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Kirillov’s orbit method: the case of discrete series representations
Paul-Emile PARADAN *
September 1, 2017

Abstract
Let $\pi$ be a discrete series representation of a real semi-simple Lie group $G'$ and let $G$ be a semi-simple subgroup of $G'$. In this paper, we give a geometric expression of the $G$-multiplicities in $\pi|_G$ when the representation $\pi$ is $G$-admissible.

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1 Introduction

This paper is concerned by a central problem of non-commutative harmonic analysis: given a unitary irreducible representation \( \pi \) of a Lie group \( G \), how does \( \pi \) decomposes when restricted to a closed subgroup \( G \subset G' \)? We analyse this problem for Harish-Chandra discrete series representations of a connected real semi-simple Lie group \( G' \) with finite center, relatively to a connected real semi-simple subgroup \( G \) (also with finite center).

We start with Harish-Chandra parametrization of the discrete series representations. We can attach an unitary irreducible representation \( \pi_{G,G'} \) of the group \( G' \) to any regular admissible elliptic coadjoint orbit \( O' \subset (g')^* \), and Schmid proved that the representation \( \pi_{G,G'} \) could be realize as the quantization of the orbit \( O' \) [34, 35]. This is a vast generalization of Borel-Weil-Bott’s construction of finite dimensional representations of compact Lie groups. In the following, we denote \( G_d' \) and \( G_d \) the sets of regular admissible elliptic coadjoint orbits of our connected real semi-simple Lie groups \( G \) and \( G' \).

One of the rule of Kirillov’s orbit method [13] is concerned with the functoriality relatively to inclusion \( G \hookrightarrow G' \) of closed subgroups. It means that, starting with discrete series representations representations \( \pi_{G,G} \) and \( \pi_{G,G'} \) attached to regular admissible elliptic orbits \( O \subset g^* \) and \( O' \subset (g')^* \), one expects that the multiplicity of \( \pi_{G,G} \) in the restriction \( \pi_{G,G'} \mid_O \) can be computed geometrically in terms of the space

\[
O'/O := O' \cap p_{g,g'}^{-1}(O)/G,
\]

where \( p_{g,g'} : (g')^* \rightarrow g^* \) denotes the canonical projection. One recognises that (1.1) is a symplectic reduced space in the sense of Marsden-Weinstein, since \( p_{g,g'} : O' \rightarrow g^* \) is the moment map relative to the Hamiltonian action of \( G \) on \( O' \).

In other words, Kirillov’s orbit method tells us that the branching laws \( \left[ \pi_{G,G} : \pi_{G,G'} \right] \) should be compute geometrically. So far, the following special cases have been achieved:

1. \( G \subset G' \) are compact. In the 1980s, Guillemin and Sternberg [8] studied the geometric quantization of general \( G \)-equivariant compact Kähler
manifolds. They proved the ground-breaking result that the multiplicities of this $G$-representation are calculated in terms of geometric quantizations of the symplectic reduced spaces. This phenomenon, which has been the center of many research and generalisations [22, 23, 37, 24, 21, 26, 33, 31, 10], is called nowadays “quantization commutes with reduction” (in short, “[Q,R]=0”).

2. $G$ is a compact subgroup of $G'$. In [25], we used the Blattner formula to see that the [Q,R]=0 phenomenon holds in this context when $G$ is a maximal compact subgroup. Duflo-Vergne have generalized this result for any compact subgroup [7]. Recently, Hochs-Song-Wu have shown that the [Q,R]=0 phenomenon holds for any tempered representation of $G'$ relatively to a maximal compact subgroup [11].

3. $\pi^{G'}_{\mathcal{O}}$ is a holomorphic discrete series. We prove that the [Q,R]=0 phenomenon holds with some assumption on $G$ [29].

However, one can observe that the restriction of $\pi^{G'}_{\mathcal{O}}$ with respect to $G$ may have a wild behavior in general, even if $G$ is a maximal reductive subgroup in $G'$ (see [15]).

In [15, 16, 17] T. Kobayashi singles out a nice class of branching problems where each $G$-irreducible summand of $\pi|_G$ occurs discretely with finite multiplicity: the restriction $\pi|_G$ is called $G$-admissible.

So we focus our attention to a discrete series $\pi^{G'}_{\mathcal{O}}$ that admit an admissible restriction relatively to $G$. It is well-known that we have then an Hilbertian direct sum decomposition

$$\pi^{G'}_{\mathcal{O}}|_G = \sum_{\mathcal{O} \in \tilde{G}_d} m^{G'}_{\mathcal{O}} \pi^{G'}_{\mathcal{O}}$$

where the multiplicities $m^{G'}_{\mathcal{O}}$ are finite.

We will use the following geometrical characterization of the $G$-admissibility obtained by Duflo and Vargas [5, 6].

**Proposition 1.1** The representation $\pi^{G'}_{\mathcal{O}}$ is $G$-admissible if and only if the restriction of the map $p_{\mathfrak{g}'}$ to the coadjoint orbit $\mathcal{O}'$ is a proper map.

Let $(\mathcal{O}', \mathcal{O}) \in \tilde{G}'_d \times \tilde{G}_d$. Let us explain how we can quantize the compact symplectic reduced space $\mathcal{O}'/\mathcal{O}$ when the map $p_{\mathfrak{g}'} : \mathcal{O}' \to \mathfrak{g}^*$ is proper.

If $\mathcal{O}$ belongs to the set of regular values of $p_{\mathfrak{g}'} : \mathcal{O}' \to \mathfrak{g}^*$, then $\mathcal{O}'/\mathcal{O}$ is a compact symplectic orbifold equipped with a spin$^c$ structure. We denote $Q^{spin}(\mathcal{O}'/\mathcal{O}) \in \mathbb{Z}$ the index of the corresponding spin$^c$ Dirac operator.
In general, we consider an elliptic coadjoint $O$, closed enough\(^1\) to $O$, so that $O'/O_\epsilon$ is a compact symplectic orbifold equipped with a spin$^c$ structure. Let $Q^{\text{spin}}(O'/O_\epsilon) \in \mathbb{Z}$ be the index of the corresponding spin$^c$ Dirac operator. The crucial fact is that the quantity $Q^{\text{spin}}(O'/O_\epsilon)$ does not depend on the choice of generic and small enough $\epsilon$. Then we take

$$Q^{\text{spin}}(O'/O) := Q^{\text{spin}}(O'/O_\epsilon)$$

for generic and small enough $\epsilon$.

The main result of this article is the following

**Theorem 1.2** Let $\pi_{G'}^G$ be a discrete series representation of $G'$ attached to a regular admissible elliptic coadjoint orbits $O'$. If $\pi_{G'}^G$ is $G$-admissible we have the Hilbertian direct sum

$$\pi\big|_{G'} = \sum_{O \in G_d} Q^{\text{spin}}(O'/O) \pi_{G'}^G.$$  \hspace{1cm} (1.2)

In other words the multiplicity $[\pi_{G'}^G : \pi_{G'}^G]$ is equal to $Q^{\text{spin}}(O'/O)$.

In a forthcoming paper we will study Equality (1.2) in further details when $G$ is a symmetric subgroup of $G'$.

Theorem 1.2 give a positive answer to a conjecture of Duflo-Vargas.

**Theorem 1.3** Let $\pi_{G'}^G$ be a discrete series representation of $G'$ that is $G$-admissible. Then all the representations $\pi_{G'}^G$ which occurs in $\pi_{G'}^G$ belongs to a unique family of discrete series representations of $G$.

2  Restriction of discrete series representations

Let $G$ be a connected real semi-simple Lie group $G$ with finite center. A discrete series representation of $G$ is an irreducible unitary representation that is isomorphic to a sub-representation of the left regular representation in $L^2(G)$. We denote $\hat{G}_d$ the set of isomorphism class of discrete series representation of $G$.

We know after Harish-Chandra that $\hat{G}_d$ is non-empty only if $G$ has a compact Cartan subgroup. We denote $K \subset G$ a maximal compact subgroup and we suppose that $G$ admits a compact Cartan subgroup $T \subset K$. The Lie algebras of the groups $T, K, G$ are denoted respectively $\mathfrak{t}, \mathfrak{k}$ and $\mathfrak{g}$.

In this section we recall well-know facts concerning restriction of discrete series representations.

\(^1\)The precise meaning will be explain in Section 5.2.
2.1 Admissible coadjoint orbits

Here we recall the parametrization of $\hat{G}_d$ in terms of regular admissible elliptic coadjoint orbits. Let us fix some notations. We denote $\Lambda \subset \mathfrak{t}^*$ the weight lattice: any $\mu \in \Lambda$ defines a 1-dimensional representation $C_\mu$ of the torus $T$.

Let $\mathcal{R}_c \subset \mathcal{R} \subset \Lambda$ be respectively the set of (real) roots for the action of $T$ on $\mathfrak{t} \otimes \mathbb{C}$ and $\mathfrak{g} \otimes \mathbb{C}$. The non-compact roots are those belonging to the set $\mathcal{R}_n := \mathcal{R} \setminus \mathcal{R}_c$. We choose a system of positive roots $\mathcal{R}_c^+$ for $\mathcal{R}_c$, we denote by $\mathfrak{t}_c^+$ the corresponding Weyl chamber. Recall that $\Lambda \cap \mathfrak{t}_c^+$ is the set of dominant weights.

We denote by $B$ the Killing form on $\mathfrak{g}$. It induces a scalar product (denoted by $(-,-)$) on $\mathfrak{t}$, and then on $\mathfrak{t}^*$. An element $\lambda \in \mathfrak{t}^*$ is called $G$-regular if $(\lambda, \alpha) \neq 0$ for every $\alpha \in \mathcal{R}$, or equivalently, if the stabilizer subgroup of $\lambda$ in $G$ is $T$. For any $\lambda \in \mathfrak{t}^*$ we denote

$$\rho(\lambda) := \frac{1}{2} \sum_{\alpha \in \mathcal{R}_c, (\alpha,\lambda) > 0} \alpha.$$  

We denote also $\rho_c := \frac{1}{2} \sum_{\alpha \in \mathcal{R}_c^+} \alpha$.

