Formal Geometric Quantization III, Functoriality in the spin-c setting

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1 Introduction

Let $(M, \mathcal{S})$ be a K-manifold of even dimension, oriented, and equipped with a K-equivariant spin^c bundle $\mathcal{S}$. The orientation induces a decomposition $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, and the corresponding spin^c Dirac operator is a first order elliptic operator $\mathcal{D}_{\mathcal{S}} : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-)$ [2, 4, 6].

When M is compact, an important invariant is the equivariant index $\mathcal{Q}_K(M, \mathcal{S}) \in R(K)$ of the operator $\mathcal{D}_{\mathcal{S}}$, that can be understood as the quantization of the data $(M, \mathcal{S}, K)$ [3, 7].

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The determinant line bundle of the spin^c-bundle $\mathcal{S}$ is the line bundle $\det(\mathcal{S}) \to M$ defined by the relation

$$\det(\mathcal{S}) := \hom_{\mathrm{Cl}}(\overline{\mathcal{S}}, \mathcal{S})$$

where $\overline{\mathcal{S}}$ is the Clifford module with opposite complex structure (see [20]).

The choice of an invariant Hermitian connection ∇ on $\det(\mathcal{S})$ determines an equivariant map $\Phi_{\mathcal{S}} : M \to \mathfrak{k}^*$ and a 2-form $\Omega_{\mathcal{S}}$ on M by means of the Kostant relations

$$\mathcal{L}(X) - \nabla_{X_M} = 2i\langle \Phi_{\mathcal{S}}, X\rangle \quad \text{and} \quad \nabla^2 = -2i\Omega_{\mathcal{S}} \tag{1.1}$$

for every $X \in \mathfrak{k}$. Here $\mathcal{L}(X)$ denotes the infinitesimal action on the sections of $\det(\mathcal{S})$. We say that $\Phi_{\mathcal{S}}$ is the *moment map* for $\mathcal{S}$ (it depends however of the choice of a connection).

Assume now that M is non-compact but that the moment map $\Phi_{\mathcal{S}}$ is *proper*. In this case the formal geometric quantization of $(M, \mathcal{S}, K)$ is well-defined:

$$\mathcal{Q}_K^{-\infty}(M, \mathcal{S}) \in \hat{R}(K)$$

as an index localized on the zeros of the Kirwan vector field [15, 17, 11, 9]. We will explain the construction in section 3.

Consider now a closed connected subgroup $H \subset K$. Let $\mathrm{p} : \mathfrak{k}^* \to \mathfrak{h}^*$ be the canonical projection. The map $\mathrm{p} \circ \Phi_{\mathcal{S}}$ corresponds to the moment map for $\mathcal{S}$ relative to the H-action.

The main result of our note is the following

Theorem 1.1 *Suppose that* $\mathrm{p} \circ \Phi_{\mathcal{S}}$ *is proper. Then the following holds:*

- *The* K*-module* $\mathcal{Q}_K^{-\infty}(M, \mathcal{S})$ *is* H*-admissible.*
- *We have* $\mathcal{Q}_K^{-\infty}(M, \mathcal{S})|_H = \mathcal{Q}_H^{-\infty}(M, \mathcal{S})$.

We obtained a similar result in the symplectic setting in [16]. Here our method uses the compactifications of reductive groups à la de Concini-Procesi and the multiplicative property of the functor $\mathcal{Q}_K^{-\infty}$ that has been proved recently by Hochs and Song [9].

Notations

Throughout the paper :

- K denotes a compact connected Lie group with Lie algebra $\mathfrak{k}$.

- We denote by $R(K)$ the representation ring of K : an element $E \in R(K)$ can be represented as a finite sum $E = \sum_{\mu\in\widehat{K}} \mathrm{m}_\mu \pi_\mu$, with $\mathrm{m}_\mu \in \mathbb{Z}$. The multiplicity of the trivial representation is denoted $[E]^K$.
- We denote by $\hat{R}(K)$ the space of $\mathbb{Z}$-valued functions on $\widehat{K}$. An element $E \in \hat{R}(K)$ can be represented as an infinite sum $E = \sum_{\mu\in\widehat{K}} \mathrm{m}(\mu)\pi_\mu$, with $\mathrm{m}(\mu) \in \mathbb{Z}$.
- An element $\xi \in \mathfrak{k}^*$ is called regular if the stabilizer subgroup $K_\xi := \{k \in K, k \cdot \xi = \xi\}$ is a maximal torus of K.
- When K acts on a manifold M, we denote $X_M(m) := \frac{d}{dt}|_{t=0} e^{-tX} \cdot m$ the vector field generated by $-X \in \mathfrak{k}$. Sometimes we will also use the notation $X_M(m) = -X \cdot m$. The set of zeroes of the vector field X_M is denoted M^X.

2 The $[Q, R] = 0$ theorem in the spin^c setting

In this section we suppose that M is compact and we recall the results of [20] concerning the multiplicities of $\mathcal{Q}_K(M, \mathcal{S}) \in R(K)$.

For any $\xi \in \mathfrak{k}^*$, we denote $\mathfrak{k}_\xi$ the Lie algebra of the stabilizer subgroup K_ξ. A coadjoint orbit $\mathcal{P} = K\eta$ is of type $(\mathfrak{k}_\xi)$ if the conjugacy classes $(\mathfrak{k}_\eta)$ and $(\mathfrak{k}_\xi)$ are equal.

Definition 2.1 *Let $(\mathfrak{k}_M)$ be the generic infinitesimal stabilizer for the K-action on M. We says that the K-action on M is* nice *if there exists $\xi \in \mathfrak{k}^*$ such that*

$$([\mathfrak{k}_M, \mathfrak{k}_M]) = ([\mathfrak{k}_\xi, \mathfrak{k}_\xi]). \tag{2.2}$$

The first result of [20] is the following

Theorem 2.2 *If the K-action on M is not nice, then $\mathcal{Q}_K(M, \mathcal{S}) = 0$ for any* spin^c *bundle $\mathcal{S}$.*

We suppose now that the K-action on M is nice. The conjugacy class $(\mathfrak{k}_\xi)$ satisfying (2.2) is unique and is denoted $(\mathfrak{h}_M)$ (see Lemma 3 in [19]).

Definition 2.3 *A coadjoint orbit $\mathcal{P} \subset \mathfrak{k}^*$ is admissible if $\mathcal{P}$ carries a* spin^c*-bundle $\mathcal{S}_\mathcal{P}$ such that the corresponding moment map is the inclusion $\mathcal{P} \hookrightarrow \mathfrak{k}^*$. We denote simply by $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ the element $\mathcal{Q}_K(\mathcal{P}, \mathcal{S}_\mathcal{P}) \in R(K)$.*

We can check easily [19] that $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P})$ is either 0 or an irreducible representation of K, and that the map

$$\mathcal{O} \mapsto \pi_{\mathcal{O}}^K := \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{O})$$

defines a bijection between the regular admissible orbits and the dual $\widehat{K}$. When $\mathcal{O}$ is a regular admissible orbit, an admissible coadjoint orbit $\mathcal{P}$ is called an ancestor of $\mathcal{O}$ (or a K-ancestor of $\pi_{\mathcal{O}}^K$) if $\mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}) = \pi_{\mathcal{O}}^K$.

