpha_2, \dots, \alpha_n\rangle_{\mathbb{R}}.\end{aligned}$$

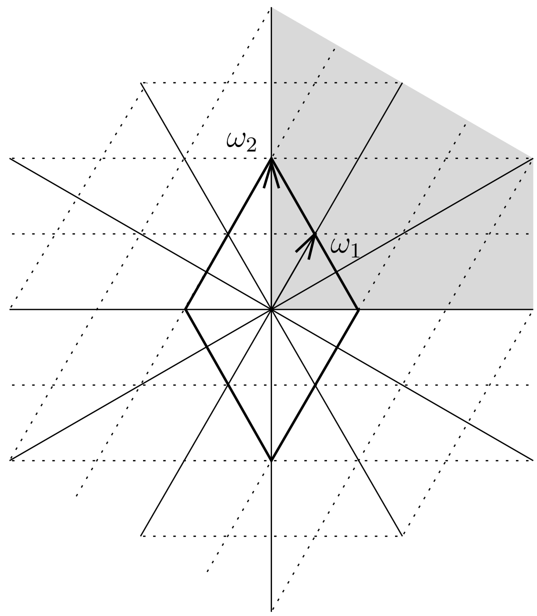

FIG. 8

Hence, $n_{-\omega_1} = (n+1)\omega_1$ and $\Sigma_{-\omega_1}$ is Fano.

CASE $\lambda = -\omega_j,\ 1 < j < n$

It follows from Remark 2.10 that

$$W_{-\omega_j}(-\omega_1) = \Big\{-\omega_1 = -\varepsilon_1 + \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i, \dots, -\varepsilon_j + \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i\Big\}$$
$$W_{-\omega_j}(-\omega_n) = \Big\{\varepsilon_{j+1} - \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i, \dots, \varepsilon_{n+1} - \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i = -\omega_n\Big\}.$$

We deduce as before that $-\omega_i \notin \mathrm{Prim}(\sigma_{-\omega_j})$ if $i \neq 1, n$. Therefore,

$$\mathrm{Prim}(\sigma_{-\omega_j}) = W_{-\omega_j}(-\omega_1) \cup W_{-\omega_j}(-\omega_n)$$
$$\big\langle \mathrm{Prim}(\sigma_{-\omega_j})\big\rangle_{\mathrm{aff}} = -\omega_1 + \big\langle \alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_n, \omega_n - \omega_1\big\rangle_{\mathbb{R}}.$$

If follows that $n_{-\omega_i} = a\omega_j$ and verifies the additional condition $\langle n_\lambda, \omega_n - \omega_1\rangle = 0$, that is $0 = \langle n_\lambda, \omega_n - \omega_1\rangle = \langle n_\lambda, (-n+2j-1)\alpha_j\rangle = a(-n+2j-1)$. It follows that if $n \neq 2j-1$, then Σ_λ is not Fano, and that if $n = 2j-1$ then $n_\lambda = \omega_j$ is such that $\langle n_\lambda, -\varepsilon_k + \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i\rangle = \frac{-3j+1}{2j-1}$. Hence, $\Sigma_{-\omega_j}$ is $\mathbb{Q}$–Gorenstein-Fano — note that in this case $\Sigma_{-\omega_j}$ is not smooth.

CASE $\lambda = -\omega_s - \omega_r,\ s < r$

SUB-CASE $1 < s < r < n$

It follows from Remark 3.11 applied to $\mu = -\omega_r$ (resp. $-\omega_s$) that $-\omega_1$ (resp. $-\omega_n$) belongs to $\mathrm{Prim}(\sigma_\lambda)$. Since

$$W_\lambda\cdot(-\omega_1) = \Big\{\omega_1 = -\varepsilon_1 + \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i, \dots, -\varepsilon_s + \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i\Big\}$$

$$W_\lambda\cdot(-\omega_n) = \Big\{\varepsilon_{r+1} - \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i, \dots, \varepsilon_{n+1} - \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i = -\omega_n\Big\},$$

it follows that $\{-\omega_2,\dots,-\omega_s,-\omega_r,\dots,-\omega_{n-1}\} \subset \mathbb{R}^+\big\langle W_\lambda\cdot(-\omega_1)\cup W_\lambda\cdot(-\omega_n)\big\rangle$, and $\mathrm{Prim}(\sigma_\lambda) \subset W_\lambda\cdot\{-\omega_1,-\omega_{s+1},\dots,-\omega_{r-1},-\omega_n\}$. We affirm that equality holds, that is $-\omega_h \in \mathrm{Prim}(\sigma_\lambda)$ for $h = s+1,\dots,r-1$.

If $s<j<r$ then

$$W_\lambda\cdot(-\omega_j) = \Big\{-\sum_{i=1}^{s}\varepsilon_i - \sum_{i=s+1}^{r} a_i\varepsilon_i + \frac{j}{n+1}\sum_{i=1}^{n+1}\varepsilon_i : a_i = 0,1,\ \sum_{i=s+1}^{r} a_i = j-s\Big\}.$$

Let $v_j = \varepsilon_i - \frac{1}{n+1}\sum_{i=1}^{n+1}\varepsilon_i$, $j = 1,\dots,n$ and w_j, $j = s+1,\dots,r-1$, $k = 1,\dots,\binom{r-s}{j-s}$ be such that

$$\Big\{v_{jk} : k = 1,\dots,\tbinom{r-s}{j-s}\Big\} =$$
$$\Big\{-\sum_{i=1}^{s}\varepsilon_i - \sum_{i=s+1}^{r} a_{jki}\varepsilon_i + \frac{j}{n+1}\sum_{i=1}^{n+1}\varepsilon_i : a_{jki} = 0,1,\ \sum_{i=s+1}^{r} a_{jki} = j-s\Big\}.$$

Assume that for some $h \in \{s+1,\dots,r-1\}$,

$$-\omega_h = \sum_{j=1}^{s} b_j(-v_j) + \sum_{j=s+2,k=1}^{r-1,\binom{r-s}{j-s}} b_{jk}v_{jk} + \sum_{j=r+1}^{n} b_j v_k,$$

with $b_j, b_{jk} \geq 0$. Looking at the coefficients of ε_i, $i = 1,\dots,n+1$ we deduce that

$$(1, i{=}1,\dots,s)\quad -b_i + \sum_{j=1}^{s}\tfrac{b_j}{n+1} - \sum_{j=r+1}^{n+1}\tfrac{b_j}{n+1} + \sum_{j=s+1,k=1}^{r-1,\binom{r-s}{j-s}}\tfrac{b_{jk}(-n+j-1)}{n+1} = \tfrac{-n-1+h}{n+1}$$

$$(2, i{=}s+1,\dots,h)\quad \sum_{j=1}^{s}\tfrac{b_j}{n+1} - \sum_{j=r+1}^{n+1}\tfrac{b_j}{n+1} + \sum_{jk:a_{jki}=1}\tfrac{b_{jk}(-n+j-1)}{n+1} + \sum_{jk:a_{jki}=0}\tfrac{jb_{jk}}{n+1} = \tfrac{-n-1+h}{n+1}$$

$$(3, i{=}h+1,\dots,r)\quad \sum_{j=1}^{s}\tfrac{b_j}{n+1} - \sum_{j=r+1}^{n+1}\tfrac{b_j}{n+1} + \sum_{jk:a_{jki}=1}\tfrac{b_{jk}(-n+j-1)}{n+1} + \sum_{jk:a_{jki}=0}\tfrac{jb_{jk}}{n+1} = \tfrac{h}{n+1}$$

$$(4, i{=}r+1,\dots,n+1)\quad b_i + \sum_{j=1}^{s}\tfrac{b_j}{n+1} - \sum_{j=r+1}^{n+1}\tfrac{b_j}{n+1} + \sum_{j=s+1,k=1}^{r-1,\binom{r-s}{j-s}}\tfrac{jb_{jk}}{n+1} = \tfrac{h}{n+1}$$