**Definition 2.1**

1. A coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ is elliptic if $\mathcal{O} \cap \mathfrak{t}^* \neq \emptyset$.

2. An elliptic coadjoint orbit $\mathcal{O}$ is admissible\(^2\) when $\lambda - \rho(\lambda) \in \Lambda$ for any $\lambda \in \mathcal{O} \cap \mathfrak{t}^*$.

Harish-Chandra has parametrized $\hat{G}_d$ by the set of regular admissible elliptic coadjoint orbits of $G$. In order to simplify our notation, we denote $\hat{G}_d$ the set of regular admissible elliptic coadjoint orbits. For an orbit $\mathcal{O} \in \hat{G}_d$ we denote $\pi_\mathcal{O}$ the corresponding discrete series representation of $G$.

Consider the subset $(\mathfrak{t}_c^+)_se := \{ \xi \in \mathfrak{t}_c^+, (\xi, \alpha) \neq 0, \forall \alpha \in \mathcal{R}_n \}$ of the Weyl chamber. The subscript means strongly elliptic, see Section 5.1. By definition any $\mathcal{O} \in \hat{G}_d$ intersects $(\mathfrak{t}_c^+)_se$ in a unique point.

**Definition 2.2** The connected component $(\mathfrak{t}_c^+)_se$ are called chambers. If $\mathcal{C}$ is a chamber, we denote $\hat{G}_d(\mathcal{C}) \subset \hat{G}_d$ the subset of regular admissible elliptic orbits intersecting $\mathcal{C}$.

---

\(^2\)Duflo has defined a notion of admissible coadjoint orbits in a much broader context [4].
Notice that the Harish-Chandra parametrization has still a meaning when $G = K$ is a compact connected Lie group. In this case $\tilde{K}$ corresponds to the set of regular admissible coadjoint orbits $\mathcal{P} \subset \mathfrak{k}^*$, i.e. those of the form $\mathcal{P} = K \mu$ where $\mu - \rho_c \in \Lambda \cap \mathfrak{k}_+^*$: the corresponding representation $\pi^K_{\mathcal{P}}$ is the irreducible representation of $\tilde{K}$ with highest weight $\mu - \rho_c$.

2.2 Spinor representation

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$: the Killing form of $\mathfrak{g}$ defines a $K$-invariant Euclidean structure on it. Note that $\mathfrak{p}$ is even dimensional since the groups $G$ and $K$ have the same rank.

We consider the two-fold cover $\text{Spin}(\mathfrak{p}) \to \text{SO}(\mathfrak{p})$ and the morphism $K \to \text{SO}(\mathfrak{p})$. We recall the following basic fact.

**Lemma 2.3** There exists a unique covering $\tilde{K} \to K$ such that

1. $\tilde{K}$ is a compact connected Lie group,
2. the morphism $K \to \text{SO}(\mathfrak{p})$ lifts to a morphism $\tilde{K} \to \text{Spin}(\mathfrak{p})$.

Let $\xi \in \mathfrak{t}^*$ be a regular element and consider

$$(2.3) \quad \rho_n(\xi) := \frac{1}{2} \sum_{\alpha \in \mathfrak{h}^*, (\alpha, \xi) > 0} \alpha.$$

Note that

$$(2.4) \quad \tilde{\Lambda} = \Lambda \bigcup \{\rho_n(\xi) + \Lambda\}$$

is a lattice that does not depend on the choice of $\xi$.

Let $T \subset K$ be a maximal torus and $\tilde{T} \subset \tilde{K}$ be the pull-back of $T$ relatively to the covering $\tilde{K} \to K$. We can now precise Lemma 2.3.

**Lemma 2.4** Two situations occur:

1. if $\rho_n(\xi) \in \Lambda$ then $\tilde{K} \to K$ and $\tilde{T} \to T$ are isomorphisms, and $\tilde{\Lambda} = \Lambda$.
2. if $\rho_n(\xi) \notin \Lambda$ then $\tilde{K} \to K$ and $\tilde{T} \to T$ are two-fold covers, and $\tilde{\Lambda}$ is the lattice of weights for $\tilde{T}$.

Let $S_p$ the spinor representation of the group $\text{Spin}(\mathfrak{p})$. Let $c : \text{Cl}(\mathfrak{p}) \to \text{End}_{\mathbb{C}}(S_p)$ be the Clifford action. Let $o$ be an orientation on $\mathfrak{p}$. If $e_1, e_2, \cdots, e_{\dim \mathfrak{p}}$ is an oriented orthonormal base of $\mathfrak{p}$ we define the element

$$e_o := (i)^{\frac{\dim \mathfrak{p}}{2}} e_1 e_2 \cdots e_{\dim \mathfrak{p}} \in \text{Cl}(\mathfrak{p}) \otimes \mathbb{C}. $$
that depends only of the orientation. We have $\epsilon_o^2 = -1$ and $\epsilon_ov = -ve$ for any $v \in \mathfrak{p}$. The element $c(\epsilon_o)$ determines a decomposition $S_\mathfrak{p} = S_\mathfrak{p}^+oo \oplus S_\mathfrak{p}^-oo$ into irreducible representations $S_\mathfrak{p}^{\pm, o} = \ker(c(\epsilon_o) \pm Id)$ of $\text{Spin}(\mathfrak{p})$. We denote

$$S^{o}_\mathfrak{p} := S_\mathfrak{p}^+oo \oplus S_\mathfrak{p}^-oo$$

the corresponding virtual representation of $\hat{K}$.

**Remark 2.5** If $o$ and $o'$ are two orientations on $\mathfrak{p}$, we have $S^o_\mathfrak{p} = \pm S^{o'}_\mathfrak{p}$, where the sign $\pm$ is the ratio between $o$ and $o'$.

**Example 2.6** Let $\lambda \in \mathfrak{k}$ such that the map $\text{ad}(\lambda) : \mathfrak{p} \to \mathfrak{p}$ is one to one. We get a symplectic form $\Omega_\lambda$ on $\mathfrak{p}$ defined by the relations $\Omega_\lambda(X,Y) = \langle \lambda, [X,Y] \rangle$ for $X,Y \in \mathfrak{p}$. We denote $o(\lambda)$ the orientation of $\mathfrak{p}$ defined by the top form $\Omega_{\lambda \dim \mathfrak{p}/2}$.

### 2.3 Restriction to the maximal compact subgroup

We start with a definition.

**Definition 2.7** We denote $\hat{R}(G,d)$ the group formed by the formal (possibly infinite) sums

$$\sum_{\mathcal{O} \in \hat{G}_d} a_\mathcal{O} \pi^{\mathcal{O}}_\mathfrak{p}$$

where $a_\mathcal{O} \in \mathbb{Z}$.

- Similarly we denote $\hat{R}(K)$ the group formed by the formal (possibly infinite) sums $\sum_{P \in \hat{K}} a_P \pi^{P}_\mathfrak{p}$ where $a_P \in \mathbb{Z}$.

The following technical fact will be used in the proof of Theorem 1.2.

**Proposition 2.8** Let $o$ be an orientation on $\mathfrak{p}$.

- The restriction morphism $V \to V|_K$ defines a map $\hat{R}(G,d) \to \hat{R}(K)$.
- The map $r^o : \hat{R}(G,d) \to \hat{R}(\hat{K})$ defined by $r^o(V) := V|_K \otimes S^o_\mathfrak{p}$ is one to one.

**Proof.** When $\mathcal{O} = G\lambda \in \hat{G}_d$, with $\lambda \in t^*$, we denote $e^{\mathcal{O}}_\mathfrak{p} = \|\lambda + \rho(\lambda)\|$. Similarly when $P = K\mu \in \hat{K}$, with $\mu - \rho_c \in \Lambda \cap t^*_+$, we denote $e^K_\mathfrak{p} = \|\mu + \rho_c\|$.

Note that for each $r > 0$ the set $\{\mathcal{O} \in \hat{G}_d, e^{\mathcal{O}}_\mathfrak{p} \leq r\}$ is finite.
Consider now the restriction of a discrete series representation $\pi^G$ relatively to $K$. The Blattner’s formula [9] tells us that the restriction $\pi^G_{|K}$ admits a decomposition

$$\pi^G_{|K} = \sum_{P \in \mathcal{K}} m_{\mathcal{O}(P)} \pi^K_{P}$$

where the (finite) multiplicities $m_{\mathcal{O}(P)}$ are non-zero only if $c^K_{P} \geq c^G$. Consider now an element $V = \sum_{\mathcal{O} \in \mathcal{G}_d} a_\mathcal{O} \pi^G_{|K} \in \hat{R}(G, d)$. The multiplicity of $\pi^K_{P}$ in $V_{|K}$ is equal to $\sum_{\mathcal{O} \in \mathcal{G}_d} a_\mathcal{O} m_{\mathcal{O}(P)}$.

Here the sum admits a finite number of non-zero terms since $m_{\mathcal{O}(P)} = 0$ if $c^K_{P} > c^K_{P}$. So we have proved that the $K$-multiplicities of $V_{|K} := \sum_{\mathcal{O} \in \mathcal{G}_d} a_\mathcal{O} \pi^G_{|K}$ are finite. The first point is proved.

The irreducible representation of $\hat{K}$ are parametrized by the set $\hat{\mathcal{K}}$ of regular $\hat{\mathcal{K}}$-admissible coadjoint orbits $P \subset \mathfrak{k}^*$, i.e. those of the form $P = K\mu$ where $\mu - \rho_c \in \Lambda \cap \mathfrak{k}_+$. It contains the set $\mathcal{K}$ of regular $K$-admissible coadjoint orbits. We define

$$\mathcal{K}_{\text{out}} \subset \hat{\mathcal{K}}$$

as the set of coadjoint orbits $P = K\mu$ where $\mu - \rho_c \in \Lambda \cap \mathfrak{k}_+$. Here $\mathfrak{g}$ is any regular element of $\mathfrak{t}^*$ and $\rho_c(\xi)$ is defined by (2.3).