Denote by $\mathcal{A}((\mathfrak{h}_M))$ the set of admissible orbits of type $(\mathfrak{h}_M)$. The following important fact is proved in section 5 of [20].

Proposition 2.4 *Let* $\mathcal{P} \in \mathcal{A}((\mathfrak{h}_M))$.

- *If* $\mathcal{P}$ *belongs to the set of regular values of* $\Phi_{\mathcal{S}}$*, the reduced space*

$$M_{\mathcal{P}} = \Phi_{\mathcal{S}}^{-1}(\mathcal{P})/K$$

  is an oriented orbifold equipped with a spin^c *bundle. The index of the corresponding Dirac operator on the orbifold* $M_{\mathcal{P}}$ *is denoted* $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}) \in \mathbb{Z}$ *[10].*

- *In general, the* spin^c *index* $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}) \in \mathbb{Z}$ *associated to the (possibly singular) reduced space* $M_{\mathcal{P}}$ *is defined by a deformation procedure.*

The $[Q, R] = 0$ Theorem in the spin^c setting takes the following form.

Theorem 2.5 ([20]) *Let* $\mathcal{O}$ *be a regular admissible orbit.*
The multiplicity of the representation $\pi_{\mathcal{O}}^K$ *in* $\mathcal{Q}_K(M, \mathcal{S})$ *is equal to*

$$\sum_{\mathcal{P}} \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})$$

where the sum runs over the ancestors of $\mathcal{O}$ *of type* $(\mathfrak{h}_M)$*. In other words*

$$\mathcal{Q}_K(M, \mathcal{S}) = \sum_{\mathcal{P} \in \mathcal{A}((\mathfrak{h}_M))} \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})\, \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}).$$

3 Formal geometric quantization in the spin^c setting

In this section the manifold M is not necessarily compact, but the moment map $\Phi_{\mathcal{S}}$ is supposed to be proper.

We choose an invariant scalar product in $\mathfrak{k}^*$ that provides an identification $\mathfrak{k} \simeq \mathfrak{k}^*$.

Definition 3.1 • *The* Kirwan vector field *associated to* $\Phi_{\mathcal{S}}$ *is defined by*

$$\kappa_{\mathcal{S}}(m) = -\Phi_{\mathcal{S}}(m) \cdot m, \quad m \in M. \tag{3.3}$$

• *We denote by* $Z_{\mathcal{S}}$ *the set of zeroes of* $\kappa_{\mathcal{S}}$. *Thus* $Z_{\mathcal{S}}$ *is a* K*-invariant closed subset of* M.

The set $Z_{\mathcal{S}}$, which is not necessarily smooth, admits an easy description. Choose a Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ in the dual of the Lie algebra of a maximal torus T of K. We see that

$$Z_{\mathcal{S}} = \coprod_{\beta \in \mathcal{B}_{\mathcal{S}}} Z_\beta \tag{3.4}$$

where Z_β corresponds to the compact set $K(M^\beta \cap \Phi_{\mathcal{S}}^{-1}(\beta))$, and $\mathcal{B}_{\mathcal{S}} = \Phi_{\mathcal{S}}(Z_{\mathcal{S}}) \cap \mathfrak{t}^*_+$. The properness of $\Phi_{\mathcal{S}}$ insures that for any compact subset $C \subset \mathfrak{t}^*$ the intersection $\mathcal{B}_{\mathcal{S}} \cap C$ is finite.

The principal symbol of the Dirac operator $D_{\mathcal{S}}$ is the bundle map $\sigma(M, \mathcal{S}) \in \Gamma(\mathrm{T}^*M, \hom(\mathcal{S}^+, \mathcal{S}^-))$ defined by the Clifford action

$$\sigma(M, \mathcal{S})(m, \nu) = \mathbf{c}_m(\tilde{\nu}) : \mathcal{S}|_m^+ \to \mathcal{S}|_m^-.$$

where $\nu \in \mathrm{T}^*M \simeq \tilde{\nu} \in \mathrm{T}M$ is an identification associated to an invariant Riemannian metric on M.

Definition 3.2 *The symbol* $\sigma(M, \mathcal{S}, \Phi_{\mathcal{S}})$ *shifted by the vector field* $\kappa_{\mathcal{S}}$ *is the symbol on* M *defined by*

$$\sigma(M, \mathcal{S}, \Phi_{\mathcal{S}})(m, \nu) = \sigma(M, \mathcal{S})(m, \tilde{\nu} - \kappa_{\mathcal{S}}(m))$$

for any $(m, \nu) \in \mathrm{T}^*M$.

For any K-invariant open subset $\mathcal{U} \subset M$ such that $\mathcal{U} \cap Z_{\mathcal{S}}$ is compact in M, we see that the restriction $\sigma(M, \mathcal{S}, \Phi_{\mathcal{S}})|_{\mathcal{U}}$ is a transversally elliptic symbol on $\mathcal{U}$, and so its equivariant index is a well defined element in $\hat{R}(K)$ (see [1, 18]).

Thus we can define the following localized equivariant indices.

Definition 3.3 • *A closed invariant subset* $Z \subset Z_{\mathcal{S}}$ *is called a* component *of* $Z_{\mathcal{S}}$ *if it is a union of connected components of* $Z_{\mathcal{S}}$.

• *If* Z *is a* compact component *of* $Z_{\mathcal{S}}$, *we denote by*

$$\mathcal{Q}_K(M, \mathcal{S}, Z) \in \hat{R}(K)$$

the equivariant index of $\sigma(M, \mathcal{S}, \Phi)|_{\mathcal{U}}$ *where* $\mathcal{U}$ *is an invariant neighbourhood of* Z *so that* $\mathcal{U} \cap Z_{\mathcal{S}} = Z$.

By definition, $Z = \emptyset$ is a component of $Z_{\mathcal{S}}$ and $\mathcal{Q}_K(M, \mathcal{S}, \emptyset) = 0$. For any $\beta \in \mathcal{B}_{\mathcal{S}}$, Z_β is a compact component of $Z_{\mathcal{S}}$.

When the manifold M is compact, the set $\mathcal{B}_{\mathcal{S}}$ is finite and we have the decomposition

$$\mathcal{Q}_K(M, \mathcal{S}) = \sum_{\beta \in \mathcal{B}_{\mathcal{S}}} \mathcal{Q}_K(M, \mathcal{S}, Z_\beta).$$

See [14, 18].

