Han's conjecture and Hochschild homology for null-square projective algebras

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Abstract

Let $\mathcal{H}$ be the class of algebras verifying Han's conjecture. In this paper we analyse two types of algebras with the aim of providing an inductive step towards the proof of this conjecture. Firstly we show that if an algebra Λ is triangular with respect to a system of non necessarily primitive idempotents, and if the algebras at the idempotents belong to $\mathcal{H}$, then Λ is in $\mathcal{H}$. Secondly we consider a 2×2 matrix algebra, with two algebras on the diagonal, two projective bimodules in the corners, and zero corner products. They are not triangular with respect to the system of the two diagonal idempotents. However, the analogous result holds, namely if both algebras on the diagonal belong to $\mathcal{H}$, then the algebra itself is in $\mathcal{H}$.

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1 Introduction

In this paper, a *smooth* finite dimensional algebra over a field k is a finite dimensional algebra of finite global dimension. The word "smooth" is originated in commutative algebra and is convenient for brevity. Observe that in [11], for finite dimensional algebras, "smooth" correspond to algebras of global dimension at most one, that is, hereditary or semisimple algebras.

In 2006, Y. Han conjectured in [13] that a finite dimensional algebra whose Hochschild homology vanishes in large enough degrees is smooth. In the same paper Y. Han proved the conjecture for monomial algebras, while in [4] P.A. Bergh and D. Madsen proved it in characteristic zero for graded local algebras, Kozsul algebras and graded cellular algebras. Recently, the same authors showed in [6] that trivial extensions of selfinjective algebras, local algebras and graded algebras have infinite Hochschild homology, a result which confirms Han's conjecture for these algebras. Observe that in the 90's, the work of the Buenos Aires Cyclic Homology Group [8], and of L. Avramov and M. Vigué-Poirrier [1] provided the result for finitely generated commutative algebras.

In relation with Han's conjecture, lower bounds are obtained in [5] for the dimension of the Hochschild homology groups of fiber products of algebras, trivial extensions, path algebras of quivers containing loops and quantum complete intersections. Note that P.A. Bergh and K. Erdmann proved in [2] that quantum

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complete intersections - not at a root of unity - satisfy Han's conjecture. In [19] A. Solotar and M. Vigué-Poirrier proved Han's conjecture for a generalization of quantum complete intersections and for a family of algebras which are in a sense opposite to these. Moreover in [18], A. Solotar, M. Suárez-Alvarez and Q. Vivas considered quantum generalized Weyl algebras and proved Han's conjecture for these algebras (out of a few exceptional cases).

In this paper we consider null-square algebras over a field k, that is algebras Λ of the form

$$\begin{pmatrix} A & N \\ M & B \end{pmatrix}$$

where A and B are k-algebras, M and N are bimodules, and the product is given by matrix multiplication subject to $MN = 0 = NM$. For these algebras, $I = M \oplus N$ is a two-sided ideal verifying $I^2 = 0$ and $C = A \times B$ is a subalgebra. Actually $\Lambda = C \oplus I$, that is, Λ is a cleft singular extension (see [16, p. 284]).

Hochschild homology is a functor HH_* from k-algebras to graded vector spaces. Hence for a null-square algebra, $HH_*(C)$ is a direct summand of $HH_*(\Lambda)$. Moreover, note that $HH_*(C) = HH_*(A) \oplus HH_*(B)$.

In relation to Han's conjecture, this paper treats two opposite cases, one corresponds to quivers without cycles, while in the other case the quiver contains cycles. Both of them aim to provide an inductive step towards proving the conjecture. In Section 2 we consider algebras which are E-triangular, that is, they do not have oriented cycles with respect to a complete system E of non necessarily primitive orthogonal idempotents - for brevity we call such a set E a "system". In Sections 3 and 4, on the contrary, we study a case where there is an oriented cycle. In this last case our analysis requires the involved bimodules to be projective.

A null-square algebra with $N = 0$ will be called a corner algebra. For these algebras $HH_*(\Lambda) = HH_*(C)$ by a direct computation that we briefly recall in Section 2, see also [15] or [10]. Moreover we show that if a corner algebra is finite dimensional, with A and B smooth, then the corner algebra is also smooth. This leads to our first result, namely corner algebras built on the class of algebras $\mathcal{H}$ verifying Han's conjecture, also belong to $\mathcal{H}$. Note that no extra assumption on M is required in the foregoing.

Based on the previous results, we go further. To a system E of a k-algebra Λ, we associate its E-quiver: the set of vertices is E, and for $x \neq y$ elements of E, there is an arrow from x to y if $y\Lambda x \neq 0$. If the E-quiver has no oriented cycles then Λ is called E-triangular. For instance the E-quiver of a corner algebra with respect to the system E given by the two diagonal idempotents is an arrow if $M \neq 0$. We show that if Λ is E-triangular, then there is a decomposition $HH_*(\Lambda) = \bigoplus_{x\in E} HH_*(x\Lambda x)$. Moreover for a finite dimensional E-triangular algebra Λ such that $x\Lambda x$ is smooth for all $x \in E$, the algebra is also smooth. We infer that finite dimensional E-triangular algebras built on the class $\mathcal{H}$ also belong to $\mathcal{H}$, without requiring additional assumptions on the bimodules $y\Lambda x$.

In Section 3, we consider null-square algebras Λ with non zero bimodules M and N, in other words the E-quiver with respect to the two diagonal idempotents is $\cdot \rightleftarrows \cdot$. If M and N are projective bimodules, Λ is called a null-square projective algebra. We provide a long exact sequence computing $HH_*(\Lambda)$, which is associated to the short exact sequence obtained from the product map $\Lambda \otimes_C \Lambda \to \Lambda$. We obtain a projective resolution of the kernel $K^1_C(\Lambda)$ of this map, which enables us to compute $\mathrm{Tor}_*^{\Lambda\text{-}\Lambda}(K^1_C(\Lambda),\ \Lambda)$ through invariants or coinvariants of a natural action of cyclic groups on the zero degree Hochschild homology of tensor powers of $N \otimes_A M$ and $M \otimes_B N$. We thus obtain the long exact sequence of Theorem 3.14.

In Section 4, we first consider a specific complex for the Hochschild homology of a null-square algebra Λ, without any further restrictions neither on the algebras nor on the bimodules. This complex is obtained through the separable subalgebra $k \times k$ given by the diagonal idempotents and allows to prove that if $HH_*(\Lambda)$ vanishes in large enough degrees, then the Hochschild homology in degree 0 which coefficients are tensor powers of $N \otimes_A M$ and of $M \otimes_B N$, vanish for large enough exponents. If A and B are basic over a perfect field, then we prove that actually the homology in degree 0 with coefficients in any tensor power of $N \otimes_A M$ and of $M \otimes_B N$ vanish. The long exact sequence obtained before provides $HH_*(\Lambda) = HH_*(A \times B)$. We infer that the tensor powers of $N \otimes_A M$ and of $M \otimes_B N$ vanish in large enough degrees. Observe that $H_0\left(A, (N \otimes_B M)^{\otimes_A *}\right)$ is related to 2-truncated cycles, namely cycles in the Gabriel quiver of a basic algebra in which the product of any two consecutive arrows is zero, as considered in [3] in order to guarantee that Hochschild homology is infinite dimensional.

Another important result that we obtain in this section is the following: for a perfect field k, let Λ be a finite dimensional null-square projective k-algebra, where A and B are basic and smooth. Assuming the bimodules verify $(N \otimes_B M)^{\otimes_A *} = 0$ for large enough exponents, the algebra Λ is also smooth. The proof relies on the construction of an explicit projective resolution obtained through successive cones of the identity. A main result of this paper follows, namely a finite dimensional null-square projective algebra built on the class of basic algebras in $\mathcal{H}$ also belongs to $\mathcal{H}$.

In the last section we give a presentation by quiver and relations of a null-square projective algebra, starting from the same type of presentations of A and B. This is useful for producing examples where our results apply.

2 Han's conjecture for corner and E-triangular algebras

In this section we first consider null-square algebras and their category of representations. Next, we will study corner algebras which are particular cases of null-square algebras, in relation with Han's conjecture. The results that we obtain in this section for corner (and then for triangular algebras) do not require a projectivity hypothesis on the bimodules considered in the definition of a null-square algebra below.

Definition 2.1 *Let k be a field and let A and B be k-algebras. Let M and N be respectively a $B-A$-bimodule and an $A-B$-bimodule. The corresponding* null-square algebra *is*

$$\begin{pmatrix} A & N \\ M & B \end{pmatrix}$$

where the product is given by matrix multiplication using the products of A and B, the bimodule structures of M and N, and setting $mn = 0$ and $nm = 0$ for all $m \in M$ and $n \in N$.

Remark 2.2 *A* square algebra *is an algebra*

$$\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$$

as before, with two bimodule maps $\alpha : N \otimes_B M \to A$ and $\beta : M \otimes_A N \to B$ verifying the obvious "associativity" conditions that ensure the associativity of the corresponding matrix product on Λ. A null-square algebra is a square algebra where $\alpha = 0 = \beta$. Observe that in [7], R.-O. Buchweitz studies square algebras which are called "(generalised) Morita context" or "pre-equivalence", and focus on the case where α or β are surjective.

Example 2.3 *Let Λ be a k-algebra with a decomposition $\Lambda = P \oplus Q$ as a right Λ-module. Then Λ is a square algebra of the form*

$$\begin{pmatrix} \mathsf{End}_\Lambda P & \mathsf{Hom}_\Lambda(Q,P) \\ \mathsf{Hom}_\Lambda(P,Q) & \mathsf{End}_\Lambda Q \end{pmatrix}$$

If for all $f \in \mathsf{Hom}_\Lambda(P,Q)$ and for all $g \in \mathsf{Hom}_\Lambda(Q,P)$ the compositions gf and fg are zero, the algebra is null-square.

Remark 2.4 *Any square algebra Λ is obtained as above by considering the right module decomposition:*

$$\begin{pmatrix} A & N \\ M & B \end{pmatrix} = \begin{pmatrix} A & N \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ M & B \end{pmatrix}.$$

Recall that a *cleft singular extension algebra* (see [16, p. 284]) is an algebra Λ with a decomposition $\Lambda = C \oplus I$, where C is a subalgebra and I is a two-sided ideal of Λ verifying $I^2 = 0$. A null-square algebra $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ is an instance of a cleft singular extension with $C = A \times B$ and $I = M \oplus N$. Indeed, I is a two-sided ideal precisely because $MN = NM = 0$.

We will next consider systems of idempotents of an arbitrary algebra in order to recall the representation theory of a null-square algebra.