We notice that $\mathcal{K}_{\text{out}} = \mathcal{K}$ when $\hat{K} \simeq K$ and that $\hat{\mathcal{K}} = \hat{K} \cup \mathcal{K}_{\text{out}}$ when $\hat{K} \to K$ is a two-fold cover.

We will use the following basic facts.

**Lemma 2.9**

1. $\mathcal{O} \mapsto \mathcal{O}_K := \mathcal{O} \cap \mathfrak{k}^*$ defines an injective map between $\mathcal{G}_d$ and $\mathcal{K}_{\text{out}}$.

2. We have $\pi^G_{|K} \otimes \mathcal{S}_{\mathfrak{c}} = \pm \pi^K_{\mathcal{O}_K}$ for all $\mathcal{O} \in \mathcal{G}_d$.

**Proof.** Let $\mathcal{O} := G\lambda \in \mathcal{G}_d$ where $\lambda$ is a regular element of the Weyl chamber $\mathfrak{t}_+^*$. Then $\mathcal{O}_K = K\lambda$ and the term $\lambda - \rho_c$ is equal to the sum $\lambda - \rho(\lambda) + \rho_n(\lambda)$ where $\lambda - \rho(\lambda) \in \Lambda$ and $\rho_n(\lambda) \in \hat{\Lambda}$ (see (2.4)), so $\lambda - \rho_c = \rho_n(\xi) \cup \Lambda$. The element $\lambda \in \mathfrak{t}_+^*$ is regular and admissible for $\hat{K}$: this implies that $\lambda - \rho_c \in \mathfrak{t}_+^*$. We have proved that $\mathcal{O}_K \in \mathcal{K}_{\text{out}}$.

\[\text{a}\] The set $\{\rho_n(\xi) \cup \Lambda\} \cap \mathfrak{t}_+^*$ does not depend on the choice of $\xi$. 8
The second point is a classical result (a generalisation is given in Theorem 5.7). Let us explain the sign \(\hat{\cdot}\) in the relation. Let \(O \in \tilde{G}_d\) and \(\lambda \in O \cap \mathfrak{t}^\ast\). Then the sign \(\hat{\cdot}\) is the ratio between the orientations \(o\) and \(o(-\lambda)\) of the vector space \(p\) (see Example 2.6).

We can now finish the proof of the second point of Proposition 2.8. If \(V = \sum_{O \in \tilde{G}_d} a_O \pi_O^G \in \tilde{R}(G, d)\), then \(r^\circ(V) = \sum_{O \in \tilde{G}_d} a_O \pi_{O,K}^\tilde{K}\). Hence \(r^\circ(V) = 0\) only if \(V = 0\). □

### 2.4 Admissibility

Let \(\pi_O^G\) be a discrete series representation of \(G\) attached to a regular admissible elliptic orbit \(O' \subset (g')^\ast\).

We denote \(\text{As}(O') \subset (g')^\ast\) the asymptotic support of the coadjoint orbit \(O'\): by definition \(\xi \in \text{As}(O')\) if \(\xi = \lim_{n \to \infty} t_n \xi_n\) with \(\xi_n \in O'\) and \((t_n)\) is a sequence of positive number tending to 0.

We consider here a closed connected semi-simple Lie subgroup \(G \subset G'\). We choose maximal compact subgroups \(K \subset G\) and \(K' \subset G'\) such that \(K' \subset K\). We denote \(\mathfrak{t}^\perp \subset (\mathfrak{t}')^\ast\) the orthogonal (for the duality) of \(\mathfrak{t} \subset \mathfrak{t}'\).

The moment map relative to the \(G\)-action on \(O'\) is by definition the map \(\Phi_G : O' \to g^\ast\) which is the composition of the inclusion \(O' \hookrightarrow (g')^\ast\) with the projection \((g')^\ast \to g^\ast\). We use also the moment map \(\Phi_K : O' \to \mathfrak{t}^\ast\) which is the composition of \(\Phi_G\) with the projection \(g^\ast \to \mathfrak{t}^\ast\).

Let \(pr_{g'} : (g')^\ast \to (\mathfrak{t}')^\ast\) be the canonical projection. The main objective of this section is the proof of the following result that refines Proposition 1.1.

**Theorem 2.10** The following facts are equivalent:

1. The representation \(\pi_O^G\) is \(G\)-admissible.
2. The moment map \(\Phi_G : O' \to g^\ast\) is proper.
3. \(pr_{g'}(\text{As}(O')) \cap \mathfrak{t}^\perp = \{0\}\).

Theorem 2.10 is a consequence of different equivalences. We start with the following result that is proved in [5, 29].

**Lemma 2.11** The map \(\Phi_G : O' \to g^\ast\) is proper if and only if the map \(\Phi_K : O' \to \mathfrak{t}^\ast\) is proper.

We have the same kind of equivalence for the admissibility.
Lemma 2.12 The representation \( \pi_{\mathcal{O}'}^{G} \) is \( G \)-admissible if and only if it is \( K \)-admissible.

Proof. The fact that \( K \)-admissibility implies \( G \)-admissibility is proved by T. Kobayashi in [15]. The opposite implication is a consequence of the first point of Proposition 2.8.

At this stage, the proof of Theorem 2.10 is complete if we show that the following facts are equivalent:

(a) The representation \( \pi_{\mathcal{O}'}^{G} \) is \( K \)-admissible.
(b) The moment map \( \Phi_{K} : \mathcal{O}' \to \mathfrak{t}^{*} \) is proper.
(c) \( p_{\mathfrak{t}, \mathfrak{g'}}(\text{As}(\mathcal{O}')) \cap \mathfrak{t}^\perp = \{0\} \).

We start by proving the equivalence (b) \( \iff \) (c).

Proposition 2.13 ([29]) The map \( \Phi_{K} : \mathcal{O}' \to \mathfrak{t}^{*} \) is proper if and only

\[ p_{\mathfrak{t}, \mathfrak{g'}}(\text{As}(\mathcal{O}')) \cap \mathfrak{t}^\perp = \{0\} \]

Proof. The moment map \( \Phi_{K'} : \mathcal{O}' \to (\mathfrak{t}')^{*} \) relative to the action of \( K' \) on \( \mathcal{O}' \) is a proper map that corresponds to the restriction of the projection \( p_{\mathfrak{t}, \mathfrak{g'}} \) to \( \mathcal{O}' \).

Let \( T' \) be a maximal torus in \( K' \) and let \( (t')^{*}_{+} \subset (\mathfrak{t}')^{*} \) be a Weyl chamber. The convexity theorem [14, 20] tells us that \( \Delta_{K'}(\mathcal{O}') = p_{\mathfrak{t}, \mathfrak{g'}}(\mathcal{O}') \cap (t')^{*}_{+} \) is a closed convex polyedral subset. We have proved in [29][Proposition 2.10], that \( \Phi_{K} : \mathcal{O}' \to \mathfrak{t}^{*} \) is proper if and only

\[ K' \cdot \text{As}(\Delta_{K'}(\mathcal{O}')) \cap \mathfrak{t}^\perp = \{0\} \]

A small computation shows that \( K' \cdot \text{As}(\Delta_{K'}(\mathcal{O}')) = p_{\mathfrak{t}, \mathfrak{g'}}(\text{As}(\mathcal{O}')) \) since \( K' \cdot \Delta_{K'}(\mathcal{O}') = p_{\mathfrak{t}, \mathfrak{g'}}(\mathcal{O}') \). The proof of Proposition 2.13 is completed. \( \square \)

We denote \( \text{AS}_{K'}(\pi_{\mathcal{O}'}^{G}) \subset (\mathfrak{t}')^{*} \) the asymptotic support of the following subset of \( (\mathfrak{t}')^{*} \):

\[ \{ \mathfrak{p}' \in \widehat{K}', [\pi_{\mathfrak{p}'_{K'}}^{\mathfrak{g'}} : \pi_{\mathcal{O}'}^{G}] \neq 0 \} \]

The following important fact is proved by T. Kobayashi (see Section 6.3 in [18]).

Proposition 2.14 The representation \( \pi_{\mathcal{O}'}^{G} \) is \( K \)-admissible if and only if

\[ \text{AS}_{K'}(\pi_{\mathcal{O}'}^{G}) \cap \mathfrak{t}^\perp = \{0\} \]
We will use also the following result proved by Barbasch and Vogan (see Propositions 3.5 and 3.6 in [2]).

**Proposition 2.15** Let \( \pi_{O'}^{G'} \) be a representation of the discrete series of \( G' \) attached to the regular admissible elliptic orbit \( O' \). We have

\[
\text{AS}_{K'}(\pi_{O'}^{G'}) = p_{\mathfrak{g}'}(\text{As}(O')).
\]

Propositions 2.14 and 2.15 give the equivalence \( (a) \iff (c) \). The proof of Theorem 2.10 is completed. \( \square \)

In fact Barbasch and Vogan proved also in [2] that the set \( \text{As}(O') \) does not depend on \( O' \) but only on the chamber \( C' \) such that \( O' \in \widehat{G}''_o(C') \). We obtain the following corollary.

**Corollary 2.16** The \( G \)-admissibility of a discrete series representation \( \pi_{O'}^{G'} \) does not depend on \( O' \) but only on the chamber \( C' \) such that \( O' \in \widehat{G}''_o(C') \).

## 3 Spin\(^c\) quantization of compact Hamiltonian manifolds

### 3.1 Spin\(^c\) structures

Let \( N \) be an even dimensional Riemannian manifold, and let \( \text{Cl}(N) \) be its Clifford algebra bundle. A complex vector bundle \( \mathcal{E} \to N \) is a \( \text{Cl}(N) \)-module if there is a bundle algebra morphism \( c : \text{Cl}(N) \to \text{End}(\mathcal{E}) \).