Subtracting equations (1, i) and (2, j) (resp. (4, i) and (3, j)) for any possible choice of i, j and using the fact that the coefficients b_j, b_{jk} are non negative, we easily deduce that $b_i = 0$ for $i = 1,\dots,s,r+1,\dots,n$, and that $b_{jk} = 0$ if $a_{jki} = 0$ for some $i = s+1,\dots,r$ or $a_{jki} = 1$ for some $i = r+1,\dots,n$. We deduce that $b_{jk} = 0$ unless $v_{jk} = -\omega_h$, and it follows that $-\omega_h \in \mathrm{Prim}(\sigma_\lambda)$.

Hence,

$$\mathcal{C}\cap \operatorname{Prim}(\sigma_\lambda)=\big\{-\omega_1,-\omega_{s+1},\dots,-\omega_{r-1},-\omega_n\big\}$$

$$\big\langle \operatorname{Prim}(\sigma_\lambda)\big\rangle_{\text{aff}}=-\omega_1+\left\langle \begin{array}{c}\alpha_1,\dots,\alpha_{s-1},\alpha_{s+1},\dots,\alpha_{r-1},\alpha_{r+1},\dots,\alpha_n,\\ \omega_{s+1}-\omega_1,\dots,\omega_{r-1}-\omega_1,\omega_n-\omega_1\end{array}\right\rangle_{\mathbb{R}}$$

It follows that $n_\lambda=a\omega_s+b\omega_r$ satisfies the additional conditions

$$\langle n_\lambda,\omega_{s+1}-\omega_1\rangle=\cdots=\langle n_\lambda,\omega_{r-1}-\omega_1\rangle=\langle n_\lambda,\omega_n-\omega_1\rangle=0$$

Equivalently, $n_\lambda=a\omega_s+b\omega_r$ satisfies the system:

(1)
$$\left\{\begin{array}{lcccc} a\big[(n-j+1)s-(n-s+1)\big] & + & b(n-r+1)(j-1) & = & 0 \qquad j=s+1,\dots,r-1\\ a(n-2s+1) & + & b(n-2r+1) & = & 0\end{array}\right.$$

Sub-case $r=s+1$

In this case the first line of equation (1) is empty. Therefore, a,b are such that $(n-2s+1)a+(n-2s-1)b=0$. Thus, if $n\neq 2s$ then $ab<0$ and it follows that $\Sigma_{-\omega_s-\omega_{s+1}}$ is not Fano. On the other hand, if $n=2s$ then $\langle\omega_s+\omega_{s+1},v\rangle=-1$ for all $v\in\operatorname{Prim}(\sigma_{-\omega_s-\omega_{s+1}})$, and therefore $\Sigma_{-\omega_s-\omega_{s+1}}$ is Fano — note that in this case Σ_λ is smooth.

Sub-case $s+2>r$

The first line of equation (1) for $j=s+1$ together with the last line of the same equation conform the system

$$\left\{\begin{array}{llcccc} (i) & a\big[(n-s)s-(n-s+1)\big] & + & b(n-r+1)s & = & 0\\ (ii) & a(n-2s+1) & + & b(n-2r+1) & = & 0\end{array}\right.$$

The coefficient of b in equation (i) above is greater or equal than zero, whereas the coefficient for a is the polynomial $p(s)=-s^2+(n+1)s-(n+1)$, with roots $\frac{-(n+1)\pm\sqrt{(n+1)(n-3)}}{-2}$. If follows that $\sigma_{-\omega_s-\omega_r}$ is not Fano if $s\geq 2$.

Assume now that $s=1$; then equation (ii) is verified for some $a,b>0$ if and only if $r=n$; in this case $a=b=1$ is a solution of Equation (1), and it follows that $\Sigma_{-\omega_1-\omega_n}$ is Gorenstein-Fano. Note that $\Sigma_{-\omega_1-\omega_n}$ is smooth if and only if $n=2$.

Case $\lambda=-\omega_s-\omega_{s+1}-\omega_{s+2}$

Since $I_{-\omega_s-\omega_{s+2}}\subset I_\lambda$, in view of Remark 3.11, it follows that

$$\operatorname{Prim}(\sigma_\lambda)=\Big\{-\omega_1,\dots,-\varepsilon_s+\frac{1}{n+1}\sum\varepsilon_i,-\omega_{s+1},\varepsilon_{s+2}-\frac{1}{n+1}\sum\varepsilon_i,\dots,-\omega_n\Big\}$$

$$\big\langle\operatorname{Prim}(\sigma_\lambda)\big\rangle_{\text{aff}}=-\omega_1+\big\langle\alpha_1,\dots,\alpha_{s-1},\alpha_{s+3},\dots,\alpha_n,\omega_{s+1}-\omega_1,\omega_n-\omega_1\big\rangle_{\mathbb{R}}$$

It follows that $n_\lambda=a\omega_s+b\omega_{s+1}+c\omega_{s+2}$ satisfies the additional conditions

$$\left\{\begin{array}{rl}\langle n_\lambda,\omega_n-\omega_1\rangle= & 0\\ \langle n_\lambda,\omega_{s+1}-\omega_1\rangle= & 0\end{array}\right.$$

That is,

$$\left\{\begin{array}{rl}(-n+2s-1)a+(-n+2s+1)b+(-n+2s+3)c= & 0\\ ((s-1)(n-s)-1)a+s(n-s)b+s(n-s-1)c= & 0\end{array}\right.$$

It follows that if $2s+1 \geq n$ then $ac \leq 0$. By symmetry, if $2(n-s-2+1)+1 \geq n$ — that is if $n \leq 2s+1$ – then $ac \leq 0$. It follows that Σ_λ is not Fano.

