MODULI SPACES OF (1,7)-POLARIZED ABELIAN SURFACES AND VARIETIES OF SUMS OF POWERS

MICHELE BOLOGNESI AND ALEX MASSARENTI

Abstract. We study the geometry of some varieties of sums of powers related to the Klein quartic. Thanks to preceding results of Mukai and of ourselves, this allows us to describe the birational geometry of certain moduli spaces of abelian surfaces. In particular we show that $\mathcal{A}_2(1,7)^-_{sym}$ is unirational by showing that it has a conic bundle structure.

Contents

1. Introduction

We investigate the birational geometry of some moduli spaces of abelian surfaces related to $\mathcal{A}_2(1,7)$, the moduli space of abelian surfaces with a polarization of type $(1,7)$ and a $(1,7)$-level structure. In particular, if we endow the abelian surfaces in $\mathcal{A}_2(1,7)$ with a symmetric theta structure and a theta characteristic (odd or even), we obtain two new moduli spaces, $\mathcal{A}_2(1,7)^-_{sym}$ and $\mathcal{A}_2(1,7)^+_{sym}$, that are finite covers of degree 6 and 10 respectively of $\mathcal{A}_2(1,7)$. For a general introduction to these spaces see our paper [BM16, Sections 6.1.1]. On the other hand, we introduce also the moduli space $\mathcal{A}_2(1,7;2,2)$ parametrizing abelian surfaces with a polarization of type $(1,7)$, a $(1,7)$-level structure and a $(2,2)$-level structure.

The relation of $\mathcal{A}_2(1,7)$ with varieties of sums of powers (VPS for short) dates back to the work of S. Mukai [Muk92]. In this paper we introduce new types of varieties of sums of powers and showcase rational maps between them and our moduli spaces of abelian surfaces. For the definition of the new VPS, we refer to Section 3.

Proposition 1.1. *The moduli spaces $\mathcal{A}_2(1,7)^-_{sym}$ and $\mathcal{A}_2(1,7)^+_{sym}$ are birational to the varieties $\mathrm{VSP}_6(F_4,6)$ and $\mathrm{VSP}^6(F_4,6)$ respectively, where $F_4 \in k[x_0,x_1,x_2]_4$ is the Klein quartic. The moduli space $\mathcal{A}_2(1,7;2,2)$ is birational to $\mathrm{VSP}_{ord}(F_4,6)$.*

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The theory of apolar varieties developed in [DK93] allows us to produce in Section 3.7 a 3-fold conic bundle dominating $\mathrm{VSP}_6(F_4, 6)$, and to conclude that it is unirational. In Section 4 we develop some birational geometry of the moduli spaces of abelian surfaces that we are considering. In particular, as a corollary of the above result, we get the following.

Theorem 1.2. *The moduli space $\mathcal{A}_2(1,7)^-_{sym}$ is unirational, and hence its Kodaira dimension is $-\infty$. Furthermore, the moduli space $\mathcal{A}_2(1,7;2,2)$ admits a rational fibration over $\mathbb{P}^2$ whose general fiber is a curve of general type.*

Throughout the paper we will work over the complex field.

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2. Moduli of polarized abelian surfaces

In this section we recall a couple of very specific results from [BM16] that are needed here. For general results see [BL04], [GP01], [Bol07] or the first four sections of [BM16]. Since all the abelian surfaces we will deal with will be endowed with a polarization of type $(1,7)$ we will not mention any more this datum in the rest of the paper. Let $\mathbb{H}_2$ be the Siegel half space for abelian surfaces. The following proposition descends from [BM16, Sections 4].

Proposition 2.1. *There exist arithmetic subgroups $\Gamma_2(1,7)^+$ and $\Gamma_2(1,7)^-$ such that there are quasi-projective moduli spaces $\mathcal{A}_2(1,7)^+_{sym} := \mathbb{H}_2/\Gamma_2(1,7)^+$ and $\mathcal{A}_2(1,7)^-_{sym} := \mathbb{H}_2/\Gamma_2(1,7)^-$ that parametrize abelian surfaces with a $(1,7)$-structure, a symmetric theta structure and respectively an even or an odd theta characteristic.*