Definition 2.5 *Let Λ be a k-algebra. A* system *of Λ is a finite set E of non zero orthogonal idempotents which is complete,* i.e. $\sum_{x\in E} x = 1$. *The system is trivial if $E = \{1\}$.*

Observe that in the above definition we do not require the idempotents to be primitive. To a system E of a k-algebra Λ we associate a k-category $\mathcal{C}_{\Lambda,E}$ as follows: its objects are the elements of E while the vector space ${}_y(\mathcal{C}_{\Lambda,E})_x$ of morphisms from x to y is $y\Lambda x$. The composition is provided by the product of Λ. Of course Λ is recovered as the direct sum of all the morphisms spaces of $\mathcal{C}_{\Lambda,E}$, endowed with the matrix product. It is well known and easy to prove that the k-categories of left Λ-modules, and of k-functors from $\mathcal{C}_{\Lambda,E}$ to k-vector spaces, are isomorphic.

Let now $\mathcal{C}$ be a small k-category, with set of objects $\mathcal{C}_0$. Notice that a k-functor $\mathcal{M}$ from $\mathcal{C}$ to k-vector spaces is given by a family of vector spaces $\{{}_x\mathcal{M}\}_{x\in\mathcal{C}_0}$ and a collection of linear maps

$${}_y\mathcal{C}_x \otimes {}_x\mathcal{M} \xrightarrow{\;{}_ym_x\;} {}_y\mathcal{M}$$

such that, for any objects x, y and z, the following diagram commutes:

$$\begin{array}{ccc} {}_z\mathcal{C}_y \otimes {}_y\mathcal{C}_x \otimes {}_x\mathcal{M} & \xrightarrow{1\otimes {}_ym_x} & {}_z\mathcal{C}_y \otimes {}_y\mathcal{M} \\ \downarrow{\scriptstyle c\otimes 1} & & \downarrow{\scriptstyle {}_zm_y} \\ {}_z\mathcal{C}_x \otimes {}_x\mathcal{M} & \xrightarrow[{}_zm_x]{} & {}_z\mathcal{M} \end{array}$$

Next we define a k-category, which will be isomorphic to the category of left modules over a square algebra.

Definition 2.6 *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a square algebra. The objects of the linear category* $\mathcal{S}(\Lambda)$ *are* $X \underset{\nu}{\overset{\mu}{\rightleftharpoons}} Y$, *where* X *is an* A*-module,* Y *is a* B*-module,* $X \overset{\mu}{\rightharpoonup} Y$ *stands for a map of* B*-modules* $\mu : M \otimes_A X \to Y$ *and analogously* $X \underset{\nu}{\leftharpoondown} Y$ *stands for a map of* A*-modules* $\nu : N \otimes_B Y \to X$ *which verify*

$$\nu(1_N \otimes \mu) = \alpha \otimes 1_X \text{ and } \mu(1_M \otimes \nu) = \beta \otimes 1_Y. \tag{2.1}$$

Note that we identify the vector spaces $A \otimes_A X$ *and* X *through the canonical isomorphism, as well as* $Y \otimes_B B$ *and* Y.

A morphism in $\mathcal{S}(\Lambda)$ *from* $X \underset{\nu}{\overset{\mu}{\rightleftharpoons}} Y$ *to* $X' \underset{\nu'}{\overset{\mu'}{\rightleftharpoons}} Y'$ *is a couple* (φ, ψ) *where* $\varphi : X \to X'$ *is a morphism of* A*-modules,* $\psi : Y \to Y'$ *is a morphism of* B*-modules such that the following diagrams commute:*

$$\begin{array}{ccc} M \otimes_A X & \xrightarrow{\mu} & Y \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ M \otimes_A X' & \xrightarrow[\mu']{} & Y' \end{array} \qquad \begin{array}{ccc} X & \xleftarrow{\nu} & N \otimes_B Y \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle 1\otimes\psi} \\ X' & \xleftarrow[\nu']{} & N \otimes_B Y' \end{array}$$

Proposition 2.7 *Let* Λ *be a square algebra. The category of left* Λ*-modules is isomorphic to* $\mathcal{S}(\Lambda)$.

Proof. Consider the complete set of orthogonal idempotents $E = \{e, 1-e\}$ of Λ, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The result is an immediate consequence of the previous observations. $\diamond$

In what follows the categories of the above proposition will be identified. Note that for a null-square algebra, the equalities (2.1) become

$$\nu(1_N \otimes \mu) = 0 \text{ and } \mu(1_M \otimes \nu) = 0. \tag{2.2}$$

Lemma 2.8 *Let* Λ *be a square algebra and let* P *be a projective* A*-module. The* Λ*-module* $\left(P \underset{\alpha}{\overset{1}{\rightleftharpoons}} M \otimes_A P\right)$ *is projective.*

Proof. Let Λ_1 be the $\Lambda - A$-bimodule given by the first column of Λ, that is, $\Lambda_1 = \begin{pmatrix} A & 0 \\ M & 0 \end{pmatrix} = \left(A \underset{\alpha}{\overset{1}{\rightleftharpoons}} M\right)$. Note that $\Lambda_1 = \Lambda e$ is a projective Λ-module. Moreover, if X is an A-module, $\Lambda_1 \otimes_A X = \left(X \underset{\alpha\otimes 1_X}{\overset{1}{\rightleftharpoons}} M \otimes_A X\right)$. Since $\Lambda_1 \otimes_A A$ is isomorphic to Λ_1, we infer that $\Lambda_1 \otimes_A P$ is a projective Λ-module. $\diamond$

The analogous result holds for B-modules.

From now on we focus on null-square algebras.

Proposition 2.9 *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a null-square algebra where* A, B, M *and* N *are finite dimensional. A simple* Λ*-module is isomorphic to*

$$S \rightleftharpoons 0 \text{ or } 0 \rightleftharpoons T$$

where S *and* T *are simple* A *and* B*-modules respectively.*

Proof. We assert that the Jacobson radical of Λ is $\begin{pmatrix} \operatorname{rad} A & N \\ M & \operatorname{rad} B \end{pmatrix}$ where $\operatorname{rad} A$ and $\operatorname{rad} B$ are the Jacobson radicals of A and B. Indeed, this vector space is a nilpotent two-sided ideal, and the quotient of Λ by it is semisimple. $\diamond$

Definition 2.10 *A* corner algebra Λ *is a square algebra with* $N = 0$*. In this case, the objects of* $\mathcal{S}(\Lambda)$ *are denoted by* $X \overset{\mu}{\rightharpoonup} Y$*.*

In this Section we consider Han's conjecture for corner algebras first, and secondly for E-triangular algebras which will be defined below. We emphasize that in what follows, that is, for corner algebras we do not make any hypothesis on the projectivity of M. First we recall the following result.

Proposition 2.11 *[12, Proposition 10, p.86] Let* A *and* B *be finite dimensional smooth* k*-algebras. The* k*-algebra* $A \otimes B$ *is smooth.*

Theorem 2.12 *Let* $\Lambda = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ *be a corner finite dimensional algebra, where* M *is a* $B-A$*-bimodule. If* A *and* B *are smooth, then* Λ *is smooth.*

Proof. It is well known that if a finite dimensional algebra A is smooth the same holds for A^{op}. By the previous proposition, $B \otimes A^{\mathsf{op}}$ is smooth.

Let $0 \to Q_q \to \cdots \to Q_1 \to Q_0 \to M \to 0$ be a finite resolution of M by projective $B-A$-bimodules.

Firstly let $S \rightharpoonup 0$ be a simple Λ-module where S a simple A-module. Let $0 \to P_p \to \cdots \to P_1 \to P_0 \to S \to 0$ be a resolution of S by projective A-modules. Observe that the following sequence of Λ-modules obtianed by tensoring the previous resolution by Λ_1

$$0 \to (P_p \overset{1}{\rightharpoonup} M \otimes_A P_p) \to \cdots \to (P_0 \overset{1}{\rightharpoonup} M \otimes_A P_0) \to (S \rightharpoonup 0) \to 0$$

is not exact in general since we do not assume that M is a projective A-module. Instead we consider the double complex obtained by tensoring both resolutions over A:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Q_1 \otimes_A P_2 & \longrightarrow & Q_1 \otimes_A P_1 & \longrightarrow & Q_1 \otimes_A P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Q_0 \otimes_A P_2 & \longrightarrow & Q_0 \otimes_A P_1 & \longrightarrow & Q_0 \otimes_A P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M \otimes_A P_2 & \longrightarrow & M \otimes_A P_1 & \longrightarrow & M \otimes_A P_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The total complex of this double complex is exact, since each column is obtained by tensoring an exact complex by a projective module. Hence we obtain a finite exact sequence of Λ-modules:

$$
\begin{array}{ccccccccc}
\vdots & & & & \vdots & & & & \\
P_2 & \rightharpoonup & M \otimes_A P_2 & \oplus & Q_0 \otimes_A P_1 & \oplus & Q_1 \otimes_A P_0 & & \\
\downarrow & & \downarrow & \swarrow & \downarrow & \swarrow & & & \\
P_1 & \rightharpoonup & M \otimes_A P_1 & \oplus & Q_0 \otimes_A P_0 & & & & \\
\downarrow & & \downarrow & \swarrow & & & & & \\
P_0 & \rightharpoonup & M \otimes_A P_0 & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
S & \rightharpoonup & 0 & & & & & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}
$$

We assert that this is a projective resolution of $S \rightharpoonup 0$. Indeed the i-th module is

$$(P_i \rightharpoonup M \otimes P_i) \oplus (0 \rightharpoonup Q_0 \otimes_A P_{i-1}) \oplus \cdots \oplus (0 \rightharpoonup Q_{i-1} \otimes_A P_0)\,.$$

The first summand $\Lambda_1 \otimes_A P_i$ is projective by Remark 2.8. For the other summands, we first notice that if Q is a projective $B-A$-bimodule and X is any A-module, $Q \otimes_A X$ is a projective B-module. Moreover, for a corner algebra Λ, if W is a projective left B-module, then the left Λ-module $0 \rightharpoonup W$ is projective.

Secondly let T be a simple B-module and let $0 \rightharpoonup T$ the corresponding simple Λ-module. Let $R_\bullet \to T$ be a finite B-projective resolution of T, then $(0 \rightharpoonup R_\bullet) \to (0 \rightharpoonup T)$ is a finite resolution of $0 \rightharpoonup T$ by projective Λ-modules. $\diamond$

Now, we will define E-triangular algebras with respect to a chosen system E. We define first a quiver inferred from the Peirce decomposition $\Lambda = \bigoplus_{x,y\in E} y\Lambda x$.