CASE $\lambda = \sum_{i=s_1}^{r_1} -\omega_i + \cdots + \sum_{i=s_\ell}^{r_\ell} -\omega_i$, $s_i \leq r_i$, $r_i + 1 < s_{i+1}$, $\# I_\lambda \geq 3$, $r_\ell - s_1 > 2$

Since $I_{-\omega_{s_1}-\omega_{r_\ell}} \subset I_\lambda$, it follows that

$$\mathcal{C} \cap \operatorname{Prim}(\sigma_\lambda) = \big\{-\omega_1, -\omega_{s_1+1}, \dots, -\omega_{r_\ell-1}, -\omega_n\big\}$$

$$\big\langle \operatorname{Prim}(\sigma_\lambda)\big\rangle_{\text{aff}} = -\omega_1 + \left\langle \begin{array}{c} \alpha_1, \dots, \alpha_{s_1-1}, \alpha_{s_1+2}, \dots, \alpha_{s_\ell-2}, \alpha_{\ell+1}, \dots, \alpha_n, \\ \omega_{s_1+1} - \omega_1, \omega_{r_\ell-1} - \omega_1, \omega_n - \omega_1 \end{array} \right\rangle_{\mathbb{R}},$$

were we used the fact that $\omega_{i-1} - 2\omega_i + \omega_{i+1} = \alpha_i$. It follows that if $r_\ell - s_1 > 4$ then $n_\lambda = \lambda = \sum_{i=s_1}^{r_1} a_i\omega_i + \cdots + \sum_{i=s_\ell}^{r_\ell} a_i\omega_i$ is such that $a_i = 0$ for $i = s_1+2, \dots, r_\ell - 2$. Hence, Σ_λ is not Fano unless

$$\lambda = \begin{cases} -\omega_s - \omega_{s+1} - \omega_{s+2+j} - \omega_{s+3+j} & j \geq 0 \\ -\omega_s - \omega_{s+1} - \omega_{s+3+j} & j \geq 0 \\ -\omega_s - \omega_{s+2+j} - \omega_{s+3+j} & j \geq 0 \end{cases}$$

SUB-CASE $\lambda = -\omega_s - \omega_{s+1} - \omega_{s+2+j} - \omega_{s+3+j}$, $j \geq 0$

In this case, $n_\lambda = a_s\omega_s + a_{s+1}\omega_{s+1} + a_{s+2+j}\omega_{s+2+j} + a_{s+3+j}\omega_{s+3+j}$ satisfies (among others) the additional condition $a_s((n-s)(s-1)-1)+a_{s+1}(n-s)s+a_{s+2+j}(n-s-j-1)s+a_{s+3+j}(n-s-j-2)s=\langle n_\lambda,\omega_{s+1}-\omega_1\rangle=0$

It follows that if $s \neq 1$, then Σ_λ is not Fano.

If $s = 1$ we consider the conditions

$$\begin{cases} (1) & a_1(-1)+a_2(n-1)+a_{3+j}(n-j-2)+a_{4+j}(n-j-3) & =\langle n_\lambda,\omega_2-\omega_1\rangle=0 \\ (2) & a_1(-n+1)+a_2(-n+3)+a_{3+j}(-n+2j+5)+a_{4+j}(-n+2j+7) & =\langle n_\lambda,\omega_n-\omega_1\rangle=0 \end{cases}$$

It easily follows that Σ_λ is not Fano (e.g. considering the linear combination $(-n+1)(1)+(2)$).

SUB-CASE $\lambda = -\omega_s - \omega_{s+1} - \omega_{s+3+j}$, $j \geq 0$

In this case, $n_\lambda = a_s\omega_s + a_{s+1}\omega_{s+1} + a_{s+3+j}\omega_{s+3+j}$ satisfies (among others) the additional condition

$$a_s\big((n-s)(s-1)-1\big) + a_{s+1}\big((n-s)s\big) + a_{s+3+j}\big(s(n-s-2)\big) = \langle n_\lambda, \omega_{s+1} - \omega_1\rangle = 0$$

If $s \neq 1$, since $n \geq s+3$, it follows that Σ_λ is not Fano.

If $s = 1$, then n_λ verifies (among others) the conditions

$$\begin{cases} (1) & -a_1 + (n-1)a_2 + (n-3)a_{3+j} = & \langle n_\lambda, \omega_2 - \omega_1\rangle = 0 \\ (2) & (-n+1)a_1 + (-n+3)a_2 + (-n+4+2j)a_{3+j} = & \langle n_\lambda, \omega_n - \omega_1\rangle = 0 \end{cases}$$

Since $n \geq j+3$, from the linear combination $(-j-1)(1)+(2)$ we deduce that Σ_λ is not Fano.

SUB-CASE $\lambda = -\omega_s - \omega_{s+2+j} - \omega_{s+3+j}$, $j \geq 0$

We deduce by symmetry that in this case Σ_λ is not Fano.

6.3. **Explicit calculations for B_n.**

CASE $\lambda = -\omega_n$

In this case, $\mathrm{Prim}(\sigma_{-\omega_n}) = W_{-\omega_n}(-\omega_1) = \{-\varepsilon_1, \dots, -\varepsilon_n\}$ and $\langle \mathrm{Prim}(\sigma_{-\omega_n})\rangle_{\mathrm{aff}} = -\omega_1 + \langle \alpha_1, \dots, \alpha_{n-1}\rangle_{\mathbb{R}}$.

It follows that $n_{-\omega_n} = \omega_n$, and $\Sigma_{-\omega_n}$ is Gorenstein-Fano. Since the supporting hyperplanes are orthogonal to the roots ε_i, it follows that the minimal pair associated to $\Sigma_{-\omega_n}$ is $(A_1 \times \cdots \times A_1, (-\omega_1, \dots, \omega_1))$, with lattice $\Lambda_{A_1^n} \subsetneq \Lambda = \langle \varepsilon_1, \dots, \varepsilon_n, \frac{1}{2}\sum \varepsilon_i\rangle_{\mathbb{Z}} \subsetneq \Lambda_{P'}$.

CASE $\lambda = -\omega_1$

By Remark 2.10, $\# W_{-\omega_1}(-\omega_n) = 2^{n-1}$. Easy calculations show then that

$$W_{-\omega_1}(-\omega_n) = \Big\{ {}^1\!/\!_2\big(-\varepsilon_1 + \sum_{i=2}^{n} \pm\varepsilon_i\big)\Big\}.$$

It follows that $-\omega_i \in \mathbb{R}^+\langle W_{-\omega_1}(-\omega_n)\rangle$ for $i = 1, \dots, n-1$; thus $\mathrm{Prim}(\sigma_{-\omega_1}) = W_{-\omega_1}(-\omega_n) = \Big\{ {}^1\!/\!_2\big(-\varepsilon_1 + \sum_{i=2}^{n} \pm\varepsilon_i\big)\Big\}$ and $\langle \mathrm{Prim}(\sigma_{-\omega_1})\rangle_{\mathrm{aff}} = -\omega_n + \langle \varepsilon_2, \dots, \varepsilon_n\rangle_{\mathbb{R}}$.

In particular, $n_{-\omega_1} = 2\omega_1$, and $\Sigma_{-\omega_1}$ is Gorenstein-Fano. If $n = 2$, then $\Sigma_{-\omega_1}$ is smooth. Since the roots ε_i are not orthogonal to any facet of $\sigma_{-\omega_1}$, it follows that they are not orthogonal to any facet of $\Sigma_{-\omega_1}$. Therefore, the minimal pair associated to $\Sigma_{-\omega_1}$ is $(B_n, -\omega_1)$, with lattice $\Lambda = \Lambda_{P'}$. Indeed, $\frac{1}{2}(\sum_{i=1}^{n-1} \varepsilon_i - \varepsilon_n) = \sum_{i=1}^{n-1} \varepsilon_i - \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i$.