The Theta-Null maps (see [BM16, Section 5.1] for details and definitions) for abelian surfaces with a theta characteristic are defined as follows. Let (A, L, ψ) be a polarized abelian surface A, with an even (respectively odd) line bundle L representing the polarization, and a symmetric theta structure ψ,

$$\begin{array}{rcl} Th^+_{(1,7)} : \mathcal{A}_2(1,7)^+_{sym} & \to & \mathbb{P}^3_+, \\ (A, L, \psi) & \mapsto & \Psi^+(\Theta_{1,7}(0)); \\ Th^-_{(1,7)} : \mathcal{A}_2(1,7)^-_{sym} & \to & \mathbb{P}^2_-, \\ (A, L, \psi) & \mapsto & \Psi^-(\Theta_{1,7}(0)); \end{array}$$

where Ψ^+ (respectively Ψ^-) is the identification between the invariant part of the space of global sections $\Theta_{1,7}$ of L with the invariant (respectively anti-invariant) part of a certain Schrödinger representation of the Heisenberg group.

Let $D \in Mat_2(\mathbb{Z})$ be a diagonal 2×2 matrix with integer entries. We define the subgroup $\Gamma_D \subset Mat_4(\mathbb{Z})$ as:

$$\Gamma_D := \left\{ R \in M_4(\mathbb{Z}) \mid R \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} R^t = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \right\}, \tag{2.1}$$

and the subgroup $\Gamma_D(D) \subset \Gamma_D$ as:

$$\Gamma_D(D) := \left\{ \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \in \Gamma_D \mid \mathbf{A} - I \equiv \mathbf{B} \equiv \mathbf{C} \equiv \mathbf{D} - I \equiv 0 \mod (D) \right\}, \tag{2.1}$$

where $\mathbf{M} \equiv 0 \mod (D)$ if and only if $\mathbf{M} \in D \cdot Mat_g(\mathbb{Z})$. See [BM16, Section 4] for details on these groups and further references. Let us now denote $\Gamma_{(1,7)}(1,7)$ the group $\Gamma_D(D)$ corresponding to $D = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$.

Definition 2.2. We define

$$\Gamma_{(1,7)}(1,7;2,2) := \left\{ \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \in \Gamma_{(1,7)}(1,7) \mid \mathbf{A} - I \equiv \mathbf{B} \equiv \mathbf{C} \equiv \mathbf{D} - I \equiv 0 \mod \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}.$$

By [BB66] and [BL04, Section 8.3], the quasi-projective variety

$$\mathcal{A}_2(1,7;2,2) := \mathbb{H}_2/\Gamma_{(1,7)}(1,7;2,2)$$

is the moduli space of abelian surfaces with a polarization of type $(1,7)$, a level $(1,7)$-structure and a level $(2,2)$-structure.

In the rest of the paper, we will abuse slightly our notation. While we will not change notation, all the moduli spaces considered will be non-singular models of suitable compactifications of the quasi-projective ones.

3. Ordered varieties of sums of powers

Varieties of sums of powers parametrize decompositions of a general homogeneous polynomial $F \in k[x_0, ..., x_n]$ as sums of powers of linear forms. They have been widely studied from both the biregular [IR01], [Muk92], [RS00] and the birational viewpoint [MM13], [Mas16].
Let $\nu_d^n : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$, with $N(n,d) = \binom{n+d}{d} - 1$ be the Veronese embedding induced by $\mathcal{O}_{\mathbb{P}^n}(d)$, and let $V_d^n = \nu_d^n(\mathbb{P}^n)$ be the corresponding Veronese variety. Let $F \in k[x_0, ..., x_n]_d$ be a general homogeneous polynomial of degree d.

Definition 3.1. Let $F \in \mathbb{P}^{N(n,d)}$ be a general point of V_d^n. Let h be a positive integer and $\mathrm{Hilb}_h(\mathbb{P}^{n*})$ the Hilbert scheme of sets of h points in $(\mathbb{P}^{n*})$. We define

$$\mathrm{VSP}(F,h)^o := \{\{L_1, ..., L_h\} \in \mathrm{Hilb}_h(\mathbb{P}^{n*}) \mid F \in \langle L_1^d, ..., L_h^d \rangle\} \subseteq \mathrm{Hilb}_h(\mathbb{P}^{n*})\},$$

and $\mathrm{VSP}(F,h) := \overline{\mathrm{VSP}(F,h)^o}$ by taking the closure of $\mathrm{VSP}(F,h)^o$ in $\mathrm{Hilb}_h(\mathbb{P}^{n*})$.