Definition 2.13 *Let Λ be a k-algebra and let E be a system of Λ. The E-*quiver *Q_E has set of vertices E; for x and y different elements of E there is an arrow from x to y in case $y\Lambda x \neq 0$. Note that Q_E contains no loops.*

Definition 2.14 *An algebra Λ is E-*triangular *with respect to a non trivial system E if Q_E has no oriented cycles.*

Remark 2.15 *In case $|E| = 2$, the E-quiver of an E-triangular algebra is an arrow, and the algebra is a corner algebra. Observe that a finite dimensional algebra which is E-triangular with respect to a system E may have oriented cycles in its Gabriel quiver.*

Lemma 2.16 *Let Λ be a k-algebra which is E-triangular. There exists a system F of two idempotents such that Λ is a corner algebra.*

Proof. The E-quiver has no oriented cycles, it is finite and it has at least two vertices. Then there exists a source vertex e, that is, a vertex with no arrows ending at it. The idempotent $f = \sum_{x\neq e} x$ is not zero. Since $e\Lambda f = 0$, the algebra Λ is a corner algebra with respect to the system $F = \{e, f\}$. $\diamond$

Corollary 2.17 *Let Λ be a finite dimensional k-algebra which is E-triangular with respect to a system E. If $x\Lambda x$ is smooth for every $x \in E$, then Λ is smooth.*

Proof. We proceed by induction on the number of vertices. Let e be a source vertex of Q_E, let $f = \sum_{x\neq e} x = 1 - e$ and let F be the system $\{e, f\}$.

Let $E' = E \setminus \{e\}$, which is a system of the algebra $f\Lambda f$. The E'-quiver of $f\Lambda f$ has no oriented cycles since $y(f\Lambda f)x = y\Lambda x$ for every $x, y \in E'$. By hypothesis the algebras $x(f\Lambda f)x = x\Lambda x$ are smooth for every $x \in E'$. By induction $f\Lambda f$ is smooth. Theorem 2.12 provides the result since $e\Lambda e$ is smooth and Λ is a corner algebra with respect to F, as in the proof of the previous lemma. $\diamond$

By definition the Hochschild homology vector spaces of a k-algebra Λ with coefficients in a Λ-bimodule Z are

$$H_*(\Lambda, Z) = \mathrm{Tor}_*^{\Lambda\otimes\Lambda^{\mathrm{op}}}(\Lambda, Z)$$

where the later is also denoted by $\mathrm{Tor}_*^{\Lambda-\Lambda}(\Lambda, Z)$.

Next we recall the computation of the Hochschild homology of a corner algebra, see for instance [15, 10]. The following well known result will be required; we provide a sketch of its proof for the convenience of the reader.

Lemma 2.18 *Let Λ be a k-algebra, let D be a separable subalgebra of Λ and let Z be a Λ-bimodule. The homology of the complex*

$$\cdots \xrightarrow{b} Z \otimes_{D-D} (\Lambda \otimes_D \Lambda \otimes_D \Lambda) \xrightarrow{b} Z \otimes_{D-D} (\Lambda \otimes_D \Lambda) \xrightarrow{b} Z \otimes_{D-D} \Lambda \xrightarrow{b} Z \to 0$$

is $H_*(\Lambda, Z)$, where $\otimes_{D-D}$ stands for $\otimes_{D\otimes D^{\mathrm{op}}}$ and where the maps b are given by the usual formulas for computing Hochschild homology:

$$\begin{aligned} b(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = & \; x_0x_1 \otimes x_2 \otimes \dots x_n \\ & + \sum_0^{n-1} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n \\ & + (-1)^n x_n x_0 \otimes x_2 \otimes \cdots \otimes x_{n-1}. \end{aligned}$$

Proof. Consider the complex with differential b' defined by the usual formulas for the canonical resolution of Λ over the ground field

$$\cdots \xrightarrow{d} \Lambda \otimes_D \Lambda \otimes_D \Lambda \xrightarrow{d} \Lambda \otimes_D \Lambda \xrightarrow{d} \Lambda \to 0.$$

The map s given by $s(x_1 \otimes \cdots \otimes x_n) = 1 \otimes x_1 \otimes \cdots \otimes x_n$ is well defined and verifies $ds+sd = 1$, this proves that the complex is acyclic. Since D is separable, $D \otimes D^{op}$ is also separable and any D-bimodule is projective. Consequently the acyclic complex above is a projective resolution of Λ by projective Λ-bimodules. The statement of the lemma is obtained by applying the functor $Z \otimes_{D-D} -$ to this resolution and observing that $Z \otimes_{\Lambda-\Lambda} (\Lambda \otimes_D X \otimes_D \Lambda)$ is canonically isomorphic to $Z \otimes_{D-D} X$ for any D-bimodule X. $\diamond$

Theorem 2.19 *[15, 10] Let $\Lambda = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ be a corner algebra, where A and B are k-algebras and M is a B–A-bimodule. There is a decomposition*

$$HH_*(\Lambda) = HH_*(A) \oplus HH_*(B)$$

Proof. Let e be the idempotent $\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$ and let $f = 1 - e = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$. Let $D = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = ke \times kf$; note that D is a separable subalgebra of Λ. We assert that the complex of the previous lemma is actually the direct sum of the complexes that compute $HH_*(A)$ and $HH_*(B)$. Indeed, notice that the D-bimodule decomposition $\Lambda = A \oplus B \oplus M$ provides a direct sum decomposition

$$\Lambda \otimes_{D-D} (\Lambda \otimes_D \cdots \otimes_D \Lambda) = (A \otimes \cdots \otimes A) \oplus (B \otimes \cdots \otimes B)$$

since

$$0 = M \otimes_{D-D} B = M \otimes_{D-D} A = M \otimes_{D-D} M = B \otimes_{D-D} M = A \otimes_{D-D} M$$

and $A \otimes_{D-D} A = A \otimes A$ while $B \otimes_{D-D} B = B \otimes B$. Observe that in degree 0 we obtain $\Lambda \otimes_{D-D} D = A \oplus B$. $\diamond$

Corollary 2.20 *For any k-algebra Λ which is E-triangular with respect to a system E, there is a decomposition*

$$HH_*(\Lambda) = \bigoplus_{x \in E} HH_*(x\Lambda x).$$

Proof. The idea of the proof is similar to the proof of Corollary 2.17. It follows by induction once a source vertex of Q_E is chosen. $\diamond$

Next we turn to Han's conjecture that we recall: if A is a finite dimensional algebra over a field such that $HH_n(A) = 0$ for n large enough, then A is smooth.

Theorem 2.21 *Finite dimensional corner k-algebras built on the class of k-algebras $\mathcal{H}$ verifying Han's conjecture also belong to $\mathcal{H}$.*

Proof. Let $\Lambda = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$ be a finite dimensional corner algebra and suppose $HH_*(\Lambda) = 0$ for large enough degrees. Theorem 2.19 shows that the same holds for A and B. Since A and B belong to $\mathcal{H}$, they are smooth. By Theorem 2.12, Λ is smooth. $\diamond$

Corollary 2.22 *Let Λ be a finite dimensional k-algebra which is E-triangular with respect to a system E of Λ. If for every $x \in E$ the algebras $x\Lambda x$ belong to $\mathcal{H}$, then Λ belongs to $\mathcal{H}$.*

Proof. The proof follows from Corollaries 2.17 and 2.20. $\diamond$

Remark 2.23 *Let Λ be a smooth finite dimensional algebra such that $\Lambda/\operatorname{rad}\Lambda$ is a product of copies of the ground field k and which admits a Wedderburm decomposition $\Lambda = D \oplus \operatorname{rad}\Lambda$ where D is a subalgebra of Λ. Note that if k is perfect a Wedderburm decomposition always exists. If Λ is smooth, it is proven by B. Keller in [14, 2.5] that there is a K-theoretical equivalence between Λ and D. In particular the cyclic homologies of these algebras are isomorphic, as well as the Hochschild homologies due to the Connes' long exact sequences relying cyclic and Hochschild homologies, for Λ and $\Lambda/\operatorname{rad}\Lambda$, see for instance [20]. Consequently the Hochschild homology of Λ is concentrated in degree zero. In this situation, it follows from Han's conjecture that if the Hochschild homology vanishes in large enough degrees, then it actually vanishes in all positive degrees.*

We observe that in the situation of Corollary 2.22, the result that we have proven agrees with the previous observation. Indeed, we have shown using Corollary 2.20 that Hochschild homology is the direct sum of the Hochschild homologies at the idempotents of the system.

3 Hochschild homology of null-square projective algebras

In this section we consider a *null-square projective algebra* Λ, that is, a null-square algebra $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ where M and N are projective B–A and A–B-bimodules respectively; we recall that $MN = NM = 0$. We will provide a long exact sequence which computes $HH_*(\Lambda)$.

First we consider a cleft extension algebra $\Lambda = C \oplus I$, where C is a subalgebra and I is a two-sided ideal, see [16, p. 284]. Let

$$K_C^1(\Lambda) = \operatorname{Ker}\left(\Lambda \otimes_C \Lambda \xrightarrow{d} \Lambda\right)$$

where d is given by the product of Λ. In case I is projective as a C-bimodule we will provide a resolution of $K^1_C(\Lambda)$ by projective Λ-bimodules. This resolution specialized to a null-square projective algebra will allow to compute $\mathrm{Tor}_*^{\Lambda-\Lambda}(K^1_C(\Lambda),\Lambda)$. The mentioned long exact sequence will be obtained as the Tor exact sequence associated to the short exact sequence of Λ-bimodules

$$0 \longrightarrow K^1_C(\Lambda) \longrightarrow \Lambda \otimes_C \Lambda \longrightarrow \Lambda \longrightarrow 0.$$

Lemma 3.1 *Let $\Lambda = C \oplus I$ be a cleft extension algebra. The following complex is acyclic:*

$$\cdots \xrightarrow{d} \Lambda \otimes_C I \otimes_C I \otimes_C \Lambda \xrightarrow{d} \Lambda \otimes_C I \otimes_C \Lambda \xrightarrow{d} \Lambda \otimes_C \Lambda \xrightarrow{d} \Lambda \longrightarrow 0$$

with differentials for $n \geq 3$

$$\begin{aligned} d(l_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes l_n) = \; & l_1 x_2 \otimes x_3 \otimes \ldots x_{n-1} \otimes l_n \\ & + \sum_{2}^{n-2} (-1)^{i+1} l_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes l_n \\ & + (-1)^n l_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} l_n \end{aligned}$$

and, for $n = 2$, the product of the algebra is denoted by d as before.