CASE $\lambda = -\omega_j$, $1 < j < n$

In this case,

$$\mathrm{Prim}(\sigma_{-\omega_j}) = W_{-\omega_j}(-\omega_1) \cup W_{-\omega_j}(-\omega_n) = \Big\{-\varepsilon_1, \dots, -\varepsilon_j, {}^1\!/\!_2\big(-\omega_j + \sum_{i=j+1}^{n} \pm\varepsilon_i\big)\Big\}$$

$$\langle \mathrm{Prim}(\sigma_{-\omega_j})\rangle_{\mathrm{aff}} = -\omega_1 + \langle \alpha_1, \dots, -\alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_{n-1}, \alpha_n, \omega_n - \omega_1\rangle_{\mathbb{R}}.$$

Therefore, $n_{-\omega_j} = a\omega_j$ verifies the additional condition $\langle n_{-\omega_j}, \omega_n - \omega_1\rangle = 0$. If $j \neq 2$ then $n_{-\omega_j} = 0$ – that is $\dim\langle \mathrm{Prim}(\sigma_{-\omega_j})\rangle_{\mathrm{aff}} = n$, and $\Sigma_{-\omega_j}$ is not Fano. If $j = 2$, then $n_{\omega_2} = \omega_2$ and $\Sigma_{-\omega_2}$ is Gorenstein-Fano.

CASE $\lambda = -\omega_s - \omega_r$, $s < r$

It follows from Remark 3.11 applied to $\mu = -\omega_r$ (resp. $-\omega_s$) that $-\omega_1$ (resp. $-\omega_n$) belongs to $\mathrm{Prim}(\sigma_\lambda)$. It is clear that $W_{-\omega_s-\omega_r} \cdot (-\omega_1) = \{-\varepsilon_1, \dots, -\varepsilon_s\}$ and $W_{-\omega_s-\omega_r} \cdot (-\omega_n) = \Big\{ {}^1\!/\!_2\big(-\omega_r + \sum_{i=r+1}^{n} \pm\varepsilon_i\big)\Big\}$. Thus, $\{-\omega_2, \dots, -\omega_s\} \subset \mathbb{R}^+\langle W_{-\omega_s-\omega_r} \cdot (-\omega_1)\rangle$ and $\{-\omega_r, \dots, -\omega_{n-1}\} \subset \mathbb{R}^+\langle W_{-\omega_s-\omega_r} \cdot (-\omega_n)\rangle$. It follows that

$$\mathrm{Prim}(\sigma_\lambda) \subset W_\lambda(-\omega_1) \cup W_\lambda(-\omega_{s+1}) \cup \cdots \cup W_\lambda(-\omega_{r-1}) \cup W_\lambda(-\omega_n) \tag{2}$$

If $s<j<r$, then

$$W_{-\omega_s-\omega_r}\cdot(-\omega_j)=\Big\{-\omega_s-\sum_{i=s+1}^{r}a_i\varepsilon_i : a_i=0,1,\sum a_i=j-s\Big\}.$$

Let $\{v_{j1}\dots,v_{\binom{r-s}{j}}\}=W_{-\omega_s-\omega_r}(-\omega_j)$, $j=s+1,\dots,r-1$, $\{w_1,\dots,w_{2^{n-r}}\}=W_{-\omega_s-\omega_r}(-\omega_n)$, and denote $v_{jk}=-\omega_s-\sum_{i=s+1}^{r}c_{jki}\varepsilon_i$, $w_j={}^1\!/\!_2\big(-\omega_r+\sum_{i=r+1}^{n}d_{ji}\varepsilon_i\big)$. If $h=s+1,\dots,r-1$, let

$$-\omega_h=\sum_{j=1}^{s}-b_j\varepsilon_j+\sum_{j=s+1,k=1}^{r-1,\binom{r-s}{j}}b_{jk}v_{jk}+\sum_{j=r+1}^{n}d_jw_j,$$

with $b_j,b_{jk},d_j\geq 0$. Then

$$\begin{cases}(1,\,i=1,\dots,s) & b_i-\sum_{jk}b_{jk}+\sum_j\frac{d_j}{2}= & 1\\ (2,\,i=s+1,\dots,h) & \sum_{jk:c_{jki}=1}b_{jk}+\sum_j\frac{d_j}{2}= & 1\\ (3,\,i=h+1,\dots,r) & \sum_{k:c_{jki}=1}b_{jk}+\sum_j\frac{d_j}{2}= & 0\\ (4,\,i=r+1,\dots,n) & \sum_j\frac{d_jd_{ji}}{2}= & 0\end{cases}$$

Combining equations $(1,i)$ and $(2,i')$ for all possible values of i,i', we deduce that $b_j=0$ for $j=1,\dots,s$ and that $b_{jk}=0$ if $c_{kji}=0$ for some $i=s+1,\dots,h$. From equation (3,i) we deduce that $d_j=0$ for $j=1,\dots,2^{n-r}$ and that $b_{jk}=0$ if $c_{jki}=1$ for some $i=h+1,\dots r$. If follows that $c_{jk}=0$ unless $v_{jk}=-\omega_h$. In other words, $-\omega_h\in\operatorname{Prim}(\sigma_\lambda)$.

Hence,

$$\mathcal{C}\cap\operatorname{Prim}(\sigma_\lambda)=\big\{-\omega_1,-\omega_{s+1},\dots,\omega_{r-1},-\omega_n\big\}$$

$$\big\langle\operatorname{Prim}(-\omega_s-\omega_r)\big\rangle_{\text{aff}}=-\omega_1+\left\langle\begin{matrix}\alpha_1,\dots,\alpha_{s-1},\alpha_{s+1},\dots,\alpha_{r-1},\alpha_{r+1},\dots,\alpha_n,\\ \omega_{s+1}-\omega_1,\dots,\omega_{r-1}-\omega_1,\omega_n-\omega_1\end{matrix}\right\rangle_{\mathbb{R}}$$

Easy calculations (e.g. imposing the additional necessary $\langle n_\lambda,\omega_n-\omega_1\rangle=0$ if $r=s+1$ or $\langle n_\lambda,\omega_{s+1}-\omega_1\rangle=0$ if $s+2\leq r$) show that in this case Σ_λ is not Fano.

Case $\lambda=-\omega_s-\omega_{s+1}-\omega_{s+2}$

Since $I_{-\omega_s-\omega_{s+2}}\subset I_\lambda$, in view of Remark 3.11, it follows that

$$\mathcal{C}\cap\operatorname{Prim}(\sigma_\lambda)=\big\{-\omega_1,-\omega_{s+1},-\omega_n\big\}$$

$$\big\langle\operatorname{Prim}(\sigma_\lambda)\big\rangle_{\text{aff}}=-\omega_1+\big\langle\alpha_1,\dots,\alpha_{s-1},\alpha_{s+3},\dots,\alpha_n,\omega_{s+1}-\omega_1,\omega_n-\omega_1\big\rangle_{\mathbb{R}}.$$

It follows that $n_\lambda=a\omega_s+b\omega_{s+1}+c\omega_{s+2}$. Again, easy calculation using the additional conditions satisfied by n_λ allow to verify that Σ_λ is not Fano.