**Definition 3.1** Let \( \mathcal{S} \to M \) be a \( \text{Cl}(N) \)-module such that the map \( c_{\mathcal{S}} \) induces an isomorphism \( \text{Cl}(N) \otimes_{\mathbb{R}} \mathbb{C} \to \text{End}(\mathcal{S}) \). Then we say that \( \mathcal{S} \) is a spin\(^c\)-bundle for \( N \).

**Definition 3.2** The determinant line bundle of a spin\(^c\)-bundle \( \mathcal{S} \) on \( N \) is the line bundle \( \text{det}(\mathcal{S}) \to M \) defined by the relation

\[
\text{det}(\mathcal{S}) := \text{hom}_{\text{Cl}(N)}(\overline{\mathcal{S}}, \mathcal{S})
\]

where \( \overline{\mathcal{S}} \) is the \( \text{Cl}(N) \)-module with opposite complex structure.

Basic examples of spin\(^c\)-bundles are those coming from manifolds \( N \) equipped with an almost complex structure \( J \). We consider the tangent bundle \( TN \) as a complex vector bundle and we define

\[
\mathcal{S}_J := \bigwedge_c TN.
\]
It is not difficult to see that $S_J$ is a spin$^c$-bundle on $N$ with determinant line bundle $\det(S_J) = \bigwedge_{\mathbb{C}}^{\max} TN$. If $L$ is a complex line bundle on $N$, then $S_J \otimes L$ is another spin$^c$-bundle with determinant line bundle equal to $\bigwedge_{\mathbb{C}}^{\max} TN \otimes L^{\otimes 2}$.

### 3.2 Spin$^c$-prequantization

In this section $G$ is a semi-simple connected real Lie group.

Let $M$ be an Hamiltonian $G$-manifold with symplectic form $\Omega$ and moment map $\Phi_G : M \to \mathfrak{g}^*$ characterized by the relation

$$i(X_M)\omega = -d\langle \Phi_G, X \rangle, \quad X \in \mathfrak{g},$$

where $X_M(m) := \frac{d}{dt}|_{t=0} e^{-tX} \cdot m$ is the vector field on $M$ generated by $X \in \mathfrak{g}$.

In the Kostant-Souriau framework [19, 36], a $G$-equivariant Hermitian line bundle $L_\Omega$ with an invariant Hermitian connection $\nabla$ is a prequantum line bundle over $\pi^* M, \Omega, \Phi_G$ if

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi_G, X \rangle \quad \text{and} \quad \nabla^2 = -i\Omega,$$

for every $X \in \mathfrak{g}$. Here $\mathcal{L}(X)$ is the infinitesimal action of $X \in \mathfrak{k}$ on the sections of $L_\Omega \to M$. The data $(L_\Omega, \nabla)$ is also called a Kostant-Souriau line bundle.

**Definition 3.3 ([28])** A $G$-Hamiltonian manifold $(M, \Omega, \Phi_G)$ is spin$^c$ prequantized if there exists an equivariant spin$^c$ bundle $S$ such that its determinant line bundle $\det(S)$ is a prequantum line bundle over $(M, \Omega, \Phi_G)$.

Consider the case of a regular elliptic coadjoint orbit $O = G\lambda$; here $\lambda \in \mathfrak{t}^*$ has a stabilizer subgroup equal to $T$. The tangent space $T_\lambda O \simeq \mathfrak{g}/T$ is an even dimensional Euclidean space, equipped with a linear action of $T$ and an $T$-invariant antisymmetric endomorphism $^4 \text{ad}(\lambda)$. Let $J_\lambda := \text{ad}(\lambda)(-\text{ad}(\lambda)^2)^{-1/2}$ be the corresponding $T$-invariant complex structure on $\mathfrak{g}/T$: we denote $V$ the corresponding $T$-module. It defines an integrable $G$-invariant complex structure on $O \simeq G/T$.

As we have explained in the previous section, the complex structure on $O$ defines the spin$^c$-bundle $S_o := \bigwedge_{\mathbb{C}} T^* O$ with determinant line bundle

$$\det(S_o) = \bigwedge_{\mathbb{C}}^{\max} T^* O \simeq G \times_T \bigwedge_{\mathbb{C}}^{\max} V.$$

$^4$Here we see $\lambda$ has an element of $\mathfrak{t}$, through the identification $\mathfrak{g}^* \simeq \mathfrak{g}$.  

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A small computation gives that the differential of the $T$-character $\Lambda^\text{max}_C V$ is equal to $i$ times $2\rho(\lambda)$. In other words, $\Lambda^\text{max}_C V = C_{2\rho(\lambda)}$.

In the next Lemma we see that for the regular elliptic orbits, the notion of admissible orbits is equivalent to the notion of spin$^c$-prequantized orbits.

Lemma 3.4 Let $O = G\lambda$ be a regular elliptic coadjoint orbit. Then $O$ is spin$^c$-prequantized if and only if $\lambda - \rho(\lambda) \in \Lambda$.

Proof. Any $G$-equivariant spin$^c$-bundle on $O$ is of the form $S_\phi \otimes L_\phi$, where $L_\phi = G \times_T \mathbb{C}_\lambda$ is a line bundle associated to a character $e^X \mapsto e^{i\langle \phi, X \rangle}$ of the group $T$. Then we have

$$\det(S_\phi) = \det(S_\phi) \otimes L_\phi^{\otimes 2} = G \times_T \mathbb{C}_{2\phi + 2\rho(\lambda)}.$$

By $G$-invariance we know that the only Kostant-Souriau line bundle on $(G\lambda, 2\Omega_{G\lambda})$ is the line bundle $G \times_T \mathbb{C}_\lambda$. Finally we see that $G\lambda$ is spin$^c$-prequantized by $S_\phi$ if and only if $\phi = \lambda - \rho(\lambda)$. □

If $O$ is a regular admissible elliptic coadjoint orbit, we denote $S_O := S_\phi \otimes L_{\lambda - \rho(\lambda)}$ the corresponding spin$^c$ bundle. Here we use the grading $S_O = S_O^+ \oplus S_O^-$ induced by the symplectic orientation.

3.3 Spin$^c$ quantization of compact manifolds

Let us consider a compact Hamiltonian $K$-manifold $(M, \Omega, \Phi_K)$ which is spin$^c$-prequantized by a spin$^c$-bundle $S$. The (symplectic) orientation induces a decomposition $S = S^+ \oplus S^-$, and the corresponding spin$^c$ Dirac operator is a first order elliptic operator [3]

$$D_S : \Gamma(M, S^+) \rightarrow \Gamma(M, S^-).$$

Its principal symbol is the bundle map $\sigma(M, S) \in \Gamma(T^*M, \text{hom}(p^*S^+, p^*S^-))$ defined by the relation

$$\sigma(M, S)(m, \nu) = c_{S_m^+(\hat{\nu}) : S_m^-} : S_m^+ \rightarrow S_m^-.$$

Here $\nu \in T^*M \mapsto \hat{\nu} \in TM$ is the identification defined by an invariant Riemannian structure.

Definition 3.5 The spin$^c$ quantization of a compact Hamiltonian $K$-manifold $(M, \Omega, \Phi_K)$ is the equivariant index of the elliptic operator $D_S$ and is denoted

$$Q^\text{spin}_K(M) \in R(K).$$
3.4 Quantization commutes with reduction

Now we will explain how the multiplicities of $Q_{K}^{\text{spin}}(M) \in R(K)$ can be computed geometrically.

Recall that the dual $\hat{K}$ is parametrized by the regular admissible coadjoint orbits. They are those of the form $\mathcal{P} = K\mu$ where $\mu - \rho_c \in \Lambda \cap t^*_+$. After Lemma 3.4, we know that any regular admissible coadjoint orbit $\mathcal{P} \in \hat{K}$ is spin$^c$-prequantized by a spin$^c$ bundle $S_{\mathcal{P}}$ and a small computation shows that $Q_{K}^{\text{spin}}(\mathcal{P}) = \frac{\pi_{\mathcal{P}}}{\pi_{\hat{K}}}$ (see [32]).

For any $\mathcal{P} \in K$, we define the symplectic reduced space
\[ M/\mathcal{P} := \Phi_K^{-1}(\mathcal{P})/K. \]

If $M/\mathcal{P} \neq \emptyset$, then any $m \in \Phi_K^{-1}(\mathcal{P})$ has abelian infinitesimal stabilizer. It implies then that the generic infinitesimal stabilizer for the $K$-action on $M$ is abelian.

Let us explain how we can quantize these symplectic reduced spaces (for more details see [25, 28, 33]).

**Proposition 3.6** Suppose that the generic infinitesimal stabilizer for the $K$-action on $M$ is abelian.

- If $\mathcal{P} \in \hat{K}$ belongs to the set of regular values of $\Phi_K : M \to t^*$, then $M/\mathcal{P}$ is a compact symplectic orbifold which is spin$^c$-prequantized. We denote $Q^{\text{spin}}(M/\mathcal{P}) \in \mathbb{Z}$ the index of the corresponding spin$^c$ Dirac operator [12].

- In general, if $\mathcal{P} = K\lambda$ with $\lambda \in t^*$, we consider the orbits $\mathcal{P}_\epsilon = K(\lambda + \epsilon)$ for generic small elements $\epsilon \in t^*$ so that $M/\mathcal{P}_\epsilon$ is a compact symplectic orbifold with a peculiar spin$^c$-structure. Let $Q^{\text{spin}}(M/\mathcal{P}_\epsilon) \in \mathbb{Z}$ be the index of the corresponding spin$^c$ Dirac operator. The crucial fact is that the quantity $Q^{\text{spin}}(M/\mathcal{P}_\epsilon)$ does not depend on the choice of generic and small enough $\epsilon$. Then we take
\[ Q^{\text{spin}}(M/\mathcal{P}) := Q^{\text{spin}}(M/\mathcal{P}_\epsilon) \]
for generic and small enough $\epsilon$.

The following theorem is proved in [25].