Proof. Let $l \in \Lambda$ and let $l = l_C + l_I$ be its decomposition in $C \oplus I$. Let s be the map given as follows:

$$s(l_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes l_n) = 1 \otimes (l_1)_I \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes l_n.$$

It is straightforward to check that s is well defined with respect to the tensor products over C. The verification that s is a homotopy contraction is not completely trivial, we illustrate this by checking the property in degree two:

$$\begin{aligned} ds(l \otimes x \otimes l') =& l_I \otimes x \otimes l' - 1 \otimes l_I x \otimes l' + 1 \otimes l_I \otimes x l', \\ sd(l \otimes x \otimes l') =& 1 \otimes (lx)_I \otimes l' - 1 \otimes l_I \otimes x l'. \end{aligned}$$

Note that $(lx)_I = lx = l_C x + l_I x$. Hence

$$\begin{aligned} (ds + sd)(l \otimes x \otimes l') &= l_I \otimes x \otimes l' - 1 \otimes l_I x \otimes l' + 1 \otimes (lx)_I \otimes l' \\ &= l_I \otimes x \otimes l' - 1 \otimes l_I x \otimes l' + 1 \otimes l_C x \otimes l' + 1 \otimes l_I x \otimes l' \\ &= l_I \otimes x \otimes l' + 1 \otimes l_c x \otimes l' \\ &= l_I \otimes x \otimes l' + l_C \otimes x \otimes l' \\ &= (l_I + l_C) \otimes x \otimes l' \\ &= l \otimes x \otimes l'. \end{aligned}$$

◇

Proposition 3.2 *Let $\Lambda = C \oplus I$ be a cleft extension algebra and suppose I is a projective C-bimodule. The following is a resolution of $K^1_C(\Lambda)$ by projective Λ-bimodules:*

$$\cdots \xrightarrow{d} \Lambda \otimes_C I \otimes_C I \otimes_C \Lambda \xrightarrow{d} \Lambda \otimes_C I \otimes_C \Lambda \xrightarrow{d} K^1_C(\Lambda) \longrightarrow 0.$$

Proof. The complex is acyclic by the previous result. We claim that if P and Q are projective C-bimodules, then $P \otimes_C Q$ is also a projective C-bimodule. Indeed $(C \otimes C) \otimes_C (C \otimes C)$ is a projective bimodule and the result follows. Consequently $I \otimes_C \cdots \otimes_C I$ is a projective C-bimodule. Moreover, if P is a projective C-bimodule it is clear that $\Lambda \otimes_C P \otimes_C \Lambda$ is a projective Λ-bimodule. $\diamond$

Let Λ be a k-algebra and Z be a Λ-bimodule. We recall the following

$$H_0(\Lambda, Z) \ = \ \Lambda \otimes_{\Lambda \otimes \Lambda^{\mathsf{op}}} Z \ = \ \Lambda \otimes_{\Lambda - \Lambda} Z \ = \ Z/\langle \lambda z - z\lambda \rangle$$

where $\langle \lambda z - z\lambda \rangle$ is the vector subspace of Z generated by the set $\{\lambda z - z\lambda\}$ for all $\lambda \in \Lambda$ and $z \in Z$.
Let Λ be an algebra and let C be a subalgebra. Let U be a C-bimodule and let $\Lambda \otimes_C U \otimes_C \Lambda$ be the induced Λ-bimodule. The next result gives a decomposition of the Hochschild homology in degree zero of a cleft algebra $\Lambda = C \oplus I$ with coefficients in an induced bimodule. We provide a proof for further use.

Proposition 3.3 *Let $\Lambda = C \oplus I$ be a cleft algebra and let U be a C-bimodule.*

$$H_0(\Lambda,\ \Lambda \otimes_C U \otimes_C \Lambda) = H_0(C, U) \oplus H_0(C,\ I \otimes_C U)$$

Proof. The mutual inverse isomorphisms are given by

$$\begin{array}{lllll} a \otimes u \otimes b & \mapsto & (ba)_C\, u & + & (ba)_I \otimes u, \\ u + x \otimes v & \mapsto & 1 \otimes u \otimes 1 & + & x \otimes v \otimes 1. \end{array}$$

$\diamond$

We will use next the previous result for $U = I^{\otimes_C n}$. Let

$$I(n) = H_0(C,\ I^{\otimes_C n}).$$

Corollary 3.4 *Let $\Lambda = C \oplus I$ be a cleft algebra. There is a decomposition*

$$H_0\left(\Lambda,\ \Lambda \otimes_C I^{\otimes_C n} \otimes_C \Lambda\right) = I(n) \ \oplus \ I(n+1).$$

Proposition 3.5 *Let $\Lambda = C \oplus I$ be a cleft algebra where I is a projective C-bimodule. The vector spaces $\mathrm{Tor}_*^{\Lambda-\Lambda}(K_C^1(\Lambda), \Lambda)$ are the homology spaces of the complex*

$$\cdots \xrightarrow{b} I(n) \oplus I(n+1) \xrightarrow{b} I(n-1) \oplus I(n) \xrightarrow{b} \cdots \xrightarrow{b} I(2) \oplus I(3) \xrightarrow{b} I(1) \oplus I(2) \longrightarrow 0$$

where

$$b : I(n) \oplus I(n+1) \to I(n-1) \oplus I(n)$$

is as follows:

- *If $z_1 \otimes \cdots \otimes z_n \in I(n)$, then*

$$\begin{aligned} b(z_1 \otimes \cdots \otimes z_n) = {} & z_1 \otimes \cdots \otimes z_n \\ & + \sum_1^{n-1} (-1)^i z_1 \otimes \cdots \otimes z_i z_{i+1} \otimes \ldots \ \otimes z_n \\ & + (-1)^n z_n \otimes z_1 \otimes \cdots \otimes z_{n-1} \end{aligned}$$

  where the first and the last terms belong to $I(n)$ and the middle sum belongs to $I(n-1)$.

- *If $z_0 \otimes \cdots \otimes z_n \in I(n+1)$, then*

$$\begin{aligned} b(z_0 \otimes \cdots \otimes z_n) &= z_0 z_1 \otimes \cdots \otimes z_n \\ &\quad + \sum_0^{n-1} (-1)^i z_0 \otimes \cdots \otimes z_i z_{i+1} \otimes \dots \otimes z_n \\ &\quad + (-1)^n z_n z_0 \otimes \cdots \otimes z_{n-1} \end{aligned}$$

  which belongs to $I(n)$.

Proof. The formulas are obtained by applying the functor $H_0(\Lambda, -)$ to the projective resolution of $K^1_C(\Lambda)$ of Proposition 3.2, and by translating the differentials to the present setting through the isomorphisms provided in Proposition 3.3. $\diamond$

Lemma 3.6 *Let A and B be k-algebras, let $C = A \times B$ and let I be a C-bimodule of the form $I = M \oplus N$ where M is a $B-A$-bimodule and N is a $A-B$-bimodule. For n odd, $I(n) = 0$.*

Proof. First we notice that $M \otimes_C M = 0 = N \otimes_C N$ since for instance $m \otimes m' = m(1_A, 0) \otimes m' = m \otimes (1_A, 0)m' = m \otimes 0 = 0$.

Moreover $N \otimes_C M = N \otimes_B M$ and $M \otimes_C N = M \otimes_A N$.

Consequently

$$I^{\otimes_C n} = (\cdots \otimes_A N \otimes_B M \otimes_A N \otimes_B M) \ \oplus \ (\cdots \otimes_B M \otimes_A N \otimes_B M \otimes_A N)$$

with n tensorands in each summand. In particular for n odd we have

$$I^{\otimes_C n} = (M \otimes_A \cdots \otimes_B M \otimes_A N \otimes_B M) \ \oplus \ (N \otimes_B \cdots \otimes_A N \otimes_B M \otimes_A N)$$

and we assert that $H_0(C, \ I^{\otimes_C n}) = 0$. Indeed $(1_A, 0)x = 0$ for every $x \in M$, while $x(1_A, 0) = x$, and $(0, 1_B)y = 0$, while $y(0, 1_B) = y$ for every $y \in N$. $\diamond$

Lemma 3.7 *In the same situation as in the previous lemma, for $n = 2m$,*

$$\begin{aligned} I^{\otimes_C n} &= (N \otimes_B M) \otimes_A \cdots \otimes_A (N \otimes_B M) \ \oplus \ (M \otimes_A N) \otimes_B \cdots \otimes_B (M \otimes_A N) \\ &= (N \otimes_B M)^{\otimes_A m} \ \oplus \ (M \otimes_A N)^{\otimes_B m}. \end{aligned}$$

Corollary 3.8 *Let A and B be k-algebras, let $C = A \times B$ and let I be a C-bimodule of the form $I = M \oplus N$ where M is a $B-A$-bimodule and N a $A-B$-bimodule. The following decomposition holds:*

$$I(2m) = H_0\left(A, (N \otimes_B M)^{\otimes_A m}\right) \ \oplus \ H_0\left(B, (M \otimes_A N)^{\otimes_B m}\right).$$

Definition 3.9 *Let $C_m = \langle t \mid t^m = 1 \rangle$ be a cyclic group of order m. The kC_m-module structures of $H_0\left(B, (M \otimes_A N)^{\otimes_B m}\right)$ and $H_0\left(A, (N \otimes_B M)^{\otimes_A m}\right)$ are given by the following action of t by cyclic permutation:*

$$t(x_m \otimes y_m \otimes \cdots x_2 \otimes y_2 \otimes x_1 \otimes y_1) = x_1 \otimes y_1 \otimes x_m \otimes y_m \otimes \cdots x_2 \otimes y_2,$$

$$t(y_m \otimes x_m \otimes \cdots y_2 \otimes x_2 \otimes y_1 \otimes x_1) = y_1 \otimes x_1 \otimes y_m \otimes x_m \otimes \cdots y_2 \otimes x_2.$$

Note that the above actions are not well defined neither on $M \otimes_A N$ nor on $N \otimes_B M$, on the other hand they are well defined on the 0-degree homology of these bimodules.

We provide two isomorphisms between these kC_m-modules that will be used in the proof of the next result:

$$\begin{array}{rcl} H_0\left(A,(N\otimes_B M)^{\otimes_A m}\right) & \stackrel{\sigma}{\to} & H_0\left(B,(M\otimes_A N)^{\otimes_B m}\right) \\ y_m\otimes x_m\otimes\cdots\otimes y_1\otimes x_1 & \mapsto & x_1\otimes y_m\otimes x_m\otimes\cdots\otimes y_1, \\ H_0\left(B,(M\otimes_A N)^{\otimes_B m}\right) & \stackrel{\tau}{\to} & H_0\left(A,(N\otimes_B M)^{\otimes_A m}\right) \\ x_m\otimes y_m\otimes\cdots\otimes x_1\otimes y_1 & \mapsto & y_1\otimes x_m\otimes y_m\otimes\cdots\otimes x_1. \end{array}$$

Notice that the compositions $\sigma\tau$ and $\tau\sigma$ are the actions of t on the corresponding vector spaces.