When the manifold M is not compact, but the moment map $\Phi_{\mathcal{S}}$ is proper, we can defined

$$\mathcal{Q}_K^{-\infty}(M, \mathcal{S}) := \sum_{\beta \in \mathcal{B}_{\mathcal{S}}} \mathcal{Q}_K(M, \mathcal{S}, Z_\beta)$$

The sum of the right hand side is not necessarily finite but it converges in $\hat{R}(K)$ (see [17, 11, 9]). We call $\mathcal{Q}_K^{-\infty}(M, \mathcal{S}) \in \hat{R}(K)$ the formal geometric quantization of the data $(M, \mathcal{S}, \Phi_{\mathcal{S}}, K)$.

Hochs and Song prove the following important property concerning the functoriality of $\mathcal{Q}_K^{-\infty}$ relatively to the product of manifolds.

Theorem 3.4 ([9]) *Let $(M, \mathcal{S})$ be a* spin^c *K-manifold with a proper moment map $\Phi_{\mathcal{S}}$. Let $(P, \mathcal{S}_P)$ be a compact* spin^c *K-manifold (even dimensional and oriented). Then the* spin^c *manifold $(M \times P, \mathcal{S} \boxtimes \mathcal{S}_P)$ admits a proper moment map and we have the following equality in $\hat{R}(K)$:*

$$\mathcal{Q}_K^{-\infty}(M \times P, \mathcal{S} \boxtimes \mathcal{S}_P) = \mathcal{Q}_K^{-\infty}(M, \mathcal{S}) \otimes \mathcal{Q}_K(P, \mathcal{S}_P).$$

With Theorem 3.4 in hand we can compute the multiplicities of $\mathcal{Q}_K^{-\infty}(M, \mathcal{S})$ like in the compact setting by using the shifting trick.

Let $\mathcal{O}$ be an admissible regular orbit of K. We denote by $[\pi_{\mathcal{O}}^K : \mathcal{Q}_K^{-\infty}(M, \mathcal{S})]$ the multiplicity of $\pi_{\mathcal{O}}^K$ in $\mathcal{Q}_K^{-\infty}(M, \mathcal{S}) \in \hat{R}(K)$. Let $\mathcal{O}^*$ be the admissible orbit $-\mathcal{O}$: we have $\mathcal{Q}_K(\mathcal{O}^*, \mathcal{S}_{\mathcal{O}^*}) = (\pi_{\mathcal{O}}^K)^*$. Thanks to Theorem 3.4 we get

$$\begin{aligned} [\pi_{\mathcal{O}}^K : \mathcal{Q}_K^{-\infty}(M, \mathcal{S})] &= \left[\mathcal{Q}_K^{-\infty}(M, \mathcal{S}) \otimes \mathcal{Q}_K(\mathcal{O}^*, \mathcal{S}_{\mathcal{O}^*})\right]^K \\ &= \left[\mathcal{Q}_K^{-\infty}(M \times \mathcal{O}^*, \mathcal{S} \otimes \mathcal{S}_{\mathcal{O}^*})\right]^K. \end{aligned}$$

We consider the product $M \times \mathcal{O}^*$ equipped with the spin^c-bundle $\mathcal{S} \otimes \mathcal{S}_{\mathcal{O}^*}$. The corresponding moment map is $\Phi_{\mathcal{S} \otimes \mathcal{S}_{\mathcal{O}^*}}(m, \xi) = \Phi_{\mathcal{S}}(m) + \xi$. We use the simplified notation $\Phi_{\mathcal{O}}$ for $\Phi_{\mathcal{S} \otimes \mathcal{S}_{\mathcal{O}^*}}$, $\kappa_{\mathcal{O}}$ for the corresponding Kirwan vector field on $M \times \mathcal{O}^*$, and $Z_{\mathcal{O}} := \{\kappa_{\mathcal{O}} = 0\}$.

In [20], we introduced a locally constant function $d_{\mathcal{O}} : Z_{\mathcal{O}} \to \mathbb{R}$, and we denote $Z_{\mathcal{O}}^{=0} = \{d_{\mathcal{O}} = 0\}$. Using the localization[1] done in [20][section 4.5], we get that

$$[\pi_{\mathcal{O}}^K : \mathcal{Q}_K^{-\infty}(M,\mathcal{S})] = \left[\mathcal{Q}_K^{-\infty}(M\times\mathcal{O}^*, \mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*}, Z_{\mathcal{O}}^{=0})\right]^K \tag{3.5}$$

Finally we obtain the same result like in the compact setting:

- If the K action on M is not nice, $Z_{\mathcal{O}}^{=0} = \emptyset$ and then the multiplicity $[\pi_{\mathcal{O}}^K : \mathcal{Q}_K^{-\infty}(M,\mathcal{S})]^K$ vanishes for any regular admissible orbit $\mathcal{O}$.
- If the K action on M is nice, we have $[\pi_{\mathcal{O}}^K : \mathcal{Q}_K^{-\infty}(M,\mathcal{S})]^K = \sum_{\mathcal{P}} \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})$ where the sum runs over the ancestors of $\mathcal{O}$ of type $(\mathfrak{h}_M)$.

In other words,

- if the K action on M is not nice, then $\mathcal{Q}_K^{-\infty}(M,\mathcal{S}) = 0$,
- if the K action on M is nice, we have
$$\mathcal{Q}_K^{-\infty}(M,\mathcal{S}) = \sum_{\mathcal{P}\in\mathcal{A}((\mathfrak{h}_M))} \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})\, \mathrm{Q}_K^{\mathrm{spin}}(\mathcal{P}).$$

Remark 3.5 *We will use a particular case of identity (3.5) when the generic infinitesimal stabilizer of the K-action on M is* abelian, *i.e.* $([\mathfrak{k}_M,\mathfrak{k}_M]) = 0$. *In this case* $Z_{\mathcal{O}}^{=0} = \{\Phi_{\mathcal{O}} = 0\}$ *and then*

$$\begin{aligned}[\pi_{\mathcal{O}}^K : \mathcal{Q}_K^{-\infty}(M,\mathcal{S})]^K &= \left[\mathcal{Q}_K^{-\infty}(M\times\mathcal{O}^*, \mathcal{S}\otimes\mathcal{S}_{\mathcal{O}^*}, \{\Phi_{\mathcal{O}} = 0\})\right]^K \\ &= \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{O}}).\end{aligned}$$

4 Functoriality relatively to a subgroup

We come back to the setting of K-manifold M, even dimensional and oriented, equipped with an equivariant spin^c bundle $\mathcal{S}$. We suppose that for some choice of connection on $\det(\mathcal{S})$ the corresponding moment map $\Phi_{\mathcal{S}}$ is proper. As explained earlier, the formal geometric quantization of $(M,\mathcal{S},K)$ is well-defined : $\mathcal{Q}_K^{-\infty}(M,\mathcal{S}) \in \hat{R}(K)$.