Case $\lambda=\sum_{i=s_1}^{r_1}-\omega_i+\dots+\sum_{i=s_\ell}^{r_\ell}-\omega_i,\ s_i\leq r_i,\ r_i+1<s_{i+1},\ \#I_\lambda\geq 3,\ s_1+2\leq r_\ell-1$

Since $I_{-\omega_{s_1}-\omega_{r_\ell}} \subset I_\lambda$, it follows that

$$\mathcal{C} \cap \operatorname{Prim}(\sigma_\lambda) = \big\{-\omega_1, \omega_{s_1+1}, \dots, -\omega_{r_\ell-1}, -\omega_n\big\}$$

$$\big\langle \operatorname{Prim}(\sigma_\lambda)\big\rangle_{\mathrm{aff}} = -\omega_1 + \left\langle \begin{matrix} \alpha_1, \dots, \alpha_{s_1-1}, \alpha_{s_1+2}, \dots, \alpha_{r_\ell-2}, \alpha_{r_\ell+1}, \dots, \alpha_n, \\ \varepsilon_{s_1+2}, \omega_{s_1+1} - \omega_1, \omega_n - \omega_1 \end{matrix} \right\rangle_{\mathbb{R}}$$

were we used the fact that $\omega_{i-1} - 2\omega_i + \omega_{i+1} = \alpha_i$. if $i \leq n-2$. It follows that if $r_\ell - s_1 > 4$ then $n_\lambda = \lambda = \sum_{i=s_1}^{r_1} a_i\omega_i + \cdots + \sum_{i=s_\ell}^{r_\ell} a_i\omega_i$ is such that $a_i = 0$ for $i = s_1+2, \dots, r_\ell - 2$. It follows that Σ_λ is not Fano unless

$$\lambda = \begin{cases} -\omega_s - \omega_{s+1} - \omega_{s+2+j} - \omega_{s+3+j} & j \geq 0 \\ -\omega_s - \omega_{s+1} - \omega_{s+3+j} & j \geq 0 \\ -\omega_s - \omega_{s+2+j} - \omega_{s+3+j} & j \geq 0 \end{cases}$$

Again, easy calculations show that is these remaining cases Σ_λ is not Fano.

6.4. Explicit calculations for C_n.

This case is very similar to the case B_n:

Case $\lambda = -\omega_1$

It is easy to show that $\operatorname{Prim}(\sigma_{-\omega_1}) = W_{-\omega_1}(-\omega_n) = \big\{-\varepsilon_1 + \sum_{i=2}^n \pm\varepsilon_i\big\}$ and $\big\langle \operatorname{Prim}(\sigma_{-\omega_1})\big\rangle_{\mathrm{aff}} = -\omega_w + \langle \varepsilon_2, \dots, \varepsilon_n\rangle_{\mathbb{R}}$. Therefore, $n_{-\omega_1} = \omega_1$ and in particular, $\Sigma_{-\omega_1}$ is Gorenstein-Fano. The minimal pair associated to $\Sigma_{-\omega_1}$ is $(D_n, -\omega_1)$, with lattice $\Lambda = \Lambda_{P'}$.

Case $\lambda = -\omega_n$

In this case, $\operatorname{Prim}(\sigma_{-\omega_n}) = W_{-\omega_n}(-\omega_1) = \{-\varepsilon_1, \dots, -\varepsilon_n\}$ and $\big\langle \operatorname{Prim}(\sigma_{-\omega_n})\big\rangle_{\mathrm{aff}} = -\omega_1 + \langle \alpha_1, \dots, \alpha_{n-1}\rangle_{\mathbb{R}}$. In particular, $n_{-\omega_n} = \omega_n$ and $\Sigma_{-\omega_n}$ is Gorenstein-Fano. If $n = 2$, then $\Sigma_{-\omega_2}$ is smooth — recall that $B_2 = C_2$. The minimal pair associated to $\Sigma_{-\omega_n}$ is $\big(A_1^n, (-\omega_1, \dots, \omega_1)\big)$, with lattice $\Lambda_{A_1^n} \subsetneq \Lambda = \big\langle \varepsilon_1, \dots, \varepsilon_n, \frac{1}{2}\sum \varepsilon_i\big\rangle_{\mathbb{Z}} \subsetneq \Lambda_{P'}$.

Case $\lambda = -\omega_j$, $1 < j < n$

In this case,

$$\operatorname{Prim}(\sigma_{-\omega_j}) = W_{-\omega_j}(-\omega_1) \cup W_{-\omega_j}(-\omega_n) = \{-\varepsilon_1, \dots, -\varepsilon_j, -\omega_j + \sum_{i=r+1}^{n} \pm\varepsilon_i\}.$$

As in the analogous B_n case, it follows that

$$\big\langle \operatorname{Prim}(\sigma_{-\omega_j})\big\rangle_{\mathrm{aff}} = -\omega_1 + \big\langle \alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_{n-1}, \alpha_n, \omega_n - \omega_1\big\rangle_{\mathbb{R}}.$$

Therefore, $\dim\big\langle \operatorname{Prim}(\sigma_{-\omega_j})\big\rangle_{\mathrm{aff}} = n$, and $\Sigma_{-\omega_j}$ is not Fano.

Case $\lambda = -\omega_s - \omega_r$, $s < r$

Calculations similar to the corresponding B_n case show that

$$\mathcal{C} \cap \operatorname{Prim}(\sigma_\lambda) = \big\{-\omega_1, -\omega_{s+1}, \dots, -\omega_{r-1}, -\omega_n\big\}.$$

It is easy now to see that Σ_λ is not Fano.

General case $\lambda = \sum_{i=s_1}^{r_1} -\omega_i + \cdots + \sum_{i=s_\ell}^{r_\ell} -\omega_i$, $r_i + 1 < s_{i+1}$, $s_1 + 2 < r_\ell$, $\#I \geq 3$

Since $I_{-\omega_{s_1}-\omega_{r_\ell}} \subset I_\lambda$ it follows that

$\mathcal{C} \cap \operatorname{Prim}(\sigma_\lambda) = \{-\omega_1, -\omega_{s_1+1}, \dots, \omega_{r_\ell -1}, -\omega_n\}$

$$\big\langle \operatorname{Prim}(\sigma_\lambda)\big\rangle_{\text{aff}} = -\omega_1 + \left\langle \begin{array}{c} \alpha_1, \dots, \alpha_{s_1-1}, \alpha_{s+1}, \dots, \alpha_{r_\ell -1}, \alpha_{r_\ell+1}, \dots, \alpha_n, \\ \omega_{s_1+1} - \omega_1, \dots \omega_{r_\ell-1} - \omega_1, \omega_n - \omega_1 \end{array} \right\rangle_{\text{aff}},$$

As in the corresponding B_n case, we deduce that if $r_\ell - s_1 > 4$ then $n_\lambda = \lambda = \sum_{i=s_1}^{r_1} a_i\omega_i + \dots + \sum_{i=s_\ell}^{r_\ell} a_i\omega_i$ is such that $a_i = 0$ for $i = s_1+2, \dots, r_\ell - 2$. It follows that Σ_λ is not Fano unless

$$\lambda = \begin{cases} -\omega_s - \omega_{s+1} - \omega_{s+2} - \omega_{s+3} & \\ -\omega_s - \omega_{s+1} - \omega_{s+3+j} & j \geq 0 \\ -\omega_s - \omega_{s+2+j} - \omega_{s+3+j} & j \geq 0 \end{cases}$$

Almost the same calculations made for the case the B_n imply that Σ_λ is not Fano.