Finally we recall that for a group G and a kG-module H, the invariants (or fixed points) of the action are $H^G = \{x \in H \mid sx = x \text{ for all } s \in G\}$. The coinvariants are $H_G = H/\langle sx - x\rangle$ where $\langle sx - x\rangle$ is the vector subspace of H generated by the elements of the form $sx - x$ for all $s \in G$ and $x \in H$. If G is finite and the characteristic of the field does not divide its order, then H_G and H^G are canonically isomorphic through the action of $\frac{1}{|G|}\sum_{s\in G} s$.

Theorem 3.10 *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a null-square projective algebra, and let* $I = M \oplus N$*. For* $m \geq 0$*,*

$$\mathrm{Tor}_{2m+1}^{\Lambda\text{–}\Lambda}(K_C^1(\Lambda),\ \Lambda) = H_0\left(B,(M\otimes_A N)^{\otimes_B m+1}\right)^{C_{m+1}} \quad \textit{and}$$

$$\mathrm{Tor}_{2m}^{\Lambda\text{–}\Lambda}(K_C^1(\Lambda),\ \Lambda) = H_0\left(B,(M\otimes_A N)^{\otimes_B m+1}\right)_{C_{m+1}}.$$

Proof. We recall that for a null-square projective algebra $MN = 0 = NM$, hence $I^2 = 0$. Moreover $I(n) = 0$ for n odd, by Lemma 3.6. Consequently the complex of Proposition 3.5 reduces to

$$\cdots \stackrel{b}{\to} I(6) \stackrel{0}{\to} I(4) \stackrel{b}{\to} I(4) \stackrel{0}{\to} I(2) \stackrel{b}{\to} I(2) \to 0$$

where for $n = 2m$

$$b(z_1\otimes\cdots\otimes z_n) = z_1\otimes\cdots\otimes z_n \ + \ z_n\otimes z_1\otimes\cdots\otimes z_{n-1}.$$

Furthermore, the matrix of

$$\begin{array}{rl} b: & H_0\left(A,(N\otimes_B M)^{\otimes_A m}\right) \ \oplus\ H_0\left(A,(N\otimes_B M)^{\otimes_A m}\right) \\ & \longrightarrow H_0\left(A,(N\otimes_B M)^{\otimes_A m}\right) \ \oplus\ H_0\left(A,(N\otimes_B M)^{\otimes_A m}\right) \end{array}$$

with respect to the decomposition of Proposition 3.8 is $\begin{pmatrix} 1 & \tau \\ \sigma & 1 \end{pmatrix}$. Moreover,

$$\begin{aligned} \operatorname{Ker} b &= \{(u,v) \mid u + \tau(v) = 0 = \sigma(u) + v\} \\ &= \{(u, -\sigma u) \mid u = \tau\sigma u\} \\ &= \{u \mid tu = u\} \\ &= H_0\left(A,(N\otimes_B M)^{\otimes_A m}\right)^{C_m}. \end{aligned}$$

In order to compute $\operatorname{Coker} b$, note that $(u,v) = -(\tau v, \sigma u)$ holds in $\operatorname{Coker} b$. Hence $(u,0) = (0, -\sigma u) = (\tau\sigma(u), 0)$. This shows that the map

$$H_0\left(A, (N \otimes_B M)^{\otimes_A m}\right)_{C_m} \to \operatorname{Coker} b$$

given by $u \mapsto (u,0)$ is well defined. Its inverse is given by $(u,v) \mapsto u - \tau(v)$. Hence $\operatorname{Coker} b = H_0\left(A, (N \otimes_B M)^{\otimes_A m}\right)_{C_m}$. $\diamond$

Towards describing the long exact sequence mentioned above, we consider now some tools of homological algebra to compute $\operatorname{Tor}_*^{\Lambda-\Lambda}(\Lambda \otimes_C \Lambda,\ \Lambda)$. The next result will be used for a null-square projective algebra $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ and for the inclusion of algebras $C \otimes C^{\mathrm{op}} \subset \Lambda \otimes \Lambda^{\mathrm{op}}$, where $C = A \times B$.

Lemma 3.11 *Let $F \subset D$ be an inclusion of k-algebras and suppose D is projective as a left F-module. Let U be a right F-module and let $U{\uparrow}^D = U \otimes_F D$ be the induced right module. Let Z be a left D-module and let ${}_F{\downarrow}Z$ be the left F-module obtained by restricting the action to F. The following holds:*

$$\operatorname{Tor}_*^D(U{\uparrow}^D, Z) = \operatorname{Tor}_*^F(U, {}_F{\downarrow}Z).$$

Proof. The left hand side functor in the variable Z is characterised by its universal property :

- $\operatorname{Tor}_0^D(U{\uparrow}^D, Z) = U{\uparrow}^D \otimes_D Z = U \otimes_F Z$,
- $\operatorname{Tor}_0^D(U{\uparrow}^D, Z) = 0$ if Z is projective,
- A short exact sequence of of D-modules provides a long exact sequence.

It is clear that the right hand side functor in the variable Z verifies the same properties. Note that the second property is fulfilled precisely because we assume ${}_F{\downarrow}D$ is projective. $\diamond$

Lemma 3.12 *Let Λ be a null-square projective algebra and let $C = A \times B$. The C-bimodule $\Lambda \otimes \Lambda$ is projective.*

Proof. Note first that by hypothesis M is a projective $B{-}A$-bimodule. It becomes a C-bimodule by extending the actions by zero, then M is a projective C-bimodule. The same holds for N, then $I = M \oplus N$ is a projective C-bimodule.

Consider the C-bimodule decomposition

$$\Lambda \otimes \Lambda = (C \otimes C) \oplus (C \otimes I) \oplus (I \otimes C) \oplus (I \otimes I).$$

We assert that a projective C-bimodule is also projective as a left (or right) C-module. Indeed, the free rank-one C-bimodule $C \otimes C$ is free as a left (or right) C-module. This observation makes the proof of the assertion immediate. We infer that I is projective as a left and as a right C-module.

We record that if P is a projective left C-module and Q is a projective right C-module, the C-bimodule $P \otimes Q$ is a projective C-bimodule.

Consequently the four terms of the above direct sum decomposition of the C-bimodule $\Lambda \otimes \Lambda$ are projective C-bimodules. $\diamond$

Theorem 3.13 *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a null-square projective algebra, and let* $C = A \times B$*. There is a decomposition*

$$\mathrm{Tor}_*^{\Lambda\text{–}\Lambda}(\Lambda \otimes_C \Lambda,\ \Lambda) = HH_*(A) \oplus HH_*(B).$$

Proof. We consider the inclusion $C \otimes C^{op} \subset \Lambda \otimes \Lambda^{op}$. Lemma 3.11 with $U = C$ provide the following:

$$\begin{aligned}\mathrm{Tor}_*^{\Lambda\text{–}\Lambda}(\Lambda \otimes_C \Lambda,\ \Lambda) &= \mathrm{Tor}_*^{\Lambda\text{–}\Lambda}(\Lambda \otimes_C C \otimes_C \Lambda,\ \Lambda) \\ &= \mathrm{Tor}_*^{C-C}(C,\ {}_C{\downarrow}\Lambda{\downarrow}_C) \\ &= H_*(C,\ {}_C{\downarrow}\Lambda{\downarrow}_C) \\ &= HH_*(C) \oplus H_*(C, M) \oplus H_*(C, N)\end{aligned}$$

We assert that $H_*(C, M) = H_*(C, N) = 0$. Indeed, let $P_\bullet \to A$ be a projective resolution of the A-bimodule A, and analogously for $Q_\bullet \to B$. Note that $P_\bullet \oplus Q_\bullet \to A \oplus B$ is a projective resolution of the C-bimodule C, where the C-bimodule structure of $P_\bullet$ is obtained by extending the action to B by zero, and analogously for $Q_\bullet$. The functor $M \otimes_{C\text{–}C} -$ applied to $P_\bullet \oplus Q_\bullet$ gives the zero complex by simple arguments already used in the proof of Lemma 3.6 and $H_*(C, M) = 0$. Analogously $H_*(C, N) = 0$. Note that the assertion also follows from [9, p. 173].

In order to prove $HH_*(C) = HH_*(A) \oplus HH_*(B)$, observe that the summands $A \otimes_{C\text{–}C} Q_\bullet$ and $B \otimes_{C\text{–}C} P_\bullet$ of $C \otimes_{C\text{–}C} (P_\bullet \oplus Q_\bullet)$ are zero for analogous reasons. $\diamond$

Theorem 3.14 *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a null-square projective algebra. There is a long exact sequence as follows:*

$$\begin{array}{llll}
\dots & & & \\
H_0\left(A, (N \otimes_B M)^{\otimes_A m+1}\right)^{C_{m+1}} & \to HH_{2m+1}(A) \oplus HH_{2m+1}(B) & \to HH_{2m+1}(\Lambda) & \to \\
H_0\left(A, (N \otimes_B M)^{\otimes_A m+1}\right)_{C_{m+1}} & \to HH_{2m}(A) \oplus HH_{2m}(B) & \to HH_{2m}(\Lambda) & \to \\
\dots & & & \\
H_0\left(A, (N \otimes_B M)^{\otimes_A 3}\right)^{C_3} & \to HH_5(A) \oplus HH_5(B) & \to HH_5(\Lambda) & \to \\
H_0\left(A, (N \otimes_B M)^{\otimes_A 3}\right)_{C_3} & \to HH_4(A) \oplus HH_4(B) & \to HH_4(\Lambda) & \to \\
H_0\left(A, (N \otimes_B M)^{\otimes_A 2}\right)^{C_2} & \to HH_3(A) \oplus HH_3(B) & \to HH_3(\Lambda) & \to \\
H_0\left(A, (N \otimes_B M)^{\otimes_A 2}\right)_{C_2} & \to HH_2(A) \oplus HH_2(B) & \to HH_2(\Lambda) & \to \\
H_0\left(A, (N \otimes_B M)\right) & \to HH_1(A) \oplus HH_1(B) & \to HH_1(\Lambda) & \to \\
H_0\left(A, (N \otimes_B M)\right) & \to HH_0(A) \oplus HH_0(B) & \to HH_0(\Lambda) & \to 0.
\end{array}$$

Corollary 3.15 *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a null-square projective algebra. If* $HH_n(\Lambda) = 0$ *for* n *large enough, then*

$$H_0\left(A, (N \otimes_B M)^{\otimes_A n}\right)_{C_n} = H_0\left(B, (M \otimes_A N)^{\otimes_B n}\right)^{C_n} = 0$$

for n *large enough.*

Proof. Hochschild homology is a functor from the category of algebras to the category of vector spaces. Let $\Lambda = C \oplus I$ where C is a subalgebra of Λ and I is a two-sided ideal. In other words, there is an algebra surjection $\Lambda \to C$ which splits in the category of algebras, then $HH_*(C)$ is a direct summand of $HH_*(\Lambda)$. Consequently, if $HH_n(\Lambda) = 0$ for n large enough, then the same holds for $HH_n(C)$. The long exact sequence of the previous theorem provides the result. $\diamond$

4 Han's conjecture for null-square projective algebras

Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a null-square basic finite dimensional algebra. Our first aim is to prove that if $HH_*(\Lambda)$ vanishes in large enough degrees, then $(N \otimes_B M)^{\otimes_A n} = 0$ for n large enough.