**Theorem 3.7** Let $(M, \Omega, \Phi_K)$ be a spin$^c$-prequantized compact Hamiltonian $K$-manifold. Suppose that the generic infinitesimal stabilizer for the $K$-action on $M$ is abelian. Then the following relation holds in $R(K)$:
\[ Q_{K}^{\text{spin}}(M) = \sum_{\mathcal{P} \in \hat{K}} Q_{M/\mathcal{P}}^{\text{spin}} \pi_{\mathcal{P}}^K. \]
Remark 3.8 Identity 3.7 admits generalisations when we do not have conditions on the generic stabilizer [28] and also when we allow the 2-form Ω to be degenerate [33]. In this article, we do not need such generalizations.

For \( P \) \( P \) \( P \) \( P \), we note \( P' \) the coadjoint orbit with \( P \) with opposite symplectic structure. The corresponding spin\( c \) bundle is \( S_{P^-} \). It is not difficult to see that \( Q^\text{spin}_K(P') = (\pi_K^P)^* \) (see [32]). The shifting trick tells us then that the multiplicity of \( \pi_K^P \) in \( Q^\text{spin}_K(M) \) is equal to \( [Q^\text{spin}_K(M \times P^-)]^K \).

If we suppose furthermore that the generic infinitesimal stabilizer is abelian we obtain the useful relation

\[
Q^\text{spin}(M/P) := [Q^\text{spin}_K(M \times P^-)]^K.
\]

Let \( \gamma \) that belongs to the center of \( K \): it acts trivially on the orbits \( P \in \hat{K} \). Suppose now that \( \gamma \) acts also trivially on the manifolds \( M \). We are interested by the action of \( \gamma \) on the fibers of the spin\( c \)-bundle \( S \times S_{P^-} \). We denote \( [S \times S_{P^-}]^\gamma \) the subbundle where \( \gamma \) acts trivially.

Lemma 3.9 If \( [S \times S_{P^-}]^\gamma = 0 \) then \( Q^\text{spin}(M/P) = 0 \).

Proof. Let \( D \) be the Dirac operator on \( M \times P^- \) associated to the spin\( c \) bundle \( S \times S_{P^-} \). Then

\[
\left[ Q^\text{spin}_K(M \times P^-) \right]^K = [\ker(D)]^K - [\coker(D)]^K.
\]

Obviously \([\ker(D)]^K \subset [\ker(D)]^\gamma \) and \([\ker(D)]^\gamma \) is contained in the set of smooth section of the bundle \([S \times S_{P^-}]^\gamma \). The same result holds for \([\coker(D)]^K \). Finally, if \([S \times S_{P^-}]^\gamma = 0 \), then \([\ker(D)]^K \) and \([\coker(D)]^K \) are reduced to 0. \( \square \)

4 Spin\( c \) quantization of non-compact Hamiltonian manifolds

In this section our Hamiltonian \( K \)-manifold \((M, \Omega, \Phi_K)\) is not necessarily compact, but the moment map \( \Phi_K \) is supposed to be proper. We assume that \((M, \Omega, \Phi_K)\) is spin\( c \)-prequantized by a spin\( c \)-bundle \( S \).

In the next section, we will explain how to quantize the data \((M, \Omega, \Phi_K, S)\).
4.1 Formal geometric quantization: definition

We choose an invariant scalar product in $\mathfrak{k} \overset{\ast}{\otimes} \mathfrak{k}$ that provides an identification $\mathfrak{k} \cong \mathfrak{k} \overset{\ast}{\otimes} \mathfrak{k}$.

**Definition 4.1** • The Kirwan vector field associated to $\Phi_K$ is defined by

\[
\kappa(m) = -\Phi_K(m) \cdot m, \quad m \in M.
\]

We denote by $Z_M$ the set of zeroes of $\kappa$. It is not difficult to see that $Z_M$ corresponds to the set of critical points of the function $||\Phi_K||^2 : M \to \mathbb{R}$.

The set $Z_M$, which is not necessarily smooth, admits the following description. Choose a Weyl chamber $t_\ast \subset \mathfrak{t}^\ast$ in the dual of the Lie algebra of a maximal torus $T$ of $K$. We see that

\[
Z_M = \bigsqcup_{\beta \in \mathcal{B}} Z_\beta
\]

where $Z_\beta$ corresponds to the compact set $K(M^\beta \cap \Phi_K^{-1}(\beta))$, and $\mathcal{B} = \Phi_K(Z_M) \cap t_\ast$. The properness of $\Phi_K$ insures that for any compact subset $C \subset \mathfrak{t}^\ast$ the intersection $\mathcal{B} \cap C$ is finite.

The principal symbol of the Dirac operator $D_S$ is the bundle map $\sigma(M, S) \in \Gamma(T^*M, \text{hom}(\mathcal{S}^+, \mathcal{S}^-))$ defined by the Clifford action

\[
\sigma(M, S)(m, \nu) = c_m(\tilde{\nu}) : S|_{m}^+ \to S|_{m}^-.
\]

where $\nu \in T^*M \cong \tilde{\nu} \in TM$ is an identification associated to an invariant Riemannian metric on $M$.

**Definition 4.2** The symbol $\sigma(M, S, \Phi_K)$ shifted by the vector field $\kappa$ is the symbol on $M$ defined by

\[
\sigma(M, S, \Phi_K)(m, \nu) = \sigma(M, S)(m, \tilde{\nu} - \kappa(m))
\]

for any $(m, \nu) \in T^*M$.

For any $K$-invariant open subset $\mathcal{U} \subset M$ such that $\mathcal{U} \cap Z_M$ is compact in $M$, we see that the restriction $\sigma(M, S, \Phi_K)|_{\mathcal{U}}$ is a transversally elliptic symbol on $\mathcal{U}$, and so its equivariant index is a well defined element in $\hat{R}(K)$ (see [1, 31]).

Thus we can define the following localized equivariant indices.

**Definition 4.3** • A closed invariant subset $Z \subset Z_M$ is called a component of $Z_M$ if it is a union of connected components of $Z_M$. 

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• If $Z$ is a compact component of $Z_M$, we denote by

$$Q^{\text{spin}}_K(M, Z) \in \hat{R}(K)$$

the equivariant index of $\sigma(M, S, \Phi_K)|_U$ where $U$ is an invariant neighbourhood of $Z$ so that $U \cap Z_M = Z$.

By definition, $Z = \emptyset$ is a component of $Z_M$ and $Q^{\text{spin}}_K(M, \emptyset) = 0$. For any $\beta \in \mathcal{B}$, $Z_\beta$ is a compact component of $Z_M$.

When the manifold $M$ is compact, the set $\mathcal{B}$ is finite and we have the decomposition

$$Q^{\text{spin}}_K(M) = \sum_{\beta \in \mathcal{B}} Q^{\text{spin}}_K(M, Z_\beta) \in \hat{R}(K).$$

See [24, 31]. When the manifold $M$ is not compact, but the moment map $\Phi_K$ is proper, we can define

$$\hat{Q}^{\text{spin}}_K(M) := \sum_{\beta \in \mathcal{B}} Q^{\text{spin}}_K(M, Z_\beta) \in \hat{R}(K).$$

The sum of the right hand side is not necessarily finite but it converges in $\hat{R}(K)$ (see [27, 21, 10]).

**Definition 4.4** We call $\hat{Q}^{\text{spin}}_K(M) \in \hat{R}(K)$ the spin$^c$ formal geometric quantization of the Hamiltonian manifold $(M, \Omega, \Phi_K)$.

We end up this section with the example of the coadjoint orbits that parametrize the discrete series representations. We have seen in Lemma 3.4 that any $O \in \hat{G}_d$ is spin$^c$-prequantized. Moreover, if we look at the $K$-action on $O$, we know also that the moment map $\Phi_K : O \to \mathfrak{k}^*$ is proper. The element $\hat{Q}^{\text{spin}}_K(O) \in \hat{R}(K)$ is then well-defined.

The following result can be understood as a geometric interpretation of the Blattner formula.

**Proposition 4.5** ([25]) For any $O \in \hat{G}_d$ we have the following equality in $\hat{R}(K)$:

$$\hat{Q}^{\text{spin}}_K(O) = \pi_O^G|_K.$$
4.2 Formal geometric quantization: main properties

In this section, we recall two important functorial properties of the formal geometric quantization process $Q^{\text{spin}}$.

We start with the following result of Hochs and Song.

**Theorem 4.6 ([10])** Let $(M, \Omega, \Phi_K)$ be a spin$^c$ prequantized Hamiltonian $K$-manifold. Assume that the moment map $\Phi_K$ is proper and that the generic infinitesimal stabilizer for the $K$-action on $M$ is abelian. Then the following relation holds in $\hat{R}(K)$:

\begin{equation}
\hat{Q}_{K}^{\text{spin}}(M) = \sum_{P \in \hat{K}} Q^{\text{spin}}(M/\mathcal{P}) \pi_{P}^K.
\end{equation}

**Remark 4.7** Identity (4.11) admits generalizations when we do not have conditions on the generic stabilizer and also when we allow the 2-form $\Omega$ to be degenerate (see [10]).

Like in the compact setting, consider an element $\gamma$ belonging to the center of $K$ that acts trivially on the manifold $M$. Let $\mathcal{P} \in \hat{K}$ and let $\mathcal{P}^-$ be the orbit $\mathcal{P}$ with opposite symplectic structure. We are interested by the action of $\gamma$ on the fibers of the spin$^c$-bundle $S \boxtimes S_{\mathcal{P}^-}$. We denote $[S \boxtimes S_{\mathcal{P}^-}]^\gamma$ the subbundle where $\gamma$ acts trivially.

**Lemma 4.8** If $[S \boxtimes S_{\mathcal{P}^-}]^\gamma = 0$ then $Q^{\text{spin}}(M/\mathcal{P}) = 0$.

**Proof.** The multiplicative property proved by Hochs and Song [10] tells us that the shifting trick still holds in the non compact setting: the multiplicity of $\pi_{P}^K$ in $\hat{Q}_{K}^{\text{spin}}(M)$ is equal to $[\hat{Q}_{K}^{\text{spin}}(M \times \mathcal{P}^-)]^K$. If we suppose furthermore that the generic infinitesimal stabilizer is abelian we obtain

\begin{align*}
Q^{\text{spin}}(M/\mathcal{P}) &= \left[\hat{Q}_{K}^{\text{spin}}(M \times \mathcal{P}^-)\right]^K \\
&= \left[Q_{K}^{\text{spin}}(M \times \mathcal{P}^-, Z_0)\right]^K
\end{align*}

where $Z_0 \subset M \times \mathcal{P}^-$ is the compact set $\{(m, \xi) \in M \times \mathcal{P}^-, \Phi_K(m) = \xi\}$.