6.5. **Explicit calculations for D_n, $n \geq 4$.**

Case $\lambda = -\omega_1$

First, we calculate the $W_{-\omega_1}$-orbits of $-\omega_{n-1}, -\omega_n$:

$$W_{-\omega_1} \cdot (-\omega_{n-1}) = \Big\{ {}^1\!/_2 \big(-\varepsilon_1 - \sum_{i=2}^n a_i\varepsilon_i\big) : a_i = \pm 1, \prod_i a_i = -1 \Big\}$$

and

$$W_{-\omega_1} \cdot (-\omega_n) = \Big\{ {}^1\!/_2 \big(-\varepsilon_1 - \sum_{i=2}^n a_i\varepsilon_i\big) : a_i = \pm 1, \prod_i a_i = 1 \Big\}.$$

It follows that

$$\{-\omega_1, \dots, -\omega_{n-2}\} \subset \mathbb{R}^+ \big\langle W_{-\omega_1} \cdot (-\omega_{n-1}) \cup W_{-\omega_1} \cdot (-\omega_n) \big\rangle = \mathbb{R}^+ \Big\langle {}^1\!/_2 \big(-\varepsilon_1 - \sum_{i=2}^n a_i\varepsilon_i\big) : a_i = \pm 1 \Big\rangle.$$

Hence,

$$\operatorname{Prim}(\sigma_{-\omega_1}) = W_{-\omega_1} \cdot (-\omega_{n-1}) \cup W_{-\omega_1} \cdot (-\omega_n)$$

and it easily follows that

$$\big\langle \operatorname{Prim}(\sigma_{-\omega_1})\big\rangle_{\text{aff}} = -\omega_n + \langle \alpha_2, \dots, \alpha_n\rangle_{\mathbb{R}} = -\omega_n + \big\langle \varepsilon_2, \dots, \varepsilon_n\big\rangle_{\mathbb{R}}.$$

Thus, $n_\lambda = \omega_1$, and $\Sigma_{-\omega_1}$ is Gorenstein-Fano – clearly, $\Sigma_{-\omega_1}$ is not smooth.

Case $\lambda = -\omega_{n-1}$

If $n = 4$ it is clear that $\Sigma_{-\omega_3}$ is isomorphic to $\Sigma_{-\omega_1}$; in particular, $\Sigma_{-\omega_3}$ is Gorenstein-Fano. Assume that $n > 4$. Since

$$W_{-\omega_{n-1}} \cdot (-\omega_n) = \{-\omega_n\} \cup \Big\{ -{}^1\!/_2 \big(\sum_{i=1}^{n-1} a_i\varepsilon_i\big) + \varepsilon_n : a_i = \pm 1, \sum a_i = n-3 \Big\}$$

and that $W_{-\omega_{n-1}} \cdot (-\omega_1) = \big\{-\varepsilon_1, \dots, -\varepsilon_{n-1}\big\}$, it follows that

$$\operatorname{Prim}(\sigma_{-\omega_{n-1}}) = W_{-\omega_{n-1}} \cdot (-\omega_1) \cup W_{-\omega_{n-1}} \cdot (-\omega_n),$$

and

$$\big\langle \operatorname{Prim}(\sigma_{-\omega_{n-1}})\big\rangle_{\text{aff}} = -\omega_1 + \big\langle \alpha_1, \dots, \alpha_{n-2}, \alpha_n, \omega_n - \omega_1\big\rangle_{\mathbb{R}}.$$

Hence, $n_\lambda = a\omega_{n-1}$ verifies the additional condition $\langle n_\lambda, \omega_n - \omega_1\rangle = 0$; that is $(n-4)a = 0$. Hence, $\dim F_{\mathcal{C}} = n$ and $\Sigma_{-\omega_{n-1}}$ is not Fano.

Case $\lambda = -\omega_n$

This case is dual to the $\lambda = -\omega_{n-1}$ case: if $n = 4$ then $\Sigma_{-\omega_4} \cong \Sigma_{-\omega_1}$, a Gorenstein-Fano singular toric variety. If $n \neq 4$, then $\mathcal{C} \cap \operatorname{Prim}(\sigma_{-\omega_n}) = \{-\omega_1,\text{-}\omega_{n-1}\}$, $\big\langle \operatorname{Prim}(\sigma_{-\omega_n})\big\rangle_{\text{aff}} = \mathbb{R}^n$ and therefore $\Sigma_{-\omega_n}$ is not Fano.

Case $\lambda = -\omega_j,\ 1 < j \leq n-2$

In this case, $W_{-\omega_j}\cdot(-\omega_1) = \big\{-\varepsilon_2, \dots, -\varepsilon_j\big\}$,

$$W_{-\omega_j}\cdot(-\omega_{n-1}) = \Big\{{}^1\!/\!_2\big(-\omega_j - \sum_{i=j+1}^{n} a_i\varepsilon_i\big) : a_i = \pm 1, \prod_i a_i = -1\Big\}$$

and

$$W_{-\omega_j}\cdot(-\omega_{n}) = \Big\{{}^1\!/\!_2\big(-\omega_j - \sum_{i=j+1}^{n} a_i\varepsilon_i\big) : a_i = \pm 1, \prod_i a_i = 1\Big\}.$$

It follows that

$$\operatorname{Prim}(\sigma_{-\omega_j}) = W_{-\omega_j}\cdot(-\omega_1) \cup W_{-\omega_j}\cdot(-\omega_{n-1}) \cup W_{-\omega_j}\cdot(-\omega_n)$$

and

$$\begin{aligned}\big\langle \operatorname{Prim}(\sigma_{-\omega_1})\big\rangle_{\text{aff}} = {} & -\omega_1 + \big\langle \alpha_1, \dots, \alpha_{j-1}, \varepsilon_{j+1}, \dots, \varepsilon_n, {}^1\!/\!_2(-\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)\big\rangle_{\mathbb{R}} \\ & -\omega_1 + \big\langle \alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_n, -\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n\big\rangle_{\mathbb{R}}.\end{aligned}$$

Hence, $n_\lambda = a\omega_j$, with $\langle a\omega_j, -\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n\rangle = 0$. It follows that if $j \neq 2$, then $\Sigma_{-\omega_i}$ is not Fano. If $j = 2$, then $n_\lambda = 2\omega_2$ and $\Sigma_{-\omega_2}$ is Gorenstein-Fano. Note that $X_{\Sigma_{-\omega_2}}$ is smooth if and only if $n = 4$.

Case $\lambda = -\omega_s - \omega_r,\ s < r \leq n-2$

In this case, $-\omega_1, -\omega_{n-1}, -\omega_n \in \operatorname{Prim}(\sigma_\lambda)$, with orbits

$$\begin{aligned} W_\lambda(-\omega_1) = {} & \{-\varepsilon_1, \dots, -\varepsilon_s\}, \\ W_\lambda\cdot(-\omega_{n-1}) = {} & \Big\{{}^1\!/\!_2\big(-\omega_r - \textstyle\sum_{i=r+1}^{n} a_i\varepsilon_i\big) : a_i = \pm 1, \prod_i a_i = -1\Big\}, \\ W_\lambda\cdot(-\omega_{n}) = {} & \Big\{{}^1\!/\!_2\big(-\omega_r - \textstyle\sum_{i=r+1}^{n} a_i\varepsilon_i\big) : a_i = \pm 1, \prod_i a_i = 1\Big\}, \\ W_\lambda\cdot(-\omega_{j}) = {} & \Big\{-\omega_s - \textstyle\sum_{i=s+1}^{r} a_i\varepsilon_i\big) : a_i = 0,1, 0 < \sum a_i = j-s\Big\} \quad s < j < r \end{aligned}$$

It follows that $\{-\omega_2, \dots - \omega_s, -\omega_r, \dots, -\omega_{n-2}\} \subset \mathbb{R}^+\big\langle W_\lambda(-\omega_1) \cup W_\lambda(-\omega_{n-1}) \cup W_\lambda(-\omega_n)\big\rangle$.