Consider now a closed connected subgroup $H \subset K$. Let $\mathrm{p} : \mathfrak{k}^* \to \mathfrak{h}^*$ be the canonical projection. The map $\mathrm{p}\circ\Phi_{\mathcal{S}}$ correspond to the moment map for the spin^c bundle $\mathcal{S}$ relative to the H-action.

This section is dedicated to the proof of our main result.

[1]In [20] we work in the compact setting, but exactly the same proof works in the non compact setting, as noticed by Hochs and Song [9].

Theorem 4.1 *Suppose that* $\mathrm{p}\circ\Phi_{\mathcal{S}}$ *is proper. Then the* K*-module* $\mathcal{Q}^{-\infty}_K(M,\mathcal{S})$ *is* H*-admissible, and*

$$\mathcal{Q}^{-\infty}_K(M,\mathcal{S})|_H = \mathcal{Q}^{-\infty}_H(M,\mathcal{S}). \tag{4.6}$$

We start with the following

Lemma 4.2 • $\mathcal{Q}^{-\infty}_K(M,\mathcal{S})$ *is* H*-admissible when* $\mathrm{p}\circ\Phi_{\mathcal{S}}$ *is proper.*

• *It is sufficient to prove (4.6) for manifolds with abelian generic infinitesimal stabilizers.*

Proof. We have $\mathcal{Q}^{-\infty}_K(M,\mathcal{S}) = \sum_{\mathcal{P}} \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P})$ where the sum runs over the admissible orbits of type $(\mathfrak{h}_M)$.

Thanks to the $[Q,R]=0$ Theorem we know that $\mathrm{Q}^{\mathrm{spin}}_K(\mathcal{P})|_H = \mathrm{Q}^{\mathrm{spin}}_H(\mathcal{P}) = \sum_{\mathcal{P}'} \mathrm{Q}^{\mathrm{spin}}(\mathcal{P}_{\mathcal{P}'})\mathrm{Q}^{\mathrm{spin}}_H(\mathcal{P}')$, where $\mathcal{P}_{\mathcal{P}'} = \mathcal{P}\cap \mathrm{p}^{-1}(\mathcal{P}')/H$ is the reduction of the K-coadjoint orbit $\mathcal{P}$ relatively to H-coadjoint orbit $\mathcal{P}'$.

Hence $\mathcal{Q}^{-\infty}_K(M,\mathcal{S})$ is H-admissible if for any H-coadjoint orbit $\mathcal{P}'$, the sum

$$\sum_{\mathcal{P}} \mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})\mathrm{Q}^{\mathrm{spin}}(\mathcal{P}_{\mathcal{P}'})$$

admits only a finite number of non-zero terms. We see that $\mathrm{Q}^{\mathrm{spin}}(\mathcal{P}_{\mathcal{P}'}) \neq 0$ only if $\mathcal{P}'\subset \mathrm{p}(\mathcal{P})$ and $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}}) \neq 0$ only if $\Phi_{\mathcal{S}}^{-1}(\mathcal{P}) \neq \emptyset$. Finally $\mathrm{Q}^{\mathrm{spin}}(M_{\mathcal{P}})\mathrm{Q}^{\mathrm{spin}}(\mathcal{P}_{\mathcal{P}'}) \neq 0$ only if

$$\mathcal{P} \in K\Phi_{\mathcal{S}}\left((\mathrm{p}\circ\Phi_{\mathcal{S}})^{-1}(\mathcal{P}')\right).$$

Since $\mathrm{p}\circ\Phi_{\mathcal{S}}$ is proper, we have only a finite number of K-admissible orbits contained in the compact set $K\Phi_{\mathcal{S}}\left((\mathrm{p}\circ\Phi_{\mathcal{S}})^{-1}(\mathcal{P}')\right)$. The first point is proved.

Let us check the second point. Suppose that (4.6) holds for manifolds with abelian generic infinitesimal stabilizers. Let $K\rho$ be the regular admissible orbit such that $\mathrm{Q}^{\mathrm{spin}}_K(K\rho)$ is the trivial representation.

To any spin^c manifold $(M,\mathcal{S},K)$ with proper moment map $\Phi_{\mathcal{S}}$, we associate the product $M\times K\rho$ which is a spin^c K-manifold with proper moment map $\Phi_{\mathcal{S}\boxtimes\mathcal{S}_{K\rho}}$. The multiplicative property (see Theorem 3.4) gives

$$\mathcal{Q}^{-\infty}_K(M,\mathcal{S}) = \mathcal{Q}^{-\infty}_K(M\times K\rho, \mathcal{S}\boxtimes\mathcal{S}_{K\rho}).$$

Now we remark that the K-manifold $M\times K\rho$ has abelian infinitesimal stabilizers, and that $\mathrm{p}\circ\Phi_{\mathcal{S}\boxtimes\mathcal{S}_{K\rho}}$ is proper if $\mathrm{p}\circ\Phi_{\mathcal{S}}$ is proper. Then, when the moment map $\mathrm{p}\circ\Phi_{\mathcal{S}}$ is proper, we have

$$\begin{aligned}
\mathcal{Q}^{-\infty}_K(M,\mathcal{S})|_H &= \mathcal{Q}^{-\infty}_K(M\times K\rho,\mathcal{S}\boxtimes\mathcal{S}_{K\rho})|_H \\
&= \mathcal{Q}^{-\infty}_H(M\times K\rho,\mathcal{S}\boxtimes\mathcal{S}_{K\rho}) && [1]\\
&= \mathcal{Q}^{-\infty}_H(M,\mathcal{S})\otimes\mathcal{Q}_H(K\rho,\mathcal{S}_{K\rho}) && [2]\\
&= \mathcal{Q}^{-\infty}_H(M,\mathcal{S}). && [3]
\end{aligned}$$

Here we see that [1] is the identity (4.6) applied to $M\times K\rho$, [2] is the multiplicative property relatively to the H-action, and [3] is due to the fact that $\mathcal{Q}_H(K\rho,\mathcal{S}_{K\rho})$ is the trivial H-representation. □

4.1 De Concini-Procesi compactifications

We recall that T is a maximal torus of the compact connected Lie group K, and W is the corresponding Weyl group. We define a K-*adapted* polytope in $\mathfrak{t}^*$ to be a W-invariant Delzant polytope P in $\mathfrak{t}^*$ whose vertices are regular elements of the weight lattice Λ. If $\{\lambda_1,\ldots,\lambda_r\}$ are the dominant weights lying in the union of all the closed one-dimensional faces of P, then there is a $G\times G$-equivariant embedding of $G=K_{\mathbb{C}}$ into

$$\mathbb{P}\Big(\bigoplus_{i=1}^{r}(V^K_{\lambda_i})^*\otimes V^K_{\lambda_i}\Big)$$

associating to $g\in G$ its representation on $\bigoplus_{i=1}^r V^K_{\lambda_i}$. The closure $\mathcal{X}_P$ of the image of G in this projective space is smooth and is equipped with a $K\times K$-action. The restriction of the canonical Kähler structure on $\mathcal{X}_P$ defines a symplectic 2-form $\Omega_{\mathcal{X}_P}$. We recall briefly the different properties of $(\mathcal{X}_P,\Omega_{\mathcal{X}_P})$: all the details can be found in [16].