The quantity $Q_{K}^{\text{spin}}(M \times \mathcal{P}^-, Z_0) \in \hat{R}(K)$ is computed as an index of a $K$-transversally elliptic operator $D_0$ acting on the sections of $S \boxtimes S_{\mathcal{P}^-}$. The argument used in the compact setting still work (see Lemma 1.3 in [31]): if $[S \boxtimes S_{\mathcal{P}^-}]^\gamma = 0$ then $[\ker(D_0)]^K$ and $[\coker(D_0)]^K$ are reduced to 0. □
Another important property of the formal geometric quantization procedure is the functoriality relatively to restriction to subgroup. Let $H \subset K$ be a closed connected subgroup. We denote $\Phi_H : M \to \mathfrak{h}^*$ the moment map relative to the $H$-action: it is equal to the composition of $\Phi_K$ with the projection $\mathfrak{f}^* \to \mathfrak{h}^*$.

**Theorem 4.9 ([30])** Let $(M, \Omega, \Phi_K)$ be a spin$^c$ prequantized Hamiltonian $K$-manifold. Assume that the moment map $\Phi_H$ is a proper. Then the element $\hat{Q}_K^\text{spin}(M) \in \hat{R}(K)$ is $H$-admissible and we have

$$\hat{Q}_K^\text{spin}(M)|_H = \hat{Q}_H^\text{spin}(M).$$

If we apply the previous Theorem to the spin$^c$-prequantized coadjoint orbits $O \in \hat{G}_d$, we obtain the following extension of Proposition 4.5. This result was obtained by other means by Duflo-Vergne [7].

**Corollary 4.10** Let $O \in \hat{G}_d$, and $H \subset K$ a closed connected subgroup such that $\Phi_H : O \to \mathfrak{h}^*$ is proper. Then $\pi^O_G$ is $H$-admissible and

$$\hat{Q}_H^\text{spin}(O) = \pi^O_G|_H.$$

## 5 Spin$^c$ quantization of $G$-Hamiltonian manifolds

In this section $G$ denotes a connected semi-simple Lie group, and we consider a symplectic manifold $(M, \Omega)$ equipped with an Hamiltonian action of $G$: we denote $\Phi_G : M \to \mathfrak{g}^*$ the corresponding moment map.

### 5.1 Proper$^2$ Hamiltonian $G$-manifolds

In this section we suppose that:

1. the moment map $\Phi_G$ is proper,
2. the $G$-action on $M$ is proper.

For simplicity, we says that $(M, \Omega, \Phi_G)$ is a proper$^2$ Hamiltonian $G$-manifold.

Following Weinstein [38], we consider the $G$-invariant open subset

$$(\mathfrak{g}_\text{se})^* = \{ \xi \in \mathfrak{g}^* \mid G_\xi \text{ is compact} \}$$

of strongly elliptic elements. It is non-empty if and only if the groups $G$ and $K$ have the same rank: real semi-simple Lie groups with this property are
the ones admitting discrete series. If we denote $t^*_{sc} := g^*_{sc} \cap t^*$, we see that $g^*_{sc} = G \cdot t^*_{sc}$. In other words, any coadjoint orbit contained in $g^*_{sc}$ is elliptic.

First we recall the geometric properties associated to proper Hamiltonian $G$-manifolds. We denote $K$ a maximal compact subgroup of $G$ and we denote $\Phi_K : M \to t^*$ the moment map relative to the $K$-action on $(M, \Omega)$.

**Proposition 5.1 ([29])** Let $(M, \Omega, \Phi_G)$ be a proper Hamiltonian $G$-manifold. Then:

1. the map $\Phi_K$ is proper,
2. the set $g^*_{sc}$ is non-empty,
3. the image of $\Phi_G$ is contained in $g^*_{sc}$,
4. the set $N := \Phi_G^{-1}(t^*)$ is a smooth $K$-submanifold of $M$,
5. the restriction of $\Omega$ on $N$ defines a symplectic form $\Omega_N$,
6. the map $[g, n] \mapsto gn$ defines a diffeomorphism $G \times_K N \simeq M$.

Let $T$ be a maximal torus in $K$, and let $t^*_+ \subset g^*_{sc}$ be a Weyl chamber. Since any coadjoint orbit in $g^*_{sc}$ is elliptic, the coadjoint orbits belonging to the image of $\Phi_G : N \to g^*$ are parametrized by the set

$$\Delta_G(M) = \Phi_G(M) \cap t^*_+.$$  

We remark that $t^*_+ \cap g^*_{sc}$ is equal to $(t^*_+)_se := \{ \xi \in t^*_+, (\xi, \alpha) \neq 0, \forall \alpha \in \mathfrak{h}_n \}$. The connected component $(t^*_+)_se$ are called chambers and if $C$ is a chamber, we denote $\bar{G}_d(C)$ the set of regular admissible elliptic orbits intersecting $C$ (see Definition 2.2).

The following fact was first noticed by Weinstein [38].

**Proposition 5.2** $\Delta_G(M)$ is a convex polyhedral set contained in a unique chamber $C_M \subset (t^*_+)_{se}$.

**Proof.** We denote $\Phi_G^N : N \to t^*$ the restriction of the map $\Phi_G$ on the sub-manifold $N$. It corresponds to the moment map relative to the $K$-action on $(N, \Omega_N)$: notice that $\Phi_G^N$ is a proper map.

The diffeomorphism $G \times_K N \simeq M$ shows that the set $\Delta_G(M)$ is equal to $\Delta_K(N) := \text{Image}(\Phi_G^N) \cap t^*_+$, and the Convexity Theorem [14, 20] asserts that $\Delta_K(N)$ is a convex polyhedral subset of the Weyl chamber. Finally since $\Delta_K(N)$ is connected and contained in $(t^*_+)_{se}$, it must belongs to a unique chamber $C_M$. \( \square \)
5.2 Spin*-quantization of proper Hamiltonian $G$-manifolds

Now we assume that our proper Hamiltonian $G$-manifold $(M, \Omega, \Phi_G)$ is spin*-prequantized by a $G$-equivariant spin*-bundle $S$.

Note that $p$ is even dimensional since the groups $G$ and $K$ have the same rank. Recall that the morphism $K \to SO(p)$ lifts to a morphism $\tilde{K} \to \text{Spin}(p)$, where $\tilde{K} \to K$ is either an isomorphism or a two-fold cover (see Section 2.2). We start with the

**Lemma 5.3**

- The $G$-equivariant spin bundle $S$ on $M$ induces a $\tilde{K}$-equivariant spin bundle $S_N$ on $N$ such that $\det(S_N) = \det(S)|_N$.
- The $\tilde{K}$-Hamiltonian manifold $(N, \Omega_N, \Phi_N^\tilde{K})$ is spin*-prequantized by $S_N$.

**Proof.** By definition we have $TM|_N = p \oplus T\tilde{N}$. The manifolds $M$ and $N$ are oriented by their symplectic forms. The vector space $p$ inherits an orientation $o(p, N)$ satisfying the relation $o(M) = o(p, N)o(N)$. The orientation $o(p, N)$ can be computed also as follows: takes any $\xi \in \text{Image}(\Phi_N^\tilde{K})$, then $o(p, N) = o(\xi)$ (see Example 2.6).

Let $S_p$ be the spinor representation that we see as a $\tilde{K}$-module. The orientation $o(p) = o(p, N)$ determines a decomposition $S_p = S_p^{+o(p)} \oplus S_p^{-o(p)}$ and we denote

$$ S_p^{o(p)} := S_p^{+o(p)} \oplus S_p^{-o(p)} \in R(\tilde{K}). $$

Let $S_N$ be the unique spin*-bundle, $\tilde{K}$-equivariant on $N$ defined by the relation

$$ S_N|_N = S_p^{o(p)} \boxtimes S_N. $$

Since $\det(S_p^{o(p)})$ is trivial (as $\tilde{K}$-module), we have the relation $\det(S_N) = \det(S)|_N$ that implies the second point. $\square$

For $O \in \hat{G}_d$, we consider the symplectic reduced space

$$ M/O := \Phi_G^{-1}(O)/G. $$

Notice that $M/O = \emptyset$ when $O$ does not belongs to $\hat{G}_d(C_M)$. Moreover the diffeomorphism $G \times_K N = M$ shows that $M/O$ is equal to the reduced space

$$ N/O_K := (\Phi_K^N)^{-1}(O_K)/K. $$
with $\mathcal{O}_K = \mathcal{O} \cap t^*$. Here $N/\mathcal{O}_K$ should be understood as the symplectic reduction of the $\tilde{K}$-manifold $N$ relative to the $\tilde{K}$-admissible coadjoint orbit $\mathcal{O}_K \in \tilde{K}$. Hence the quantization $Q^\text{spin}(N/\mathcal{O}_K) \in \mathbb{Z}$ of the reduced space $N/\mathcal{O}_K$ is well defined (see Proposition 3.6).

**Definition 5.4** For any $\mathcal{O} \in \hat{G}_d$, we take $Q^\text{spin}(M/\mathcal{O}) := Q^\text{spin}(N/\mathcal{O}_K)$.

The main tool to prove Theorem 1.2 is the comparison of the formal geometric quantization of three different geometric data: we work here in the setting where the $G$-action on $M$ has abelian infinitesimal stabilizers.