Assume that the general polynomial $F \in \mathbb{P}^{N(n,d)}$ is contained in a $(h-1)$-linear space h-secant to V_d^n. Then, by [Dol04, Proposition 3.2] the variety $\mathrm{VSP}(F,h)$ has dimension $h(n+1) - N(n,d) - 1$. Furthermore, if $n = 1, 2$ then for F varying in an open Zariski subset of $\mathbb{P}^{N(n,d)}$ the variety $\mathrm{VSP}(F,h)$ is smooth and irreducible.

In order to apply this object to the study of abelian surfaces, we need to construct similar varieties parametrizing the decomposition of homogeneous polynomials as sums of powers of linear forms and admitting natural generically finite rational maps onto $\mathrm{VSP}(F,h)$.

Definition 3.2. Let $F \in \mathbb{P}^{N(n,d)}$ be a general point. We define

$$\mathrm{VSP}_{ord}(F,h)^o := \{(L_1, ..., L_h) \in (\mathbb{P}^{n*})^h \mid F \in \langle L_1^d, ..., L_h^d \rangle\} \subseteq (\mathbb{P}^{n*})^h,$$

and $\mathrm{VSP}_{ord}(F,h) := \overline{\mathrm{VSP}_{ord}(F,h)^o}$ by taking the closure of $\mathrm{VSP}_{ord}(F,h)^o$ in $(\mathbb{P}^{n*})^h$.

Note that $\mathrm{VSP}_{ord}(F,h)$ is a variety of dimension $h(n+1)-N(n,d)-1$. Furthermore, two general points of $\mathrm{VSP}_{ord}(F,h)$ define the same point of $\mathrm{VSP}(F,h)$ if and only if they differ by a permutation in the symmetric group S_h. Therefore, we have a generically finite rational map $\phi_h : \mathrm{VSP}_{ord}(F,h) \dashrightarrow \mathrm{VSP}(F,h)$ of degree $h!$

Remark 3.3. Arguing as in the proof of [Dol04, Proposition 3.2], with $(\mathbb{P}^{n*})^h$ instead of the Hilbert scheme $\mathrm{Hilb}_h(\mathbb{P}^{n*})$, we can show that for a general polynomial F the variety $\mathrm{VSP}_{ord}(F,h)$ is smooth and irreducible of dimension $h(n+1)-N(n,d)-1$.

Definition 3.4. Consider the rational action of S_{h-1} on $\mathrm{VSP}_{ord}(F,h)$ given by permuting the linear forms $(L_2,\dots,L_h)$. The variety $\mathrm{VSP}_h(F,h)$ is the quotient $\mathrm{VSP}_h(F,h) = \mathrm{VSP}_{ord}(F,h)/S_{h-1}$.

If $h=2r$ is even consider the rational action of $S_r\times S_r$ on $\mathrm{VSP}_{ord}(F,h)$ where the first and the second copy of S_r act on the first and the last r linear forms respectively. Let $X_h(F) = \mathrm{VSP}_{ord}(F,h)/S_r\times S_r$. Therefore

$$X_h(F) = \{(\{L_1,\dots,L_r\},\{L_{r+1},\dots,L_h\}) \mid (L_1,\dots,L_h)\in \mathrm{VSP}_{ord}(F,h)\}$$

comes with a natural S_2-action. We define $\mathrm{VSP}^h(F,h) = X_h(F)/S_2$.

Note that $\mathrm{VSP}_h(F,h)$ admits a generically finite rational map $\chi_h : \mathrm{VSP}_h(F,h) \dashrightarrow \mathrm{VSP}(F,h)$ of degree h, and that the h points on the fiber of ψ over a general point $\{L_1,...,L_h\}\in \mathrm{VSP}(F,h)$ can be identified with the linear forms $L_1,...,L_h$ themselves. Similarly $\mathrm{VSP}^h(F,h)$ has a generically finite rational map $\chi^h : \mathrm{VSP}^h(F,h) \dashrightarrow \mathrm{VSP}(F,h)$ of degree $\frac{h!}{2(r!)^2}$.