(1) $\mathcal{X}_P$ is equipped with an Hamiltonian action of $K\times K$. Let $\Phi:=(\Phi_l,\Phi_r):\mathcal{X}_P\to\mathfrak{k}^*\times\mathfrak{k}^*$ be the corresponding moment map.

(2) The image of Φ is equal to $\{(k\cdot\xi,-k'\cdot\xi)\mid\xi\in P \text{ and } k,k'\in K\}$.

(3) The Hamiltonian $K\times K$-manifold $(\mathcal{X}_P,\Omega_{\mathcal{X}_P})$ has no multiplicities: the pull-back by Φ of a $K\times K$-orbit in the image is a $K\times K$-orbit in $\mathcal{X}_P$.

(4) The symplectic manifold $(\mathcal{X}_P,\Omega_{\mathcal{X}_P})$ is prequantized by the restriction of the hyperplane line bundle $\mathcal{O}(1)\to\mathbb{P}(\oplus_{i=1}^N(V^K_{\lambda_i})^*\otimes V^K_{\lambda_i})$ to $\mathcal{X}_P$: let us denoted L_P the corresponding $K\times K$-equivariant line bundle.

Let $\mathcal{U}_P := K \cdot P^\circ$ where P° is the interior of P. We define

$$\mathcal{X}_P^\circ := \Phi_l^{-1}(\mathcal{U}_P)$$

which is an invariant, open and dense subset of $\mathcal{X}_P$. We have the following important property concerning $\mathcal{X}_P^\circ$.

(5) There exists an equivariant diffeomorphism $\Upsilon : K \times \mathcal{U}_P \to \mathcal{X}_P^\circ$ such that $\Upsilon^*(\Phi_l)(g,\nu) = g \cdot \nu$ and $\Upsilon^*(\Phi_r)(g,\nu) = -\nu$.

The manifold $\mathcal{X}_P$ is equipped with a family of spin^c bundles

$$\mathcal{S}_P^n := \bigwedge (\mathrm{T}\mathcal{X}_P)^{1,0} \otimes L_P^{\otimes n}, \quad n \geqslant 1,$$

and we consider the corresponding $K \times K$-modules $\mathcal{Q}_{K\times K}(\mathcal{X}_P, \mathcal{S}_P^n)$.

Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a Weyl chamber and let $\Lambda \subset \mathfrak{t}^*$ be the lattice of weights: we denote by $\rho \in \mathfrak{t}^*_+$ the half sum of the positive roots.

Proposition 4.3 *We have the following decomposition*

$$\mathcal{Q}_{K\times K}(\mathcal{X}_P, \mathcal{S}_P^n) = \sum_{\mathcal{O}\cap\{n\,P^o+\rho\}\neq\emptyset} \pi_{\mathcal{O}} \otimes \pi_{\mathcal{O}^*} + \sum_{\mathcal{O}\cap\{n\,\partial P+\rho\}\neq\emptyset} a_{n,\mathcal{O}}\ \pi_{\mathcal{O}} \otimes \pi_{\mathcal{O}^*}.$$

Proof. The result is a consequence of the Meinrenken-Sjamaar $[Q,R]=0$ theorem [12, 13, 21, 14]. To explain it, we parametrize the dual $\widehat{K}$ with the highest weights. For any dominant weight $\alpha \in \Lambda \cap \mathfrak{t}^*_+$, let V_α^K be the irreducible representation of K with highest weight α. In other terms, $V_\alpha^K = \pi^K_{\alpha+\rho}$.

The symplectic $[Q,R]=0$ theorem tells us that the multiplicity of $V_\alpha^K \otimes V_\gamma^K$ in $\mathcal{Q}_{K\times K}(\mathcal{X}_P, \mathcal{S}_P^n)$ is equal to the Riemann-Roch number of the symplectic reduced space $\Phi^{-1}(K\frac{\alpha}{n} \times K\frac{\gamma}{n})/K \times K$.

Points (2) above tell us that $\Phi^{-1}(Ka \times Kb)/K \times K$ is non empty only if $Kb = -Ka$ and $Ka \cap P \neq \emptyset$. With point (4) , we see that the reduced space $\Phi^{-1}(Ka \times -Ka)/K \times K$ is a (smooth) point if $a \in P^o$. The proof is completed.

4.2 Cutting

Let M be a K-manifold of even dimension, oriented, equipped with a K-equivariant spin^c bundle $\mathcal{S}$. Let $\Phi_{\mathcal{S}}$ be the moment map associated to a hermitian connection on $\det(\mathcal{S})$. We assume that $\Phi_{\mathcal{S}}$ is *proper*.

We consider the manifold $\mathcal{X}_P$ equipped with the spin^c bundle $\mathcal{S}_P^n := \bigwedge(\mathrm{T}\mathcal{X}_P)^{1,0} \otimes L_P^{\otimes n}$. The determinant line bundle $\det(\mathcal{S}_P^n)$ is equal to

$$\det((\mathrm{T}\mathcal{X}_P)^{1,0}) \otimes L_P^{\otimes 2n}.$$

Let $\varphi_l, \varphi_r : \mathcal{X}_P \to \mathfrak{k}^*$ be the moment maps associated to the action of $K \times K$ on the line bundle $\det((\mathrm{T}\mathcal{X}_P)^{1,0})$. So the moment map relative to the action of $K \times K$ on $\det(\mathcal{S}_P^n)$ is the map $\Phi^n = (\Phi_l^n, \Phi_r^n) : \mathcal{X}_P \to \mathfrak{k}^* \times \mathfrak{k}^*$ defined by $\Phi_l^n = n\Phi_l + \varphi_l$ and $\Phi_r^n = n\Phi_r + \varphi_r$.

On the dense open subset $\mathcal{X}_P^o \simeq K \times \mathcal{U}_P$ the line bundle $\det((\mathrm{T}\mathcal{X}_P)^{1,0})$ admits a trivialization. Let $c > 0$ such that the closed ball $\{\|\xi\| \leqslant c\}$ is contained in $\mathcal{U}_P$. By partition of unity, we can choose a connection on $\det((\mathrm{T}\mathcal{X}_P)^{1,0})$ such that the corresponding moment maps satisfy

$$\varphi_l(x) = \varphi_r(x) = 0 \tag{4.7}$$

if $x \in \mathcal{X}_P$ satisfies $\|\Phi_r(x)\| \leqslant c$.