1. The formal geometric quantization of the $G$-action on $(M, \Omega, \Phi_G, S)$ is the element $\hat{Q}^\text{spin}_G(M) \in \hat{R}(G, d)$ defined by the relation
   \[
   \hat{Q}^\text{spin}_G(M) := \sum_{\mathcal{O} \in \hat{G}} Q^\text{spin}(M/\mathcal{O}) \pi^G_{\mathcal{O}}.
   \]

2. The formal geometric quantization of the $K$-action on $(M, \Omega, \Phi_K, S)$ is the element $\hat{Q}^\text{spin}_K(M) \in \hat{R}(K)$ (see Definition 4.4). As the $K$-action on $M$ has abelian infinitesimal stabilizers, we have the decomposition
   \[
   \hat{Q}^\text{spin}_K(M) = \sum_{\mathcal{P} \in \bar{\mathcal{P}}} Q^\text{spin}(M/\mathcal{P}) \pi^K_{\mathcal{P}}.
   \]

3. The formal geometric quantization of the $\tilde{K}$-action on $(N, \Omega_N, \Phi^N_{\mathcal{K}}, S_N)$ is the element $\hat{Q}^\text{spin}_{\tilde{K}}(N) \in \hat{R}(\tilde{K})$. As the $\tilde{K}$-action on $N$ has abelian infinitesimal stabilizers, we have the decomposition
   \[
   \hat{Q}^\text{spin}_{\tilde{K}}(N) = \sum_{\tilde{\mathcal{P}} \in \bar{\tilde{\mathcal{P}}}} Q^\text{spin}(N/\tilde{\mathcal{P}}) \pi^K_{\tilde{\mathcal{P}}}.
   \]

In the next section we explain the link between these three elements.

**5.3 Spin$^c$-quantization: main results**

Let $\mathcal{C}_M \subset t^*_+ \subset t^*_*$ be the chamber containing $\Theta_G(M) \cap t^*_+$.

**Definition 5.5** We defines the orientation $\sigma^+$ and $\sigma^-$ on $\mathfrak{p}$ as follows. Take $\lambda \in \mathcal{C}_M$, then $\sigma^+ := o(\lambda)$ and $\sigma^- := o(-\lambda)$ (see Example 2.6).
We denote $S_p^+, S_p^-$ the virtual representations of $\hat{K}$ associated to the spinor representation of $\text{Spin}(p)$ and the orientations $o^+$ and $o^-$. We denote $\overline{S_p^+}$ the $\hat{K}$-module with opposite complex structure. Remark that $\overline{S_p^+} \cong S_p^-$.  

Recall that the map $V \mapsto V|_K$ defines a morphism $\hat{R}(G, d) \to \hat{R}(\hat{K})$. We have also the morphism $\rho^o = \hat{R}(G, d) \to \hat{R}(\hat{K})$ defined by $\rho^o(V) = V|_K \otimes S_p^o$.

We start with the following

**Theorem 5.6** If the $G$-action on $M$ has abelian infinitesimal stabilizers then

\[
(5.15) \quad \rho^o \left( \tilde{Q}_G^{\text{spin}}(M) \right) = \epsilon^o_M \tilde{Q}_K^{\text{spin}}(N).
\]

Here $\epsilon^o_M = \pm$ is equal to the ratio between $o$ and $o^-$.

**Proof.** If the $G$-action on $M$ has abelian infinitesimal stabilizers, then the $\tilde{K}$-action on $N$ has also abelian infinitesimal stabilizers. It implies the following relation:

\[
\tilde{Q}_K^{\text{spin}}(N) = \sum_{\tilde{P} \in \hat{K}} \mathcal{Q}^{\text{spin}}(N/\tilde{P}) \pi_{\tilde{P}}^{\tilde{K}} \in \hat{R}(\hat{K}).
\]

Following the first point of Lemma 2.9, we consider the following subset $\Gamma := \{O_K := O \cap t^*, O \in \hat{G}_d \subset \hat{K}_{\text{out}} \subset \hat{K}\}$.

Thanks to the second point of Lemma 2.9 we have

\[
\rho^o \left( \tilde{Q}_G^{\text{spin}}(M) \right) = \sum_{\mathcal{O} \in \hat{G}_d} \mathcal{Q}^{\text{spin}}(M/\mathcal{O}) \pi_{\hat{G}}^{K} \otimes S_p^o,
\]

\[
= \epsilon^o_M \sum_{\mathcal{O} \in \hat{G}_d} \mathcal{Q}^{\text{spin}}(N/O_K) \pi_{\hat{G}}^{K}
\]

\[
= \epsilon^o_M \sum_{\tilde{P} \in \Gamma} \mathcal{Q}^{\text{spin}}(N/\tilde{P}) \pi_{\tilde{P}}^{\tilde{K}}.
\]

Identity (5.15) is proved if we check that $\mathcal{Q}^{\text{spin}}(N/\tilde{P}) = 0$ for any $\tilde{P} \in \hat{K}$ which does not belong to $\Gamma$.

Suppose first that $\hat{K} \simeq K$. In this case we have $\hat{K} = \hat{K}_{\text{out}} = \hat{K}$ and a coadjoint orbit $\tilde{P} = K\mu \in \hat{K}$ does not belong to $\Gamma$ if and only if $\mu$ is not contained in $g^*_\text{se}$. But the image of $\Phi_G$ is contained in $g^*_\text{se}$, so $N/\tilde{P} = \emptyset$ and then $\mathcal{Q}^{\text{spin}}(N/\tilde{P}) = 0$ if $\tilde{P} \not\in \Gamma$.

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Suppose now that $\tilde{K} \to K$ is a two-fold cover and let us denote by $\{\pm 1_{\tilde{K}}\}$ the kernel of this morphism. Here $\gamma := -1_K$ acts trivially on $N$ and $(5.14)$ shows that $\gamma$ acts by multiplication by $-1$ on the fibers of the spin$^c$ bundle $S_N$. The element $\gamma$ acts also trivially on the orbits $\tilde{P} \times \tilde{K}$:

- if $\tilde{P} \in \tilde{K}_{out}$, then $\gamma$ acts by multiplication by $-1$ on the fibers of the spin$^c$ bundle $S_{\tilde{P}}$,
- if $\tilde{P} \notin \tilde{K}_{out}$, then $\gamma$ acts trivially on the fibers of the spin$^c$ bundle $S_{\tilde{P}}$.

Our considerations show that $[S_N \boxtimes S_{\tilde{P}}]\gamma = 0$ when $\tilde{P} \in \tilde{K}\setminus\tilde{K}_{out}$.

Thanks to Lemma 4.8, it implies the vanishing of $Q^{\text{spin}}(N/\tilde{P})$ for any $\tilde{P} \in \tilde{K}\setminus\tilde{K}_{out}$.

Like in the previous case, when $\tilde{P} \in \tilde{K}_{out}\setminus\Gamma$, we have $Q^{\text{spin}}(N/\tilde{P}) = 0$ because $N/\tilde{P} = \emptyset$. □

We compare now the formal geometric quantizations of the $K$-manifolds $M$ and $N$.

**Theorem 5.7** We have the following relation

$$(5.16) \quad \hat{Q}^{\text{spin}}_K(M) \otimes S^p_n = \hat{Q}^{\text{spin}}_K(N) \otimes R(\tilde{K}).$$

When $M = O \in \hat{G}_d$ the manifold $N$ is equal to $O_K := O \cap t^*$. We have $\hat{Q}^{\text{spin}}_K(N) = \pi^{G}_{K}$ and we know also that $\hat{Q}^{\text{spin}}_K(O) = \pi^{G}_{O}|_K$ (see Proposition 4.5). Here $(5.16)$ becomes

$$(5.17) \quad \pi^{G}_{O}|_K \otimes S^p_n = \pm \pi^{G}_{K},$$

where the sign $\pm$ is the ratio between the orientations $o$ and $o^-$ of the vector space $p$.

If we use Theorems 5.6 and 5.7 we get the following

**Corollary 5.8** If the $G$-action on $M$ has abelian infinitesimal stabilizers, we have $r^o\left(\hat{Q}^{\text{spin}}_G(M)\right) = \hat{Q}^{\text{spin}}_K(M) \otimes S^p_n$.

The following conjecture says that the functorial property of $Q^{\text{spin}}$ relative to restrictions (see Theorem 4.9) should also holds for non-compact groups.
Conjecture 5.9 If the $G$-action on $M$ has abelian infinitesimal stabilizers then the following relation

$$\bar{Q}_G^\text{spin}(M)|_K = \bar{Q}_K^\text{spin}(M)$$

holds in $\hat{R}(K)$.

The remaining part of this section is devoted to the proof of Theorem 5.7.

We work with the manifold $M := G \times_K N$. We denote $\Phi_K^N : N \to \mathfrak{t}^*$ the restriction of $\Phi_G : M \to \mathfrak{g}^*$ to the submanifold $N$. We will use the $K$-equivariant isomorphism $\mathfrak{p} \times N \cong M$ defined by $(X,n) \mapsto (e^X,n)$.

The maps $\Phi_G, \Phi_K, \Phi_K^N$ are related through the relations $\Phi_G(X,n) = e^X \cdot \Phi_K^N(n)$ and $\Phi_K(X,n) = \text{pr}_\mathfrak{t}(e^X \cdot \Phi_K^N(n))$.

We consider the Kirwan vector fields on $N$ and $M$

$$\kappa_N(n) = -\Phi_K^N(n) \cdot n, \quad \kappa_M(m) = -\Phi_K(m) \cdot m.$$  

The following result is proved in [29][Section 2.2].

Lemma 5.10 An element $(X,n) \in \mathfrak{p} \times N$ belongs to $Z_M := \{\kappa_M = 0\}$ if and only if $X = 0$ and $n \in Z_N := \{\kappa_N = 0\}$.

Let us recall how are defined the characters $\bar{Q}_K^\text{spin}(M)$ and $\bar{Q}_K^\text{spin}(N)$. We start with the decomposition $Z_N = \bigsqcup_{\beta \in \mathcal{B}} Z_\beta$ where $Z_\beta = K(N^\beta \cap (\Phi_K^N)^{-1}(\beta))$, and $\mathcal{B} = \Phi_K^N(Z_N) \cap \mathfrak{t}_+$. Thanks to Lemma 5.10 the corresponding decomposition on $M$ is $Z_M := \bigsqcup_{\beta \in \mathcal{B}} \{0\} \times Z_\beta$.