In the following we drop the tensor product symbol over the field k.

Theorem 4.1 *Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$. The Hochschild homology $HH_*(\Lambda)$ is the homology of the complex whose n-chains are direct sums of the vector spaces*

$$(A \dots ANB \dots BMA \dots A) \dots (A \dots ANB \dots BMA \dots A)$$

$$(B \dots BMA \dots ANB \dots B) \dots (B \dots BMA \dots ANB \dots B)$$

which have n tensorands, and possibly empty sequences of A or B. The boundary is given by the usual formulas.

Proof. Lemma 2.18 for the separable subalgebra $D = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ provides the result, by noticing that the following vector spaces

$$M \otimes_D B,\ A \otimes_D M,\ N \otimes_D A,\ B \otimes_D N,\ A \otimes_D B,\ B \otimes_D A$$

vanish, while

$$B \otimes_D M = BM,\ B \otimes_D B = BB,\ N \otimes_D B = NB,$$

$$A \otimes_D N = AN,\ A \otimes_D A = AA,\ M \otimes_D A = MA.$$

$\diamond$

We note that in the previous proof, for each direct summand of the n-chains, the number p of M tensorands equals the number of N tensorands. We call this positive integer p the *revolution number* of the direct summand. Of course $2p \leq n$. Notice that $2p = n$ if and only if the direct summand has neither A nor B tensorands involved.

Corollary 4.2 *Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a null-square algebra. For every p, the vector space*

$$H_0\left(A, (N \otimes_B M)^{\otimes_A p}\right) \oplus H_0\left(B, (M \otimes_A N)^{\otimes_B p}\right)$$

is a direct summand of $HH_{2p}(\Lambda)$.

Proof. Since $MN = 0 = NM$, the complex displayed in the proof of the previous theorem decomposes as the direct sum of the subcomplexes obtained by fixing the revolution number. The subcomplex with revolution number p is zero in degree $n < 2p$, while in degree $2p$ its chains are

$$NM \dots NM \oplus MN \dots MN.$$

Moreover the homology of the revolution number p subcomplex in degree $2p$ is the required direct summand. ◇

Corollary 4.3 *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a null-square algebra. If* $HH_*(\Lambda) = 0$ *in large enough degrees, then*

$$H_0\left(A, (N \otimes_B M)^{\otimes_A *}\right) = 0 \text{ and } H_0\left(B, (M \otimes_A N)^{\otimes_A *}\right) = 0$$

for large enough exponents.

The previous result shows, in a sense that will be made precise when M and N are projectives, that these bimodules do not provide new arbitrarily long oriented cycles in the quiver of Λ. In what follows we will confirm that actually they do not provide new oriented cycles at all, and consequently that the length of the new paths is bounded, namely the vector spaces $(N \otimes_B M)^*$ and $(M \otimes_A N)^*$ vanish for large enough exponents.

Let A and B be finite dimensional and basic algebras. Let E and F be complete sets of primitive orthogonal idempotents of A and B respectively. If k is perfect, then

$$\mathrm{rad}\,(A \otimes B^{\mathsf{op}}) = A \otimes \mathrm{rad}\, B^{\mathsf{op}} + \mathrm{rad}\, A \otimes B^{\mathsf{op}}$$

and $\{g \otimes e\}_{(g,e)\in F\times E}$ is a complete set of primitive orthogonal idempotents of $B \otimes A^{\mathsf{op}}$. Consequently

$$\{Bg \otimes eA\}_{(g,e)\in F\times E}$$

is a complete set of representatives, without repetitions, of the isomorphism classes of projective $B{-}A$-bimodules. Of course the analog fact holds for projective $A{-}B$-bimodules.

Let

$$_BM_A = \bigoplus_{(g,e)\in F\times E} {_gm_e}\,(Bg \otimes eA) \tag{4.1}$$

be a projective finitely generated $B - A$ - bimodule, where by the Krull-Schmidt Theorem, the integers $_gm_e$ are uniquely determined by M. Similarly, let

$$_AN_B = \bigoplus_{(f,h)\in E\times F} {_fn_h}\,(Af \otimes hB) \tag{4.2}$$

be a finitely generated projective $A{-}B$ - bimodule.

Definition 4.4 *In the situation considered above, the* (N, M)-quiver *is defined as follows: its vertices are* $F \cup E$, *where we agree to distribute* E *in a first horizontal floor and* F *in a ground floor.*

There are two sort of arrows:

- *Horizontal, distributed into:*
  - first floor ones, which provides the E-quiver of A (see Definition 2.13), and
  - ground floor ones, namely the F-quiver of B.
- *Vertical, distributed into:*
  - down ones, there are ${}_g m_e$ arrows from e to g in one-to-one correspondence with the direct summands $Bg \otimes eA$ of M, and
  - up ones, defined according to N in the analogous way than for M.

We agree to write the sequence of arrows of a path from right to left, as for composition of morphisms. Recall that the *length* of a path is the length of the corresponding sequence, and that a *cycle* is a path which starts and ends at the same vertex. Next we define some particular paths in the (N, M)-quiver.

Definition 4.5 *Let γ be a path of the (N, M)-quiver,*

- *γ is* balanced *if it does not contain two consecutive horizontal arrows. In case γ starts and ends at the same floor, its* revolution number *is half of the number of the vertical arrows of the sequence of γ.*
- *γ is E-balanced if it is balanced and it starts and ends at the first floor, that is, at E-vertices. The set of E-balanced paths with revolution number m is denoted by P^E_m.*
- *γ is an E-*vertical balanced *if it is an E-balanced cycle whose first arrow is down vertical. The set of E-vertical balanced cycles with revolution number m is denoted by CV^E_m.*

Theorem 4.6 *Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a null-square projective algebra where A and B are basic finite dimensional algebras over a perfect field k, and let M and N be finitely generated projective bimodules, given as in (4.1) and (4.2). If $HH_n(\Lambda) = 0$ for n large enough, then $H_0\left(A, (N \otimes_B M)^{\otimes_A n}\right) = 0$ for all $n > 0$ and $(N \otimes_B M)^{\otimes_A n} = 0$ for n large enough.*

Proof. We assert first of all that it is easy to establish that the vector space $H_0\left(A, (N \otimes_B M)^{\otimes_A n}\right)$ is a direct sum of vector spaces indexed by CV^E_m. In order to provide an outline of the evidence, let us consider the case

$$N = (Af \otimes hB) \oplus (Af' \otimes h'B) \quad \text{and} \quad M = (Bg \otimes eA) \oplus (Bg' \otimes e'A)$$

where all the idempotents are distinct. Notice that the (M, N)-quiver has two down arrows and two up arrows. Then

$$\begin{aligned} N \otimes_B M = \quad & (Af \otimes hBg \otimes eA) \;\oplus\; (Af \otimes hBg' \otimes e'A) \;\oplus \\ & (Af' \otimes h'Bg \otimes eA) \;\oplus\; (Af' \otimes h'Bg' \otimes e'A), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} H_0\left(A, (N \otimes_B M)\right) = \quad & (eAf \otimes hBg) \;\oplus\; (e'Af \otimes hBg') \;\oplus \\ & (eAf' \otimes h'Bg) \;\oplus\; (e'Af' \otimes h'Bg'). \end{aligned} \tag{4.4}$$

For instance, the first summand is non zero if and only if $eAf \neq 0$ and $hBg \neq 0$. Note that, by definition, there are corresponding arrows in the E and F-quivers from f to e and from g to h. If the first summand above is non zero, we associate to it the following E-vertical balanced cycle with revolution number 1:

– the first vertical down arrow from e to g corresponds to $(Bg \otimes eA)$ as a projective direct summand of M,

– it is followed by the horizontal arrow at the ground floor from g to h due to $hBg \neq 0$,

– a vertical up arrow from h to f follows, it is associated to the projective bimodule $(Af \otimes hB)$,

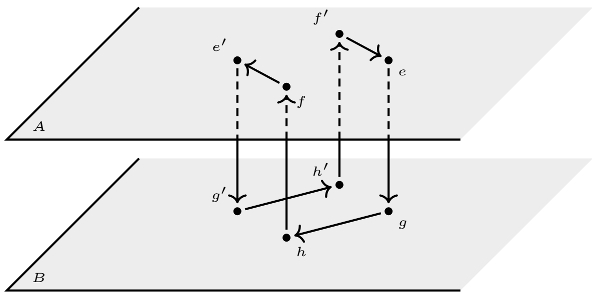
– finally there is an horizontal arrow at the first floor from f to e due to $eAf \neq 0$.

Observe that the decomposition of $H_0\left(A, (N \otimes_B M)^{\otimes_A 2}\right)$ contains the following direct summand

$$(eAf' \otimes hBg \otimes h'Bg' \otimes e'Af \otimes hBg). \tag{4.5}$$

This direct summand corresponds to the E-vertical balanced cycle γ with revolution number 2, described by the following sequence of vertices (from right to left) and drawn below:

$$e, f', h', g', e', f, h, g, e$$



Observe that the direct summands of $H_0\left(A, (N \otimes_B M)^{\otimes_A 2}\right)$ are originated by the projective bimodules of M and N. However they do not appear directly in the writing, while in the E-vertical balanced cycle the vertical arrows which are involved keep track of them. Notice for instance that the vertical arrow from e' to g' corresponds to the projective direct summand $Bg' \otimes e'A$ of M.

Note also that this specific γ is not the square of a vertical balanced cycle of revolution number 1. On the other hand, E-vertical balanced cycles which are squares do exist.