The variety $\mathrm{VSP}_h(F,h)$ can be explicitly constructed in the following way. Let us consider the incidence variety

$$\mathcal{J} := \{(l,\{L_1,...,L_h\}) \mid l\in\{L_1,...,L_h\}\in \mathrm{VSP}(F,h)^o\} \subseteq \mathbb{P}^{n*}\times \mathrm{VSP}(F,h)^o. \tag{3.5}$$

Then $\mathrm{VSP}_h(F,h)$ is the closure $\overline{\mathcal{J}}$ of $\mathcal{J}$ in $\mathbb{P}^{n*}\times\mathrm{VSP}(F,h)$.

Remark 3.6. In [Muk92] Mukai proved that if $F\in k[x_0,x_1,x_2]_4$ is a general polynomial then $\mathrm{VSP}(F,6)$ is a smooth Fano 3-fold V_{22} of index 1 and genus 12. In this case we have a generically 720 to 1 rational map $\phi_6 : \mathrm{VSP}_{ord}(F,6) \dashrightarrow \mathrm{VSP}(F,6)$, a generically 6 to 1 rational map $\chi_6 : \mathrm{VSP}_6(F,6) \dashrightarrow \mathrm{VSP}(F,6)$, and a generically 10 to 1 rational map $\chi^6 : \mathrm{VSP}^6(F,6) \dashrightarrow \mathrm{VSP}(F,6)$.

By [GP01, Corollary 5.6], under the same assumptions on F, the moduli space $\mathcal{A}_2(1,7)^{lev}$ of $(1,7)$-polarized abelian surfaces with canonical level structure is birational to $\mathrm{VSP}(F,6)$. As already observed in [MS01, Theorem 4.4] the Klein quartic

$$F_4 = x_0^3x_1 + x_1^3x_2 + x_0x_2^3 = 0 \tag{3.7}$$

is general in the sense of Mukai [Muk92], hence the variety $\mathrm{VSP}(F,6)$ is isomorphic to the VSP obtained for any other general quartic curve.

3.7. **Apolar varieties.** Let V be a complex vector space of dimension $n+1$, choose coordinates $x_0,\dots,x_n$ in V and the dual coordinates $\xi_0,\dots,\xi_n$ in V^*. For any homogeneous polynomial $G(\xi_0,\dots,\xi_n)\in\mathbb{C}[\xi_0,\dots,\xi_n]_k$ consider the linear differential operator $\Delta_G = G(\partial_1,\dots,\partial_n)$, where $\partial_i = \frac{\partial}{\partial x_i}$. We say that $G\in\mathbb{C}[\xi_0,\dots,\xi_n]_{d-k}$ is a $k-th$ *apolar* of a linear form $L\in\mathbb{C}[x_0,\dots,x_n]_1$ with respect to $F\in\mathbb{C}[x_0,\dots,x_n]_d$ if $\Delta_G(F)=L^k$.

Now, let F_4 be the Klein quartic (3.7). The apolar conic of a linear form $L = ax_0 + bx_1 + cx_2$ with respect to F_4 is given by

$$G_L = ab\xi_0^2 + a^2\xi_0\xi_1 + c^2\xi_0\xi_2 + bc\xi_1^2 + b^2\xi_1\xi_2 + ac\xi_2^2. \tag{3.8}$$

Definition 3.9. For any $s = 1, \dots, 6$ we define the *s-th apolar variety* of F_4 as the subvariety $\mathcal{P}_s$ of $(\mathbb{P}^{2*})^s$ cut out by the conditions $L_i \in G_{L_j}$ for any $i, j = 1, \dots, s$ with $i \neq j$.