By definition we have

$$\bar{Q}_K^\text{spin}(N) := \sum_{\beta \in \mathcal{B}} Q_K^\text{spin}(N,Z_\beta) \in \hat{R}(\hat{K})$$

and $\bar{Q}_K^\text{spin}(M) = \bar{Q}_K^\text{spin}(\mathfrak{p} \times N) := \sum_{\beta \in \mathcal{B}} Q_K^\text{spin}(\mathfrak{p} \times N,\{0\} \times Z_\beta) \in \hat{R}(K)$. The proof of Theorem 5.7 is completed if we show that for any $\beta \in \mathcal{B}$ we have

$$Q_K^\text{spin}(\mathfrak{p} \times N,\{0\} \times Z_\beta) \otimes \overline{S_\beta^+} = Q_K^\text{spin}(N,Z_\beta) \in R(\hat{K}).$$

Let $S$ be the $G$-equivariant spin$^c$-bundle on $M$. The $K$-equivariant diffeomorphism $M \cong \mathfrak{p} \times N$ induces a $\hat{K}$-equivariant isomorphism at the level of spin$^c$ bundles:

$$S \cong S_\mathfrak{p}^+ \otimes S_N.$$  

$^5\text{pr}_\mathfrak{t} : \mathfrak{g}^* \to \mathfrak{t}^*$ is the canonical projection.
We denote \( \text{cl}_p : p \to \text{End}(S_p) \) the Clifford action associated to the Clifford module \( S_p \). Any \( X \in p \) determines an odd linear map \( \text{cl}_p(X) : S_p \to S_p \).

For \( n \in N \), we denote \( \text{cl}_n : T_n N \to \text{End}(S_{N|n}) \) the Clifford action associated to the spin* bundle \( S_N \). Any \( v \in T_n N \) determines an odd linear map \( \text{cl}_n(v) : S_{N|n} \to S_{N|n} \).

**Lemma 5.11** Let \( U_\beta \subset N \) be a small invariant neighborhood of \( Z_\beta \) such that \( Z_N \cap \overline{U_\beta} = Z_\beta \).

- The character \( Q_{K}^{\text{spin}}(N, Z_\beta) \) is equal to the index of the \( \tilde{K} \)-transversally elliptic symbol
  \[
  \sigma_1^n(v) : S_{N|n}^+ \longrightarrow S_{N|n}^-, \quad v \in T_{n \cdot} \overline{U_\beta}
  \]
defined by \( \sigma_1^n(v) = \text{cl}_n(v + \Phi_K^N(n) \cdot n) \).

- The character \( Q_{K}^{\text{spin}}(p \times N, \{0\} \times Z_\beta) \) is equal to the index of the \( K \)-transversally elliptic symbol
  \[
  \sigma_2^{(A,n)}(X, v) : (S_p^+ \otimes S_{N|n})^+ \longrightarrow (S_p^+ \otimes S_{N|n})^-
  \]
defined by \( \sigma_2^{(A,n)}(X, v) = \text{cl}_p(X + [\Phi_K^N(n), A]) \otimes \text{cl}_n(v + \Phi_K^N(n) \cdot n) \) for \( (X, v) \in T_{(A,n)}(p \times \overline{U_\beta}) \).

**Proof.** The first point corresponds to the definition of the character \( Q_{K}^{\text{spin}}(N, Z_\beta) \).

By definition, \( Q_{K}^{\text{spin}}(p \times N, \{0\} \times Z_\beta) \) is equal to the index of the \( K \)-transversally elliptic symbol

\[
\tau_{(A,n)}(X, v) = \text{cl}_p(X + [\Phi_K(X, n), A]) \otimes \text{cl}_n(v + \Phi_K(X, n) \cdot n).
\]

It is not difficult to see that \( \tau_{(A,n)}^t(X, v) = \text{cl}_p(X + [\Phi_K(tX, n), A]) \otimes \text{cl}_n(v + \Phi_K(tX, n) \cdot n), \quad 0 \leq t \leq 1, \)
defines an homotopy of transversally elliptic symbols between \( \sigma^2 = \tau^0 \) and \( \tau = \tau^1 \); like in Lemma 5.10, we use the fact that \([\Phi_K(0, n), A] = 0 \) only if \( A = 0 \). It proves the second point. \( \square \)

We can now finish the proof of (5.18). We use here the following isomorphism of Clifford modules for the vector space \( p \times p : \)

\[
S_p^+ \otimes \overline{S_p^+} \cong \bigwedge_p C,
\]
where the Clifford action \((X, Y) \in \mathfrak{p} \times \mathfrak{p}\) on the left is \(\text{cl}_p(X) \otimes \text{cl}_p(Y)\) and on the right is \(\text{cl}_p(X + iy)\).

The product \(\sigma^2 \otimes \overline{S_p}\) corresponds to the symbol

\[
\text{cl}_p(X + [\Phi_K(X, n), A]) \otimes \text{cl}_p(0) \otimes \text{cl}_n(v + \Phi_N^*(n) \cdot n)
\]

which is homotopic to

\[
\text{cl}_p(X + [\Phi_K(X, n), A]) \otimes \text{cl}_p(A) \otimes \text{cl}_n(v + \Phi_N^*(n) \cdot n),
\]

and is also homotopic to

\[
\sigma^3 := \text{cl}_p(X) \otimes \text{cl}_p(A) \otimes \text{cl}_n(v + \Phi_N^*(n) \cdot n).
\]

We have then proved that the \(K\)-equivariant index of \(\sigma^2\) times \(\overline{S_p}\) is equal to the \(\hat{K}\)-equivariant index of \(\sigma^3\) (that we denote \(\text{Index}_{\hat{K}}^{p \times U_B}(\sigma^3)\)). The multiplicative property of the equivariant index [1] tells us that

\[
\text{Index}_{\hat{K}}^{p \times U_B}(\sigma^3) = \text{Index}_{\hat{K}}^{p}(\text{cl}_p(X + iA)) \cdot \text{Index}_{\hat{K}}^{U_B}(\sigma^1).
\]

But \(\text{cl}_p(X + iA) : \bigwedge^2 \mathfrak{p}_C \to \bigwedge^2 \mathfrak{p}_C, (X, A) \in \mathfrak{T}_p\), is the Bott symbol and its index is equal to the trivial 1-dimensional representation of \(\hat{K}\). We have finally proved that the \(K\)-equivariant index of \(\sigma^2\) times \(\overline{S_p}\) is equal to the \(\hat{K}\)-equivariant index of \(\sigma^1\). The proof of (5.18) is complete. \(\square\)

### 5.4 Proof of the main Theorem

Let \(G\) be a connected semi-simple subgroup of \(G'\) with finite center, and let \(\mathcal{O}' \in \hat{G}'\). We suppose that the representation \(\pi_{G'}^{\mathcal{O}'}\) is \(G\)-admissible. Then we have a decomposition

\[
\pi_{G'}^{\mathcal{O}'}|_G = \sum_{\mathcal{O} \in \mathcal{G}_d} m_{\mathcal{O}} \pi_{\mathcal{G}}^G.
\]

Let \(\Phi_G : \mathcal{O}' \to \mathfrak{g}^*\) be the moment map relative to the \(G\)-action on \(\mathcal{O}'\). We have proved in Theorem 2.10, that the \(G\)-admissibility of \(\pi_{G'}^{\mathcal{O}'}\) implies the properness of \(\Phi_G\). Moreover, since \(\mathcal{O}'\) is a regular orbit, the \(G\)-action on it is proper. Finally we see that \(\mathcal{O}'\) is a \(\text{spin}^c\) prequantized proper Hamiltonian \(G\)-manifold. We can consider its formal \(\text{spin}^c\) quantization \(\hat{Q}_G^{\text{spin}}(\mathcal{O}') \in \hat{R}(G, d)\), which is defined by the relation

\[
\hat{Q}_G^{\text{spin}}(\mathcal{O}') := \sum_{\mathcal{O} \in \mathcal{G}_d} \mathcal{Q}_{\text{spin}}(\mathcal{O}' \mid \mathcal{O}) \pi_{\mathcal{G}}^G.
\]
Theorem 1.2 is proved if we show that $\pi_1^G|_G$ and $\hat{Q}_{G}^{\text{spin}}(\mathcal{O}')$ are equal in $\hat{R}(G, d)$. Since the morphism $r^\phi : \hat{R}(G, d) \to \hat{R}(K)$ is one to one, it is sufficient to prove that

$$r^\phi \left( \pi_1^G|_G \right) = r^\phi \left( \hat{Q}_{G}^{\text{spin}}(\mathcal{O}') \right).$$

On one hand, the element $r^\phi \left( \pi_1^G|_G \right)$ is equal to $\pi_1^G|_K \otimes \mathcal{S}_{\mathfrak{p}}^\mathfrak{o}$. The restriction $\pi_1^G|_K \in \hat{R}(K)$, which is well defined since the moment map $\Phi_K : \mathcal{O}' \to \mathfrak{k}^*$ is proper, is equal to $\hat{Q}_{K}^{\text{spin}}(\mathcal{O}')$ (see Corollary 4.10). So we get

$$r^\phi \left( \pi_1^G|_G \right) = \hat{Q}_{K}^{\text{spin}}(\mathcal{O}') \otimes \mathcal{S}_{\mathfrak{p}}^\mathfrak{o}.$$

On the other hand, Corollary 4.10 tells us that

$$r^\phi \left( \hat{Q}_{G}^{\text{spin}}(\mathcal{O}') \right) = \hat{Q}_{K}^{\text{spin}}(\mathcal{O}') \otimes \mathcal{S}_{\mathfrak{p}}^\mathfrak{o}.$$

Hence we obtain Equality (5.19). The proof of Theorem 1.2 is completed.

References