Hence to $\gamma \in \mathsf{CV}_m^E$ we associate the family of vector spaces of the Peirce decompositions of A and B which are determined by the horizontal arrows of γ. We denote V_γ the tensor product of the vector spaces of this family, in the order prescribed by γ. Conversely, each non zero direct summand of $H_0\left(A, (N \otimes_B M)^{\otimes_A m}\right)$ determines an E-vertical balanced cycle of revolution number m. Then

$$H_0\left(A, (N \otimes_B M)^{\otimes_A m}\right) = \bigoplus_{\gamma \in \mathsf{CV}_m^E} V_\gamma.$$

The hypothesis that Hochschild homology of Λ vanishes in large enough degrees imply, by Corollary 4.3, that $H_0\left(A, (N \otimes_B M)^{\otimes_A m}\right) = 0$ for m large enough, hence $\mathsf{CV}_m^E = \emptyset$ for the same set of m's.

It is important to note that if $\mathsf{CV}^E_{n_0} \neq \emptyset$ for some n_0, then $\mathsf{CV}^E_{rn_0} \neq \emptyset$ for all $r > 0$. Hence $\mathsf{CV}^E_n = \emptyset$ for all $n > 0$, and as a consequence $H_0\left(A, (N \otimes_B M)^{\otimes_A n}\right) = 0$ for all $n > 0$.

We assert that, as before, $(N \otimes_B M)^{\otimes_A m}$ is a direct sum of non zero vector spaces which are in one-to-one correspondence with P^E_m. For instance in the decomposition (4.3), the first summand $Af \otimes hBg \otimes eA$ corresponds to all the E-balanced paths which contain the vertical down arrow from e to g and the vertical up arrow from h to f. More precisely, there is a subsequent decomposition

$$Af \otimes hBg \otimes eA = \bigoplus_{y,x \in E} yAf \otimes hBg \otimes eAx,$$

and for each non zero summand $yAf \otimes hBg \otimes eAx$, the E-balanced path is determined by the sequence of vertices y, f, h, g, e, x.

In particular $(N \otimes_B M)^{\otimes_A m} = 0$ if and only if $\mathsf{P}^E_m = \emptyset$.

We have shown before that the (N, M)-quiver has no E-balanced vertical cycles. We observe that the (N, M)-quiver is finite, hence the E-balanced paths have a maximal length. Then $\mathsf{P}^E_n = \emptyset$ for n large enough, and $(N \otimes_B M)^{\otimes_A n} = 0$ for the same set of n's. ◇

As an immediate consequence of the previous result, the long exact sequence of Theorem 3.14 provides the following.

Corollary 4.7 *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a null-square projective algebra where* A *and* B *are basic finite dimensional algebras over a perfect field* k*, and* M *and* N *are finitely generated projective bimodules. If* $HH_n(\Lambda) = 0$ *for* n *large enough, then for all* n

$$HH_n(\Lambda) = HH_n(A) \oplus HH_n(B).$$

Our next aim is to provide a tool for bounding above the global dimension of a null-square projective algebra. For this purpose we first briefly recall the *mapping cone* construction. Let $(C_\bullet, c) = \{C_n \stackrel{c_n}{\to} C_{n-1}\}_{n \in \mathbb{Z}}$ and $(D_\bullet, d) = \{D_n \stackrel{d_n}{\to} D_{n-1}\}_{n \in \mathbb{Z}}$ be complexes with differentials c and d. Let $f : C_\bullet \to D_\bullet$ be a map of complexes. Let $C_\bullet[1]$ be the complex defined by $C_n[1] = C_{n-1}$. There exists a complex $(\mathsf{co}(f)_\bullet, e)$ called the mapping cone of f, and a short exact sequence of complexes

$$0 \to C_\bullet[1] \to \mathsf{co}(f)_\bullet \to D_\bullet \to 0$$

such that the connecting homomorphism in the long exact sequence of cohomology is the morphism induced by f. In particular, f induces isomorphisms (*i.e.* f is a *quasi-isomorphism*) if and only if the mapping cone complex is acyclic. Actually $\mathsf{co}(f)_n = C_n \oplus D_{n-1}$ with differential $e = \begin{pmatrix} c & f \\ 0 & -d \end{pmatrix}$; note that the change of sign for d guarantees $e^2 = 0$, since $fc = df$.

We simplify the tensor product notation as follows: let U be a $C-B$-bimodule and let V be a $B-A$-bimodule, we will write UV instead of $U \otimes_B V$ and VU instead of $V \otimes_A U$.

Theorem 4.8 *Let* $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ *be a null-square projective algebra where* A *and* B *are* k*-algebras, and* M *and* N *are* $B-A$ *and* $A-B$*-projective bimodules respectively. Let* X *be a left* A*-module and* $P_\bullet \to X$ *be a projective resolution.*

Associated to $P_\bullet \to X$, there is a Λ-projective resolution $Q_\bullet \to (X \rightleftharpoons 0)$ such that if $P_\bullet \to X$ is finite and if $(N \otimes_B M)^{\otimes_A n} = 0$ for n large enough, then $Q_\bullet \to (X \rightleftharpoons 0)$ is finite.

Proof. We define the modules of $Q_\bullet$ as follows:

$$Q_0 = \left(P_0 \underset{0}{\overset{1}{\rightleftharpoons}} MP_0\right)$$

$$Q_1 = \left(P_1 \underset{0}{\overset{1}{\rightleftharpoons}} MP_1\right) \oplus \left(NMP_0 \underset{1}{\overset{0}{\rightleftharpoons}} MP_0\right)$$

$$Q_2 = \left(P_2 \underset{0}{\overset{1}{\rightleftharpoons}} MP_2\right) \oplus \left(NMP_1 \underset{1}{\overset{0}{\rightleftharpoons}} MP_1\right) \oplus \left(NMP_0 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)P_0\right)$$

$$Q_3 = \left(P_3 \underset{0}{\overset{1}{\rightleftharpoons}} MP_3\right) \oplus \left(NMP_2 \underset{1}{\overset{0}{\rightleftharpoons}} MP_2\right) \oplus \left(NMP_1 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)P_1\right) \oplus$$
$$\left((NM)^2P_0 \underset{1}{\overset{0}{\rightleftharpoons}} M(NM)P_0\right)$$

$$Q_4 = \left(P_4 \underset{0}{\overset{1}{\rightleftharpoons}} MP_4\right) \oplus \left(NMP_3 \underset{1}{\overset{0}{\rightleftharpoons}} MP_3\right) \oplus \left(NMP_2 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)P_2\right) \oplus$$
$$\left((NM)^2P_1 \underset{1}{\overset{0}{\rightleftharpoons}} M(NM)P_1\right) \oplus \left((NM)^2P_0 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)^2P_0\right)$$

$$\vdots$$

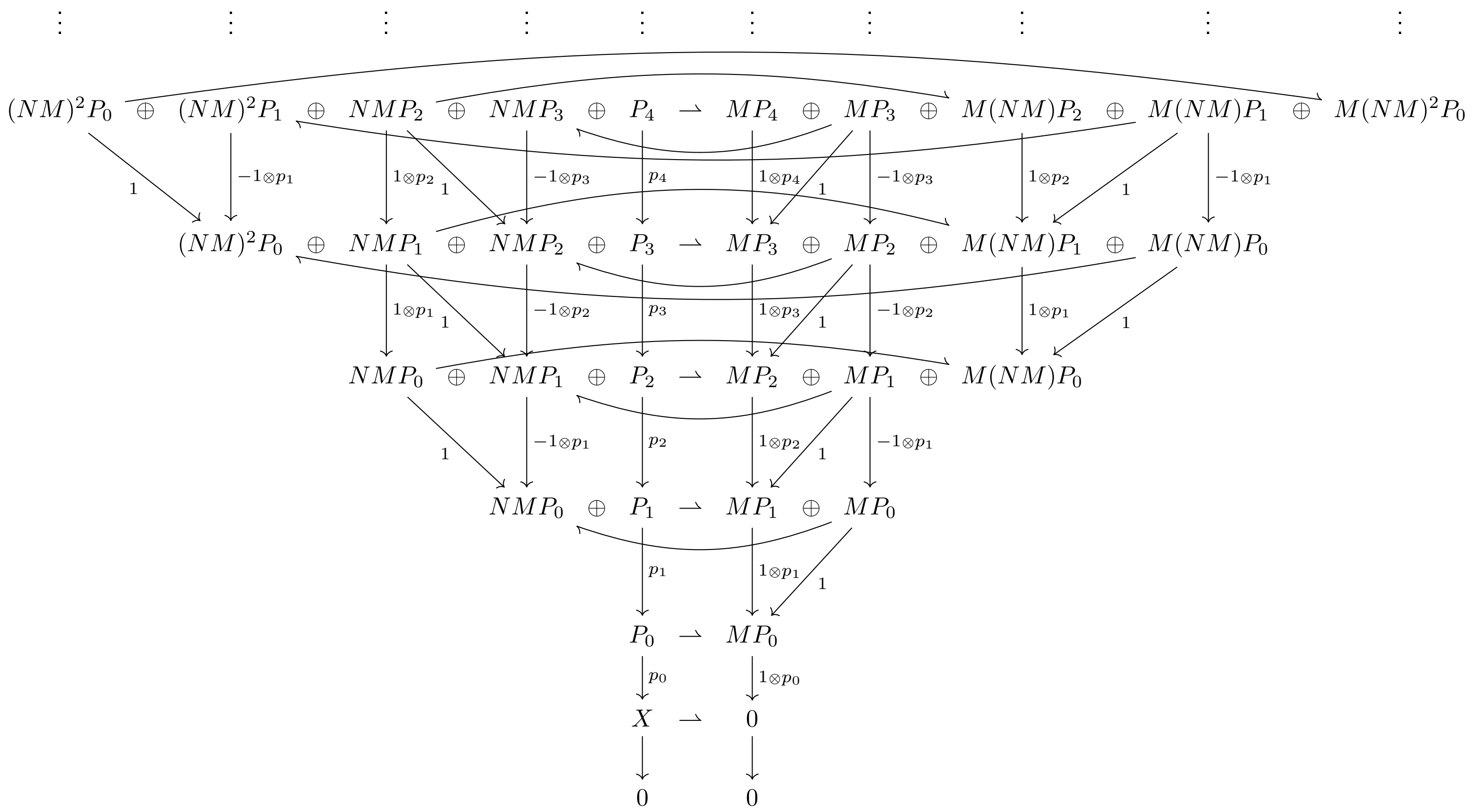
$$Q_{2m} = \left(P_{2m} \underset{0}{\overset{1}{\rightleftharpoons}} MP_{2m}\right) \oplus \left(NMP_{2m-1} \underset{1}{\overset{0}{\rightleftharpoons}} MP_{2m-1}\right) \oplus \cdots \oplus$$
$$\left((NM)^mP_0 \underset{0}{\overset{1}{\rightleftharpoons}} M(NM)^mP_0\right)$$

$$Q_{2m+1} = \left(P_{2m+1} \underset{0}{\overset{1}{\rightleftharpoons}} MP_{2m+1}\right) \oplus \left(NMP_{2m} \underset{1}{\overset{0}{\rightleftharpoons}} MP_{2m}\right) \oplus \cdots \oplus$$
$$\left((NM)^{m+1}P_0 \underset{1}{\overset{0}{\rightleftharpoons}} M(NM)^mP_0\right)$$

$$\vdots$$

We observe that the Q_i are projective Λ-modules. Indeed, first we note that the free rank one bimodule $B \otimes A$ is projective as a left (or right module), hence any projective bimodule (for instance M) is projective as a left (or right module). Consequently for any left A-module X, the left B-module $M \otimes_A X$ is projective. Finally, Lemma 2.8 shows that each direct summand of Q_i is a projective Λ-module.