Proposition 3.10. *We have that $\mathcal{P}_1 = \mathbb{P}^{2*}$, $\mathcal{P}_2 \subset (\mathbb{P}^{2*})^2$ is a smooth unirational 3-fold conic bundle defined by a polynomial of bidegree $(2,2)$, and $\mathcal{P}_3 \subset (\mathbb{P}^{2*})^3$ is an irreducible complete intersection 3-fold defined by three polynomials of multidegree $(2,2,0), (2,0,2)$ and $(0,2,2)$.*

Proof. Let $[a_i : b_i : c_i]$ be the homogeneous coordinates in the i-th factor $\mathbb{P}_i$ of $(\mathbb{P}^{2*})^s$. Therefore, a point in $\mathbb{P}_i$ with homogeneous coordinates $[a_i : b_i : c_i]$ corresponds to the linear form $L_i = a_i x_0 + b_i x_1 + c_i x_2$. Clearly $\mathcal{P}_1 = \mathbb{P}^{2*}$. Now, by (3.8) the condition $L_i \in G_{L_j}$ in Definition 3.9 is given by

$$D_{i,j} = a_j b_j a_i^2 + a_j^2 a_i b_i + c_j^2 a_i c_i + b_j c_j b_i^2 + b_j^2 b_i c_i + a_j c_j c_i^2 = 0.$$

In particular $\mathcal{P}_1 = \mathbb{P}^{2*}$. Note that the hypersurface $D_{i,j} = 0$ is a divisor in $(\mathbb{P}^{2*})^s$ of multidegree $(d_1, \dots, d_s)$ with $d_i = d_j = 2$ and $d_r = 0$ for any $r \neq i, j$. Therefore, $\mathcal{P}_2 \subset (\mathbb{P}^{2*})^2$ is cut out by the equation

$$D_{1,2} = a_1 b_1 a_2^2 + a_1^2 a_2 b_2 + c_1^2 a_2 c_2 + b_1 c_1 b_2^2 + b_1^2 b_2 c_2 + a_1 c_1 c_2^2 = 0. \tag{3.10}$$