On the other hand, $-\omega_i \in \operatorname{Prim}(\sigma_\lambda)$ for $i = s+1, \dots, r-1$. Indeed, let $\{v_1, \dots, v_{2^{n-r-1}}\} = W_\lambda\cdot(-\omega_{n-1})$, with $v_j = {}^1\!/\!_2\big(-\omega_r - \sum_{i=r+1}^{n} a_{ji}\varepsilon_i\big)$, $\{w_1, \dots, w_{2^{n-r-1}}\} = W_\lambda\cdot(-\omega_n)$, with $w_j = {}^1\!/\!_2\big(-\omega_r - \sum_{i=r+1}^{n} b_{ji}\varepsilon_i\big)$, and $\{v_{j1}, \dots, v_{j,\binom{r-s}{j}}\} = W_\lambda\cdot(-\omega_j)$, with $v_{jk} = -\omega_s - \sum_{i=s+1}^{r} a_{jki}\varepsilon_i$. Assume that $= \omega_h$, $s+1 \leq h \leq r < n-1$ is a linear combination with positive coefficients of the form

$$-\omega_h = \sum_{j=1}^{s} c_j(-\varepsilon_j) + \sum_{jk} c_{jk}v_{jk} + \sum_j d_jv_j + \sum e_jw_j$$

Then,

$$\begin{cases} (1,\ i=1,\dots,s) & c_i+\sum\limits_{jk} c_{jk}+\sum\limits_{j}\dfrac{d_j}{2}+\sum\limits_{j}\dfrac{e_j}{2} = 1\\ (2,\ i=s+1,\dots,h) & \sum\limits_{jk:a_{jki}=1} c_{jk}+\sum\limits_{j}\dfrac{d_j}{2}+\sum\limits_{j}\dfrac{e_j}{2} = 1\\ (3,\ i=h+1,\dots,r) & \sum\limits_{jk:a_{jki}=1} c_{jk}+\sum\limits_{j}\dfrac{d_j}{2}+\sum\limits_{j}\dfrac{e_j}{2} = 0\\ (4,\ i=r+1,\dots,n) & \sum\limits_{jk}\dfrac{d_ja_{jk}}{2}+\sum\limits_{jk}\dfrac{e_jb_{jk}}{2} = 0 \end{cases}$$

We deduce from equation (3,i) that $d_j = e_j = 0$ for all $j = 1,\dots,2^{n-r-1}$, and $c_{jk} = 0$ if $c_{jki} = 1$ for some $i = h+1,\dots,r$. From equations (1,i) and (2,i') we deduce that $c_j = 0$ for $j = 1,\dots,s$ and $c_{jk} = 0$ if $c_{jki} = 0$ for some $i = s+1,\dots,h$. It follows that $c_{jk} = 0$ unless $v_{jk} = \omega_j$; therefore, $-\omega_h \in \mathrm{Prim}(\sigma_\lambda)$.

Thus,

$$\mathcal{C}\cap \mathrm{Prim}(\sigma_\lambda) = \big\{-\omega_1,-\omega_{s+1},\dots,-\omega_{r-1},-\omega_{n-1},-\omega_n\big\}$$

$$\big\langle \mathrm{Prim}(\sigma_\lambda)\big\rangle_{\mathrm{aff}} = -\omega_1+\left\langle \begin{array}{c}\alpha_1,\dots,\alpha_{s-1},\alpha_{s+1},\dots,\alpha_{r-1},\alpha_{r+1},\dots,\alpha_n,\\ \omega_{s+1}-\omega_1,\dots,\omega_{r-1}-\omega_1,\omega_n-\omega_1\end{array}\right\rangle_{\mathbb{R}},$$

Therefore, Σ_λ is not Fano.

Case $\lambda = -\omega_s-\omega_{n-1},\ s\leq n-2$

In this case,

$$\begin{aligned} W_\lambda(-\omega_1) ={}& \{-\varepsilon_1,\dots,-\varepsilon_s\},\\ W_\lambda\cdot(-\omega_{n-1}) ={}& \{-\omega_{n-1}\}\\ W_\lambda\cdot(-\omega_n) ={}& \Big\{-\omega_n, -1/2\big(\omega_s+\textstyle\sum_{i=s+1}^{n-1} a_i\varepsilon_i-\varepsilon_n\big) : a_i=\pm1, \sum a_i = n-s-3\Big\},\\ W_\lambda\cdot(-\omega_j) ={}& \Big\{-\omega_s-\textstyle\sum_{i=s+1}^{n-1} a_i\varepsilon_i : a_i = 0,1, \sum a_i = j-s\Big\} \quad s<j\leq n-2 \end{aligned}$$

It follows that $-\omega_{n-2} = -\omega_n + s_{\alpha_n}(-\omega_n) \notin \mathrm{Prim}(\sigma_\lambda)$. If $s<j<n-2$, by considerations similar to the ones made in previous cases, we deduce that $-\omega_j \in \mathrm{Prim}(\sigma_\lambda)$. Therefore,

$$\mathcal{C}\cap\mathrm{Prim}(\sigma_\lambda) = \{-\omega_1,-\omega_{s+1},\dots-\omega_{n-3},-\omega_{n-1},-\omega_n\}$$

$$\big\langle\mathrm{Prim}(\sigma_\lambda)\big\rangle_{\mathrm{aff}} = -\omega_1+\left\langle\begin{array}{c}\alpha_1,\dots,\alpha_{s-1},\alpha_{s+1},\dots,\alpha_{n-2},\alpha_n,\\ \varepsilon_{s+1},\omega_n-\omega_1,\omega_{s+1}-\omega_1,\varepsilon_n\end{array}\right\rangle_{\mathbb{R}} = \mathbb{R}^n.$$

It follows that Σ_λ is not Fano.

Case $\lambda = -\omega_s-\omega_n,\ s\leq n-2$

This case is dual to the previous one: $\mathcal{C}\cap\mathrm{Prim}(\sigma_\lambda) = \{-\omega_1,-\omega_{s+1},\dots-\omega_{n-3},-\omega_{n-1},-\omega_n\}$ and Σ_λ is not Fano.

Case $\lambda = -\omega_{n-1}-\omega_n$

It is easy to see that $\mathrm{Prim}(\sigma_\lambda) = \big\{-\varepsilon_1,\dots,-\varepsilon_{n-1},-\omega_{n-1},-\omega_n\big\}$. It follows that $\big\langle\mathrm{Prim}(\sigma_\lambda)\big\rangle_{\mathrm{aff}} = -\omega_1+\big\langle\alpha_1,\dots,\alpha_{n-2},\omega_n-\omega_1,\varepsilon_n\big\rangle = \mathbb{R}^n$. Therefore, Σ_λ is not Fano.