The differentials are defined in the figure below:



It is immediate to check that the differentials are morphisms of Λ-modules, that is, the corresponding squares commute (see Definition 2.6 and Proposition 2.7).

The column with X in the bottom is the projective resolution $P_\bullet \to X$. We observe that the two columns on its right give the mapping cone of the identity of the complex $(MP_\bullet, -1\otimes p)$. Since the identity is an isomorphism, the mapping cone is exact. Similarly, the next two columns on the right provide the mapping cone of the identity for the complex $(M(NM)P_\bullet, -1\otimes p)$, and so forth.

The two columns on the left of $P_\bullet \to X$ correspond to the mapping cone of the identity of the complex $(NMP_\bullet, 1\otimes p)$. The next two columns on the left are the mapping cone of the identity of $\big((NM)^2P_\bullet, 1\otimes p\big)$, and so forth.

Consequently $Q_\bullet$ is a resolution of $X \rightleftharpoons 0$ by projective Λ-modules.

Let r be an integer such that $(NM)^i = 0$ for $i > r$. Moreover let l be an integer such that $P_j = 0$ for $j > l$. For a given m, the module Q_m is the direct sum of vector spaces of the form $(NM)^iP_j$ for $2i+j = m$ or $2i+j = m+1$, and of vector spaces of the form $M(NM)^iP_j$ for $2i+j = m$ or $2i+j = m+1$. Let $m > 2r+l$. In case $2i+j = m$ or $m+1$, either $i > r$ or $j > l$ since otherwise $2i+j \leq 2r+l < m$. Hence $Q_m = 0$ for all $m > 2r+l$. ◇

Corollary 4.9 *Let k be a perfect field and let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a finite dimensional null-square projective algebra where A and B are smooth.*

If $(NM)^n = 0$ for large enough n, then Λ is smooth.

Proof. The complete list of simple Λ-modules is $\{S \rightleftharpoons 0\} \bigcup \{0 \rightleftharpoons T\}$ where S and T are simple modules over A and B respectively, see Proposition 2.9. The previous theorem shows that $S \rightleftharpoons 0$ is of finite projective dimension. The analogous theorem holds for Λ-modules of the form $0 \rightleftharpoons Y$ where Y is a B-module. Then the simple modules $0 \rightleftharpoons T$ are also of finite projective dimension. ◇

Theorem 4.10 *Let k be a perfect field. Any finite dimensional null-square projective k-algebra built on the class $\mathcal{H}$ of basic k-algebras verifying Han's conjecture also belongs to $\mathcal{H}$.*

Proof. Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$, where A and B are finite dimensional basic k-algebras which belong to $\mathcal{H}$, and M and N are projective bimodules. Suppose $HH_n(\Lambda) = 0$ for n large enough. Then by Corollary 4.7, $HH_n(A)$ and $HH_n(B)$ vanish for the same set of n's, hence A and B are smooth. Moreover, by Theorem 4.6 we have $(NM)^n = 0$ for n large enough. The previous corollary shows that Λ is smooth. ◇

Remark 4.11 *As much as in Remark 2.23, we observe that according to Corollary 4.7 this result agrees with the property proved by B. Keller in [14, 2.5], namely the Hochschild homology of a finite dimensional smooth algebra over a perfect field is concentrated in degree zero.*

5 Gabriel quiver and relations of a null-square projective algebra

Let A be a finite dimensional algebra such that $A/\operatorname{rad} A$ is a product of copies of k, in other words A is basic - equivalently, A is Morita reduced - and sober - that is, the algebra of A-endomorphisms of each simple A-module is just k. Let E be a complete system of primitive and orthogonal idempotents. The set of vertices of the *Gabriel quiver* Q_A is E; the number of arrows from x to y is the dimension of the vector space $y(\operatorname{rad} A/\operatorname{rad}^2 A)x$. It is well known that Q_A is canonical, in the sense that Q_A does not depend on the choice of E.

Let Q be a quiver with finite set of vertices Q_0 and set of arrows Q_1. The vector space kQ_0 is endowed with a semisimple algebra structure where Q_0 is a complete set of primitive orthogonal idempotents. Note that kQ_0 is basic and sober. The vector space kQ_1 is a kQ_0-bimodule in the natural way. The *path algebra* kQ is by definition the tensor algebra $T_{kQ_0}(kQ_1)$, it has a canonical basis given by the oriented paths of Q. The universal property of kQ is as follows: any algebra map $\varphi : kQ \to X$ is determined by an algebra map $\varphi_0 : kQ_0 \to X$ - that is, a set map from Q_0 to a system of X -, and $\varphi_1 : kQ_1 \to X$, a kQ_0-bimodule map - the structure of X as kQ_0-bimodule being inferred from φ_0.

A finite dimensional algebra A as above can be *presented*, namely there exists a - non canonical - algebra surjection $kQ \to A$ such that its kernel I is an admissible two-sided ideal, that is, there exist a positive integer m such that $F^m \subset I \subset F^2$, where F is the two sided ideal generated by $(Q_A)_1$. Moreover, the ideal I decomposes as $\oplus_{x,y\in E}\ yIx$ since $(Q_A)_0$ is complete. The system of generators R of I considered in a presentation is *adapted*, that is, R is graded with respect to this decomposition, its elements are called relations. Note that any system of generators R' gives rise to a graded one, namely $R = \bigsqcup_{x,y\in E} yR'x$, where for a set of paths Z, we denote by yZx the paths of Z starting at x and ending at y.

Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a finite dimensional null-square projective algebra, where A and B are basic and sober, with respective presentations (Q_A, R_A) and (Q_B, R_B), and where the projective bimodules M and N are given as in (4.1) and (4.2).

Lemma 5.1 *The Gabriel quiver of Λ is the disjoint union of Q_A, Q_B and new arrows as follows:*

- ${}_g m_e$ *arrows from $e \in E$ to $g \in F$, which we call* down arrows*,*
- ${}_f n_h$ *arrows from $h \in F$ to $f \in E$, which we call* up arrows*.*

Proof. The description of the Jacobson radical of Λ as given in the proof of Lemma 2.9 provides immediately the result. ◇

Let $T_{A\times B}(M \oplus N)$ be the tensor algebra of the $A\times B$-bimodule $M \oplus N$, where, as already mentioned, the given actions are extended by zero in order to consider M and N as $A \times B$-bimodules. For instance we infer $M \otimes_{A\times B} M = N \otimes_{A\times B} N = 0$.

The next two results are easy to prove, using both that M and N are projective bimodules, and the universal properties of the algebras involved.

Lemma 5.2 *There is an algebra isomorphism $\varphi : T_{A\times B}(M\oplus N) \to kQ_\Lambda/\langle R_A, R_B\rangle$.*

Lemma 5.3 *Let $\psi : T_{A\times B}(M\oplus N) \to \Lambda$ be the algebra map given by the inclusions of $A \times B$ and $M \oplus N$.*

$$\operatorname{Ker}\psi = \langle (M \oplus N)^{\otimes_{(A\times B)}2} \rangle = \langle N \otimes_B M + M \otimes_A N \rangle.$$

The set of all oriented paths of Q_A generates the vector space $kQ_A/\langle R_A \rangle$, hence we can choose a subset P_A which is a basis of $kQ_A/\langle R_A \rangle$. Let also P_B be a basis of $kQ_B/\langle R_B \rangle$, where P_B is a subset of the oriented paths of Q_B.

Let u be a down arrow from e to g, and let v be an up arrow from h to f in Q_Λ. We define the sets $v \curlyvee u$ and $u \curlyvee v$ of oriented paths of Q_Λ as follows:

$$v \curlyvee u = v(hP_Bg)u \quad \text{and} \quad u \curlyvee v = u(eP_Af)v.$$

Let R be the disjoint union of R_A, R_B, and $v \curlyvee u$ and $u \curlyvee v$ for all pairs (u, v), where u is a down arrow and v is an up arrow.

Theorem 5.4 *Let $\Lambda = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a finite dimensional null-square projective algebra, where A and B are basic and sober algebras with presentations (Q_A, R_A) and (Q_B, R_B) respectively, and where the projective bimodules M and N are given as in (4.1) and (4.2). The algebra Λ is presented by (Q_Λ, R).*

Proof. The key point of the proof is the following. Consider the image of $\operatorname{Ker}\psi$ by φ (see Lemma 5.2) in $kQ_\Lambda/\langle R_A, R_B \rangle$. Let $Bg \otimes eA$ be a direct summand of M, and $Af \otimes hB$ be a direct summand of N. They provide the direct summand $Bg \otimes eAf \otimes hB$ of $M \otimes_A N \subset (T_{A\times B}(M \oplus N))_2 \subset \operatorname{Ker}\psi$. In order to consider its image by φ, let u and v be the arrows in Q_Λ associated respectively to $Bg \otimes eA$ and $Af \otimes hB$. The image of $Bg \otimes eAf \otimes hB$ in $kQ_\Lambda/\langle R_A, R_B \rangle$ is generated by $u \curlyvee v$. ◇

Example 5.5 *Let Q_A be a crown quiver with three arrows a_0, a_1 and a_2, starting respectively at e_0, e_1 and e_2. Let $R_A = \{a_2a_0\}$. It is easy to establish that $A = kQ_A/\langle R_A \rangle$ is smooth. Let (Q_B, R_B) be a presentation of a basic and sober algebra B, and let g and h be vertices of B.*

Let $M = Bh \otimes e_1A$ and $N = Ae_2 \otimes gB$, and u from e_1 to h and v from g to e_2 the corresponding arrows. Note that the (N, M)-quiver has no oriented cycles. Let Λ be the corresponding null-square projective algebra, next we describe its Gabriel quiver and a set of relations:

- $Q_\Lambda = Q_A \cup Q_B \cup \{u, v\}$.
- $R = \{a_2a_0, R_B\} \cup \{v\gamma u\}_{\gamma \in gP_Bh}$, *where P_B is basis of oriented paths of B.*

Moreover it follows from the previous results that if B is smooth, then Λ is smooth.

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