We consider now the product $M \times \mathcal{X}_P$ equipped with the spin^c bundle $\mathcal{S} \boxtimes \mathcal{S}_P^n$ and with the following $K \times K$ action:

$$(k_l, k_r) \cdot (m, x) = (k_r \cdot m, (k_l, k_r) \cdot x).$$

The moment map relative to the action of $K \times K$ on the line bundle $\det(\mathcal{S} \boxtimes \mathcal{S}_P^n)$ is

$$(m, x) \longmapsto (\Phi_l^n(x), \Phi_{\mathcal{S}}(m) + \Phi_r^n(x)).$$

We restrict the action of $K \times K$ on $M \times \mathcal{X}_P$ to the subgroup $H \times K$. We see that the corresponding moment map $(p \circ \Phi_l^n, \Phi_{\mathcal{S}} + \Phi_r^n)$ is proper, so we can consider the formal geometric quantization of the spin^c manifold $(M \times \mathcal{X}_P, \mathcal{S} \boxtimes \mathcal{S}_P^n)$ relative to the action of $H \times K$: $\mathcal{Q}_{K \times H}^{-\infty}(M \times \mathcal{X}_P, \mathcal{S} \boxtimes \mathcal{S}_P^n) \in \hat{R}(K \times H)$.

We are interested in the following H-module

$$E(n) := \left[\mathcal{Q}_{H \times K}^{-\infty}(M \times \mathcal{X}_P, \mathcal{S} \boxtimes \mathcal{S}_P^n)\right]^K.$$

The proof of Theorem 4.1 will follows from the computation of the asymptotic behaviour of $E(n)$ by two means.

4.3 First computation

Let us write $\mathcal{Q}_K^{-\infty}(M, \mathcal{S}) = \sum_{\mathcal{O}} m_{\mathcal{O}} \pi_{\mathcal{O}}^K$.

Proposition 4.4 *We have*

$$\lim_{n\to\infty} E(n) = \mathcal{Q}_K^{-\infty}(M,\mathcal{S})|_H := \sum_{\lambda\in\hat{K}} m_{\mathcal{O}}\, \pi_{\mathcal{O}}^K|_H \in \hat{R}(H)$$

Proof. We start by using the multiplicative property (Theorem 3.4):

$$\mathcal{Q}_{H\times K}^{-\infty}(M\times\mathcal{X}_P, \mathcal{S}\boxtimes\mathcal{S}_P^n) = \mathcal{Q}_K^{-\infty}(M,\mathcal{S}_M)\otimes\mathcal{Q}_{H\times K}(\mathcal{X}_P,\mathcal{S}_P^n).$$

Thanks to Proposition 4.3, we know that $\mathcal{Q}_{H\times K}(\mathcal{X}_P,\mathcal{S}_P^n)$ is equal to

$$\sum_{\mathcal{O}\cap\{nP^\circ+\rho\}\neq\emptyset} \pi_{\mathcal{O}}^K|_H\otimes(\pi_{\mathcal{O}}^K)^* + R(n)$$

with $R(n)=\sum_{\mathcal{O}} a_{n,\mathcal{O}}\,\pi_{\mathcal{O}}^K|_H\otimes(\pi_{\mathcal{O}}^K)^*$ where $a_{n,\mathcal{O}}\neq 0$ only if $\mathcal{O}\cap\{n\partial P+\rho\}\neq\emptyset$. So we see that

$$E(n) = \sum_{\mathcal{O}\cap\{nP^\circ+\rho\}\neq\emptyset} m_{\mathcal{O}}\,\pi_{\mathcal{O}}^K|_H + r(n)$$

with $r(n)=\left[\mathcal{Q}_K^{-\infty}(M,\mathcal{S})\otimes R(n)\right]^K$. It remains to check that $\lim_{n\to\infty} r(n) = 0$ in $\hat{R}(H)$.

If $\mathcal{O}$ is a K-coadjoint orbit we denote $\|\mathcal{O}\|$ the norm of any of its element. Note that there exists $d>0$ such that if $\mathcal{O}\cap\{n\partial P+\rho\}\neq\emptyset$, then $\|\mathcal{O}\|\geqslant nd$.

Let $\mathcal{O}'$ be a regular admissible H-orbit. By definition the multiplicity of $\pi_{\mathcal{O}'}^H$ in $r(n)$ decomposes as follows

$$[\pi_{\mathcal{O}'}^H : r(n)] = \sum_{\mathcal{O}} a_{n,\mathcal{O}}\, m_{\mathcal{O}}\,[\pi_{\mathcal{O}'}^H : \pi_{\mathcal{O}}^K|_H].$$

Suppose that $r(n)$ does not tends to 0 in $\hat{R}(H)$: there exists a regular admissible H-orbit $\mathcal{O}'$ such that the set $\{n\geqslant 1, [\pi_{\mathcal{O}'}^H : r(n)]\neq 0\}$ is infinite. Hence there exists a sequence $(n_k,\mathcal{O}_k)$ such that $\lim_{k\to\infty} n_k=\infty$ and $a_{n_k,\mathcal{O}_k}\, m_{\mathcal{O}_k}\,[\pi_{\mathcal{O}'}^H : \pi_{\mathcal{O}_k}^K|_H]\neq 0$.

Thanks to the $[Q,R]=0$ property, we have

$$\begin{cases} 1.\ \|\mathcal{O}_k\|\geqslant d\,n_k,\\ 2.\ \mathcal{O}_k\in\Phi_{\mathcal{S}}(M),\\ 3.\ \mathcal{O}'\subset \mathrm{p}(\mathcal{O}_k),\end{cases} \tag{4.8}$$

where $\mathrm{p}:\mathfrak{k}^*\to\mathfrak{h}^*$ is the projection. Points 2. and 3. give that

$$\mathcal{O}_k\in K\Phi_{\mathcal{S}}\left((\mathrm{p}\circ\Phi_{\mathcal{S}})^{-1}(\mathcal{O}')\right).$$

Since $\mathrm{p}\circ\Phi_{\mathcal{S}}$ is proper, we have only a finite number of K-admissible orbits contained in the compact set $K\Phi_{\mathcal{S}}\left((\mathrm{p}\circ\Phi_{\mathcal{S}})^{-1}(\mathcal{O}')\right)$. This is in contradiction with the first point. □

4.4 Reduction in stage

In this section, we explain the case of reduction in stages. Suppose that we have an action of the compact Lie group $G \times K$ on the spin^c manifold $(N, \mathcal{S}_N)$. Let $\Phi_{\mathcal{S}_N} = \Phi^G_{\mathcal{S}_N} \oplus \Phi^K_{\mathcal{S}_N} : N \to \mathfrak{g}^* \oplus \mathfrak{k}^*$ be the corresponding moment map associated to the choice of an invariant connection ∇ on $\det(\mathcal{S}_N)$. We suppose that

- 0 is a regular value of $\Phi^K_{\mathcal{S}_N}$,
- K acts freely on $Z := (\Phi^K_{\mathcal{S}_N})^{-1}(0)$,
- the set $\Phi^{-1}_{\mathcal{S}_N}(0)$ is compact.