Considering the nine standard affine charts covering $D_{1,2}$ we can prove that $D_{1,2}$ is smooth. For instance, by the following Macaulay2 [Mac92] script

```
Macaulay2, version 1.9.2
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases, PrimaryDecomposition,
ReesAlgebra, TangentCone
i1 : R = QQ[a_1,b_1,a_2,b_2]
o1 = R
o1 : PolynomialRing
i2 : I = ideal(a_2*b_2*a_1^2+a_2^2*a_1*b_1+a_1+b_2*b_1^2+b_2^2*b_1+a_2)
o2 : Ideal of R
i3 : Q = R/I
o3 = Q
o3 : QuotientRing
i4 : X = Spec(Q)
o4 = X
o4 : AffineVariety
i5 : S = singularLocus(X)
o5 = S
o5 : AffineVariety
i6 : dim S
o6 = -infinity
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we get that $D_{1,2}$ is smooth in the chart $\{c_1 \neq 0, c_2 \neq 0, c_3 \neq 0\}$. Similarly, we can show that it is smooth in the other eight affine charts as well.

Note that any of the two projections onto the factors induces on $\mathcal{P}_2$ a structure of conic bundle over $\mathbb{P}^2$. If we choose for instance the projection on the first factor then (3.10) yields that the discriminant of the conic bundle is the smooth sextic given by

$$C = \{a_1 b_1^5 + a_1^5 c_1 - 5a_1^2 b_1^2 c_1^2 + b_1 c_1^5 = 0\} \subset \mathbb{P}^{2*}.$$

Now [Mel14, Corollary 1.2] implies that $\mathcal{P}_2$ is unirational.

Let us consider the case $s=3$. We have that $\mathcal{P}_3 \subset (\mathbb{P}^{2*})^3$ is the complete intersection 3-fold defined by the equations

$$(3.11)\qquad \begin{cases} D_{1,2} = a_1b_1a_2^2 + a_1^2a_2b_2 + c_1^2a_2c_2 + b_1c_1b_2^2 + b_1^2b_2c_2 + a_1c_1c_2^2 = 0, \\ D_{1,3} = a_1b_1a_3^2 + a_1^2a_3b_3 + c_1^2a_3c_3 + b_1c_1b_3^2 + b_1^2b_3c_3 + a_1c_1c_3^2 = 0, \\ D_{2,3} = a_2b_2a_3^2 + a_2^2a_3b_3 + c_2^2a_3c_3 + b_2c_2b_3^2 + b_2^2b_3c_3 + a_2c_2c_3^2 = 0. \end{cases}$$

Finally, by a standard Macaulay2 [Mac92] script we can show that $\mathcal{P}_3$ is irreducible. □

Proposition 3.12. *For $s=2,3$ there exist generically finite dominant morphisms*

$$f_s : \mathrm{VSP}_{ord}(F_4,6) \to \mathcal{P}_s$$

of degree 24 and 6 respectively. Furthermore, there is a morphism

$$g_3 : \mathcal{P}_3 \to \mathbb{P}^2$$

whose general fiber is a smooth curve of general type. Finally, there exists a generically finite rational map

$$g_2 : \mathcal{P}_2 \dashrightarrow \mathrm{VSP}_6(F_4,6)$$

of degree 5. In particular $\mathrm{VSP}_6(F_4,6)$ is unirational.

Proof. As in (3.8) we will denote by G_L the apolar conic of a linear form $L=(ax_0+bx_1+cx_2)$ with respect to the Klein quartic F_4. Let $L_1 \in \mathbb{P}^{2*}$ be a linear form and L_2 a general linear form in $\{G_{L_1}=0\}$. By [DK93, Theorem 6.14.2] L_1, L_2 and the points $\{G_{L_1}=0\}\cap\{G_{L_2}=0\}=\{l_3,l_4,l_5,l_6\}$ give a decomposition of $F_4=\lambda_1L_1^4+\lambda_2L_2^4+\lambda_3l_3^4+\cdots+\lambda_6l_6^4$. Therefore, if $s\geq 2$ then the image of the restriction of the projection $f_s := \pi_{|\mathrm{VSP}_{ord}(F_4,6)} : \mathrm{VSP}_{ord}(F_4,6) \to (\mathbb{P}^{2*})^s$ is contained in the s-th apolar variety $\mathcal{P}_s$. On the other hand, given a general point $(L_1,L_2,\dots,L_s)\in \mathrm{Im}(f_s)$ the fiber of $f_s^{-1}(L_1,L_2,\dots,L_s)$ consists of the ordered 6-uples of linear forms $(L_1,L_2,\dots,L_s,l_1,\dots,l_{6-s})$ such that $\{l_1,\dots,l_{6-s}\}\in\{G_{L_1}=0\}\cap\{G_{L_2}=0\}\setminus\{L_3,\dots,L_s\}$. Therefore, $\mathrm{Im}(f_s)$ is an irreducible 3-fold and $\mathrm{Im}(f_s)\subseteq\mathcal{P}_s$. On the other hand, by Proposition 3.10 we have that $\mathcal{P}_2$ and $\mathcal{P}_3$ are irreducible, and this yields $\mathrm{Im}(f_s)\subseteq\mathcal{P}_s$ when $s=2,3$. Furthermore, since a point in a general fiber of f_s is determine up to a permutation of the $6-s$ points $\{l_1,\dots,l_{6-s}\}\in\{G_{L_1}=0\}\cap\{G_{L_2}=0\}\setminus\{L_3,\dots,L_s\}$ we conclude that $f_2 : \mathrm{VSP}_{ord}(F_4,6)\to\mathcal{P}_2$ is a generically finite dominant morphism of degree 24, and $f_3 : \mathrm{VSP}_{ord}(F_4,6)\to\mathcal{P}_3$ is a generically finite dominant morphism of degree 6.