CASE $\lambda = -\omega_s - \omega_{s+1} - \omega_{s+2},\ s+2 \leq n-2$

In this case, $\mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda) = \{-\omega_1, -\omega_{s+1}, -\omega_{n-1}, -\omega_n\}$ and

$$\langle \mathrm{Prim}(\sigma_\lambda)\rangle_{\mathrm{aff}} = -\omega_1 + \langle \alpha_1, \dots, \alpha_{s-1}, \alpha_{s+3}, \dots, \alpha_n, \omega_n - \omega_1, \omega_{s+1} - \omega_1, \rangle.$$

It is easily checked that Σ_λ is not Fano.

CASE $\lambda = -\omega_{n-3} - \omega_{n-2} - \omega_{n-1}$

Since $-\omega_{n-2} = -\omega_n - s_{\alpha_n}(-\omega_n)$, it follows that $\mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda) = \{-\omega_1, -\omega_{n-1}, -\omega_n\}$ and

$$\langle \mathrm{Prim}(\sigma_\lambda)\rangle_{\mathrm{aff}} = -\omega_1 + \langle \alpha_1, \dots, \alpha_{n-4}, \alpha_n, \omega_n - \omega_1, \varepsilon_n \rangle.$$

Thus, Σ_λ is not Fano.

CASE $\lambda = -\omega_{n-3} - \omega_{n-2} - \omega_n$

This case is dual to the previous one: $\mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda) = \{-\omega_1, -\omega_{n-1}, -\omega_n\}$ and Σ_λ is not Fano.

CASE $\lambda = -\omega_{n-2} - \omega_{n-1} - \omega_n$

In this case $\mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda) = \{-\omega_1, -\omega_{n-1}, -\omega_n\}$ and

$$\langle \mathrm{Prim}(\sigma_\lambda)\rangle_{\mathrm{aff}} = -\omega_1 + \langle \alpha_1, \dots, \alpha_{n-3}, \omega_n - \omega_1, \varepsilon_n \rangle.$$

It easily follows that $(n-4)a + {}^1\!/\!_2(n-4+n-2)b = 0$. Hence, Σ_λ is not Fano.

CASE $\lambda = -\omega_s - \omega_{n-1} - \omega_n,\ s \leq n-3$

In this case, $\mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda) = \{-\omega_1, -\omega_{s+1}, \dots, -\omega_{n-2}, -\omega_{n-1}, -\omega_n\}$, with $W_\lambda(-\omega_j) = \{-\omega_s - \sum_{i=s+1}^{n-1} a_i\varepsilon_i, a_i = 0, 1 : a_i = j - s\}$ if $s < j < n-2$. It follows that

$$\langle \mathrm{Prim}(\sigma_\lambda)\rangle_{\mathrm{aff}} = -\omega_1 + \langle \alpha_1, \dots, \alpha_{s-1}, \alpha_{s+2}, \dots, \alpha_{n-2}, \omega_n - \omega_1, \omega_{s+1} - \omega_1, \varepsilon_n \rangle.$$

It follows that Σ_λ is not Fano.

CASE $\lambda = \sum_{i=s_1}^{r_1} -\omega_i + \dots + \sum_{i=s_\ell}^{r_\ell} -\omega_i,\ r_i + 1 < s_{i+1},\ r_\ell - s_1 > 3,\ \#I_\lambda \geq 3,\ r_\ell \leq n-1$

We deduce from the preceding cases that

$\mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda) = \{-\omega_1, -\omega_{s+1}, \dots, -\omega_{n-3}, -\omega_{n-1}, -\omega_n\}$

$$\langle \mathrm{Prim}(\sigma_\lambda)\rangle_{\mathrm{aff}} = -\omega_1 + \left\langle \begin{array}{c} \alpha_1, \dots, \alpha_{s_1-1}, \alpha_{r_1+1}, \dots, \alpha_{s_2-1}, \alpha_{r_2+1}, \dots, \alpha_{s_\ell-1}, \\ \alpha_{r_\ell+1}, \dots, \alpha_n, \varepsilon_{s_1+2}, \dots, \varepsilon_{r_\ell-1}, \varepsilon_n, \omega_{s_1+1} - \omega_1, \omega_n - \omega_1 \end{array} \right\rangle_{\mathbb{R}}.$$

An easy calculation shows that $\langle \mathrm{Prim}(\sigma_\lambda)\rangle_{\mathrm{aff}} = \mathbb{R}^n$; therefore Σ_λ is not Fano.

CASE $\lambda = \sum_{i=s_1}^{r_1} -\omega_i + \dots + \sum_{i=s_\ell}^{r_\ell} -\omega_i,\ r_i + 1 < s_{i+1},\ r_\ell - s_1 > 3,\ \#I_\lambda \geq 3,\ r_\ell = n-1$

In this case

$$\mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda) = \{-\omega_1, -\omega_{s_1+1}, \dots, -\omega_n\}$$

$$\langle \mathrm{Prim}(\sigma_\lambda) \rangle_{\mathrm{aff}} = -\omega_1 + \left\langle \begin{matrix} \alpha_1, \dots, \alpha_{s_1-1}, \alpha_{r_1+1}, \dots, \alpha_{s_2-1}, \alpha_{r_2+1}, \dots, \alpha_{s_\ell-1}, \\ \alpha_{r_\ell+1}, \dots, \alpha_n, \varepsilon_{s_1+2}, \dots, \varepsilon_n, \omega_{s_1+1} - \omega_1, \omega_n - \omega_1 \end{matrix} \right\rangle_{\mathbb{R}} = \mathbb{R}^n,$$

It follows that Σ_λ is not Fano.

Case $\lambda = \sum_{i=s_1}^{r_1} -\omega_i + \cdots + \sum_{i=s_\ell}^{r_\ell} -\omega_i - \omega_n,\ r_i + 1 < s_{i+1},\ r_\ell - s_1 > 3,\ \#I_\lambda \geq 3,$ $r_\ell \leq n-2$

This case is dual to the previous one: Σ_λ is not Fano.

Case $\lambda = \sum_{i=s_1}^{r_1} -\omega_i + \cdots + \sum_{i=s_\ell}^{r_\ell} -\omega_i - \omega_{n-1} - \omega_n,\ r_i + 1 < s_{i+1},\ r_\ell - s_1 > 3,$ $\#I_\lambda \geq 3,\ r_\ell \leq n-2$

Since $\{s_1, n-1\} \subset I_\lambda$, it follows that

$$\mathcal{C} \cap \mathrm{Prim}(\sigma_\lambda) = \{-\omega_1, -\omega_{s_1+1}, \dots, -\omega_n\}$$

$$\langle \mathrm{Prim}(\sigma_\lambda) \rangle_{\mathrm{aff}} = -\omega_1 + \left\langle \begin{matrix} \alpha_1, \dots, \alpha_{s_1-1}, \alpha_{r_1+1}, \dots, \alpha_{s_2-1}, \alpha_{r_2+1}, \dots, \alpha_{s_\ell-1} \\ \varepsilon_{s_1+2}, \dots, \varepsilon_n, \omega_{s_1+1} - \omega_1, \omega_n - \omega_1 \end{matrix} \right\rangle_{\mathbb{R}} = \mathbb{R}^n,$$

and we deduce that Σ_λ is not Fano.

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