We denote by $\pi : Z \to N_0 := Z/K$ the corresponding G-equivariant principal fibration.

On Z, we obtain an exact sequence $0 \longrightarrow \mathrm{T}Z \longrightarrow \mathrm{T}M|_Z \stackrel{\mathrm{T}\Phi_K}{\longrightarrow} [\mathfrak{k}^*] \to 0$, where $[\mathfrak{k}^*]$ is the trivial bundle $Z \times \mathfrak{k}^*$. We have also an orthogonal decomposition $\mathrm{T}Z = \mathrm{T}_K Z \oplus [\mathfrak{k}]$ where $[\mathfrak{k}]$ is the sub-bundle identified to $Z \times \mathfrak{k}$ through the map $(p, X) \mapsto X \cdot p$. So $\mathrm{T}M|_Z$ admits the orthogonal decomposition $\mathrm{T}N|_Z \simeq \mathrm{T}_K Z \oplus [\mathfrak{k}] \oplus [\mathfrak{k}^*]$. We rewrite this as

$$\mathrm{T}N|_Z \simeq \mathrm{T}_K Z \oplus [\mathfrak{k}_\mathbb{C}] \tag{4.9}$$

with the convention $[\mathfrak{k}] = Z \times (\mathfrak{k} \otimes \mathbb{R})$ and $[\mathfrak{k}^*] = Z \times (\mathfrak{k} \otimes i\mathbb{R})$. Note that the bundle $\mathrm{T}_K Z$ is naturally identified with $\pi^*(\mathrm{T}N_0)$.

We can divide the spin^c-bundle $\mathcal{S}_N|_Z$ by the spin^c-bundle $\bigwedge \mathfrak{k}_\mathbb{C}$ for the vector space $\mathfrak{k}_\mathbb{C}$ (see Section 2.2 in [18]).

Definition 4.5 *Let $\mathcal{S}_{N_0}$ be the* spin^c*-bundle on N_0 such that*

$$\mathcal{S}_N|_Z \simeq \pi^*(\mathcal{S}_{N_0}) \otimes [\textstyle\bigwedge \mathfrak{k}_\mathbb{C}]$$

is an isomorphism of graded Clifford bundles on $\mathrm{T}N|_Z$.

We see then that the line bundle $\det(\mathcal{S}_{N_0})$ is equal to $\det(\mathcal{S}_N)|_Z/K$. Hence the connection ∇ on $\det(\mathcal{S}_N)$ induces a G-invariant connection ∇_0 on $\det(\mathcal{S}_{N_0})$. The corresponding moment map $\Phi_{\mathcal{S}_{N_0}} : N_0 \to \mathfrak{g}^*$ is the equivariant map induced by $\Phi^G_{\mathcal{S}_N} : N \to \mathfrak{g}^*$.

Proposition 4.6 *We have the following relation*

$$[\mathcal{Q}_{G\times K}(N, \mathcal{S}_N, \{\Phi_{\mathcal{S}_N} = 0\})]^K = \mathcal{Q}_G(N_0, \mathcal{S}_{N_0}, \{\Phi_{\mathcal{S}_{N_0}} = 0\}) \quad \text{in} \quad \hat{R}(G).$$

Proof. The proof is done in Section 3.4.2 of [18] in the Hamiltonian setting. The same proof works here.

4.5 Second computation

We consider $\mathcal{Q}^{-\infty}_H(M,\mathcal{S}) \in \hat{R}(H)$.

Proposition 4.7 *Suppose that the generic infinitesimal stabilizer of the K-action on M is* abelian. *Let $\mathcal{O}'$ be a regular admissible H-orbit. There exists $n_{\mathcal{O}'} \geqslant 1$ such that*

$$\left[\pi^H_{\mathcal{O}'} : E(n)\right] = \left[\pi^H_{\mathcal{O}'} : \mathcal{Q}^{-\infty}_H(M,\mathcal{S})\right]$$

when $n \geqslant n_{\mathcal{O}'}$.

Proof. First of all, since the K-action on M has generic abelian infinitesimal stabilizers, we see that the $H \times K$-action on $M \times \mathcal{X}_P$ has also generic abelian infinitesimal stabilizers.

Let us denote $\tilde{\mathcal{O}}$ the $H \times K$ regular admissible orbit $\mathcal{O}' \times K\rho$. We work with the $H \times K$ manifold

$$N := M \times \mathcal{X}_P \times \tilde{\mathcal{O}}^*$$

which is equipped with the spin^c bundles $\mathcal{S}^n_N := \mathcal{S} \boxtimes \mathcal{S}^n_P \boxtimes \mathcal{S}_{\tilde{\mathcal{O}}^*}$. The moment map associated to the action of $H \times K$ on $\det(\mathcal{S}^n_N)$ is $\Phi_{\mathcal{S}^n_N} = (\Phi^n_H, \Phi^n_K)$ where

$$\Phi^n_H(m,x,\eta,\xi) = \mathrm{p}\left(n\Phi_l(x) + \varphi_l(x)\right) + \eta,$$

and

$$\Phi^n_K(m,x,\eta,\xi) = \Phi_{\mathcal{S}}(m) + n\Phi_r(x) + \varphi_r(x) + \xi$$

for $(m,x,\eta,\xi) \in M \times \mathcal{X}_P \times (\mathcal{O}')^* \times (K\rho)^*$.

Thanks to the multiplicative property we have

$$\left[\pi^H_{\mathcal{O}'} : E(n)\right] = \left[\mathcal{Q}^{-\infty}_{H\times K}(N,\mathcal{S}^n_N)\right]^{H\times K}.$$

Using the fact that the $H \times K$-action on $M \times \mathcal{X}_P$ has generic abelian infinitesimal stabilizers, we know that

$$\left[\pi^H_{\mathcal{O}'} : E(n)\right] = \left[\mathcal{Q}_{H\times K}(N,\mathcal{S}^n_N,\{\Phi_{\mathcal{S}^n_N} = 0\})\right]^{H\times K}. \tag{4.10}$$

See Remark 3.5. Now we are going to compute the right hand side of (4.10) by using the reduction in stage (see Section 4.4).

We start with the

Lemma 4.8 *There exists $R, R' > 0$, independent of n, such that if $(m,x,\eta,\xi) \in \{\Phi_{\mathcal{S}^n_N} = 0\}$ then $\|\Phi_{\mathcal{S}}(m)\| \leqslant R$ and $\|\Phi_r(x)\| \leqslant R'/n$.*

Proof. Let $(m, x, \eta, \xi) \in \{\Phi_{\mathcal{S}_N^n} = 0\}$. We have $\mathrm{p}\,(n\Phi_l(x) + \varphi_l(x)) + \eta = 0$ and $\Phi_{\mathcal{S}}(m) + n\Phi_r(x) + \varphi_r(x) + \xi = 0$. Let $k \in K$ such that $k\Phi_r(x) = -\Phi_l(x)$ (see Point (2) in Section 4.1). We get then $\mathrm{p}(\Phi_{\mathcal{S}}(km)) + \mathrm{p}(k\varphi_r(x) + \varphi_l(x) + \xi) + \eta = 0$. The term $\mathrm{p}(k\varphi_r(x) + \varphi_l(x) + \xi) + \eta$ is bounded, and since $\mathrm{p} \circ \Phi_{\mathcal{S}}$ is proper, the variable m belongs to a compact of M (independent of n). Finally the identity $\Phi_{\mathcal{S}}(m) + n\Phi_r(x) + \varphi_r(x) + \xi = 0$ shows that $n\Phi_r(x)$ is bounded by a quantity independent of n. □

So, if n is large enough, the set $\{\Phi_{\mathcal{S}_N^n} = 0\}$ is contained in the open subset $M \times \mathcal{X}_P^o \times \tilde{\mathcal{O}}^* \subset N$ that we can identify with

$$\tilde{N} = M \times K \times \mathcal{U}_P \times \tilde{\mathcal{O}}^*$$

through the diffeomorphism $\Upsilon : K \times \mathcal{U}_P \to \mathcal{X}_P^o$ (see Point (4) in Section 4.1). Moreover, thanks to (4.7), for n large enough an element $(m, g, \nu, \eta, \xi) \in \tilde{N}$ belongs to $\{\Phi_{\mathcal{S}_N^n} = 0\}$ if and only if

$$\begin{cases} n\mathrm{p}(g\nu) + \eta = 0, \\ \Phi_{\mathcal{S}}(m) - n\nu + \xi = 0. \end{cases} \tag{4.11}$$

We use now the reduction in stage for n large enough. The map

$$(m, \eta, \xi, g) \mapsto (m, g, \frac{\Phi_{\mathcal{S}}(m) + \xi}{n}, \eta)$$

defines a diffeomorphism between $M \times \tilde{\mathcal{O}}^* \times K$ and the sub-manifold $Z := \{\Phi_K^n = 0\} \subset \tilde{N}$ and induces a diffeomorphism

$$\Psi : M \times \tilde{\mathcal{O}}^* \xrightarrow{\sim} N_0 = Z/K.$$

Through the diffeomorphism Ψ, the H-action on $N_0 = Z/K$ corresponds to the induced action of the subgroup $H \simeq \{(h, h), h \in H\} \subset H \times K$ on $M \times \tilde{\mathcal{O}}^*$. Through the diffeomorphism Ψ, the moment map $\Phi_{\mathcal{S}_{N_0}^n} : N_0 \to \mathfrak{h}^*$ becomes

$$\Phi_{\tilde{\mathcal{O}}}(m, \xi, \eta) = \mathrm{p}(\Phi_{\mathcal{S}}(m) + \xi) + \eta,$$

for $(m, \xi, \eta) \in M \times \tilde{\mathcal{O}}^*$.

Lemma 4.9 *Through the diffeomorphism Ψ, the induced* spin^c *bundle $\mathcal{S}_{N_0}^n$ corresponds to $\mathcal{S}_M \boxtimes \mathcal{S}_{\tilde{\mathcal{O}}^*}$.*

Proof. We consider the restriction of the spin^c bundle $\mathcal{S}_P^n$ to the open subset $\mathcal{X}_P^\circ$. Let $\mathcal{S}^n := \Upsilon^{-1}(\mathcal{S}_P^n|_{\mathcal{X}_P^\circ})$ be the corresponding $K \times K$-equivariant spin^c bundle on $K \times \mathcal{U}_P$. It must be of the form $\mathcal{S}^n \simeq F \times \bigwedge \mathfrak{k}_\mathbb{C} \times K \times \mathcal{U}_P$ where F is a character of $K \times K$. If we look at the value of $\mathcal{S}_P^n$ at the point $\Upsilon(1,0) \in \mathcal{X}_P$, we see that F is trivial. The Lemma follows. □

Finally, for n large enough, we get

$$\begin{aligned}
\left[\pi^H_{\mathcal{O}'} : E(n)\right] &= \left[\mathcal{Q}_{H\times K}(N, \mathcal{S}^n_N, \{\Phi_{\mathcal{S}^n_N} = 0\})\right]^{H\times K} & [1]\\
&= \left[\mathcal{Q}_H(N_0, \mathcal{S}^n_{N_0}, \{\Phi_{\mathcal{S}^n_{N_0}} = 0\})\right]^H & [2]\\
&= \left[\mathcal{Q}_H(M \times \tilde{\mathcal{O}}^*, \mathcal{S} \boxtimes \mathcal{S}_{\tilde{\mathcal{O}}^*}, \{\Phi_{\tilde{\mathcal{O}}} = 0\})\right]^H & [3]\\
&= \left[\mathcal{Q}^{-\infty}_H(M \times \tilde{\mathcal{O}}^*, \mathcal{S} \boxtimes \mathcal{S}_{\tilde{\mathcal{O}}^*})\right]^H & [4]\\
&= \left[\mathcal{Q}^{-\infty}_H(M, \mathcal{S}) \otimes \mathcal{Q}_H(\tilde{\mathcal{O}}^*, \mathcal{S}_{\tilde{\mathcal{O}}^*})\right]^H & [5]\\
&= \left[\pi^H_{\mathcal{O}'} : \mathcal{Q}^{-\infty}_H(M, \mathcal{S})\right]. & [6]
\end{aligned}$$

First we see that [1] corresponds to (4.10). Equality [2] is the reduction in stage (see Proposition 4.6) and Equality [3] is a consequence of the diffeomorphism Ψ (see Lemma 4.9). Equality [4] follows from the fact that M has abelian generic infinitesimal stabilizers (see Remark 3.5). Equality [5] is a consequence of the multiplicative property. Equality [6] follows from the identity $\mathcal{Q}_H(\tilde{\mathcal{O}}^*, \mathcal{S}_{\tilde{\mathcal{O}}^*}) = (\pi^H_{\mathcal{O}'})^*$.

The proof of Proposition 4.7 is completed. □

We can now conclude our exposition. Proposition 4.4 tell us that $\lim_{n\to\infty} E(n) = \mathcal{Q}^{-\infty}_K(M, \mathcal{S})|_H$ while Proposition 4.7 says that $\lim_{n\to\infty} E(n) = \mathcal{Q}^{-\infty}_H(M, \mathcal{S})$ when the manifold M has abelian generic infinitesimal stabilizers. So we have proved Theorem 4.1 for manifolds with abelian generic infinitesimal stabilizers. But we have checked in Lemma 4.2 that it is sufficient to get the proof of Theorem 4.1 in the general case.

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