Let $g_3 : \mathcal{P}_3 \to \mathbb{P}^2$ be the restriction to $\mathcal{P}_3\subset(\mathbb{P}^2)^3$ of any of the projections, say the first one. A standard Macaulay2 [Mac92] computations shows that the fiber of g_3 over $[a_1:b_1:c_1]=[1:1:1]$ is a smooth connected curve in $(\mathbb{P}^2)^2$. Therefore a general fiber Γ of g_3 is a smooth connected curve as well. Note that by (3.11) $\Gamma\subset(\mathbb{P}^2)^2$ is a smooth complete intersection defined by three polynomials of bi-degree $(2,0),(0,2),(2,2)$ respectively. By adjunction we get that the canonical sheaf of Γ is given by

$$\omega_\Gamma = \mathcal{O}_\Gamma(-3+2+0+2,-3+0+2+2) = \mathcal{O}_\Gamma(1,1)$$

and hence ω_Γ is ample.

Now, consider the case $s=2$. A general point in $(L_1,L_2)\in\mathcal{P}_2$ determines four additional linear forms $\{l_3,\dots,l_6\}=\{G_{L_1}=0\}\cap\{G_{L_2}=0\}$ and by [DK93, Theorem 6.14.2] $\{L_1,L_2,l_3,\dots,l_6\}$ gives an additive decomposition of F_4. Therefore, keeping in mind (3.5) we may define a rational map $g_2 : \mathcal{P}_2 \dashrightarrow \mathrm{VSP}_6(F_4,6)$ by mapping $(L_1,L_2)\in\mathcal{P}_2$ to $(L_1,\{L_1,L_2,l_3,\dots,l_6\})\in\mathrm{VSP}_6(F_4,6)$. Note that g_2 is dominant and of degree 5. Finally, since by Proposition 3.10 $\mathcal{P}_2$ is unirational we conclude that $\mathrm{VSP}_6(F_4,6)$ is unirational as well. □

4. Birational geometry of moduli spaces of (1,7)-polarized abelian surfaces

A proper variety X over an algebraically closed field is rationally connected if two general points $x_1, x_2 \in X$ can be joined by an irreducible rational curve. In [BM16, Theorem 2] we proved that the moduli space $\mathcal{A}_2(1,7)^-_{sym}$ of abelian surfaces with a $(1,7)$-level structure, a symmetric theta structure and an odd theta characteristic is rationally connected. Clearly, rationality implies unirationality, which in turns implies rational connectedness. If X is a smooth algebraic variety over an algebraically closed field of characteristic zero and $\dim(X) \leq 2$ these three notions are indeed equivalent [Har01, Remark 1.3]. On the other hand, it is well known that a smooth cubic 3-fold $X \subset \mathbb{P}^4$ is unirational but not rational [CG72]. It is a long-standing open problem whether there exist varieties which are rationally connected but not unirational [Har01, Section 1.24]. In this section, by using the techniques developed in Section 3.7 we will prove that $\mathcal{A}_2(1,7)^-_{sym}$ is unirational.

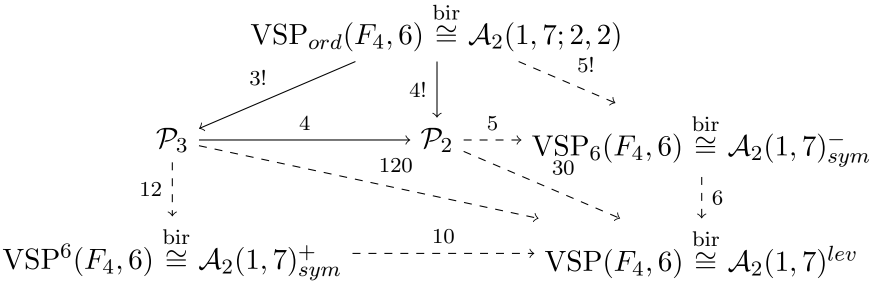
Proposition 4.1. *Let $\mathcal{A}_2(1,7)^-_{sym}$ and $\mathcal{A}_2(1,7)^+_{sym}$ be the moduli spaces of abelian surfaces with level $(1,7)$-structure, a symmetric theta structure and an odd, respectively even theta characteristic. Then $\mathcal{A}_2(1,7)^-_{sym}$ and $\mathcal{A}_2(1,7)^+_{sym}$ are birational to the varieties $\mathrm{VSP}_6(F_4,6)$ and $\mathrm{VSP}^6(F_4,6)$ respectively, where $F_4 \in k[x_0,x_1,x_2]_4$ is the Klein quartic. Furthermore, the moduli space $\mathcal{A}_2(1,7;2,2)$ is birational to $\mathrm{VSP}_{ord}(F_4,6)$.*

Proof. Let $\mathcal{A}_2(1,7)^{lev}$ be the moduli space of abelian surfaces with a $(1,7)$-level structure. Recall from [BM16, Section 6.1.1] that there exists a Theta-Null map $Th^-_{(1,7)} : \mathcal{A}_2(1,7)^-_{sym} \to \mathbb{P}^2$. By [GP01, Proposition 5.4 and Corollary 5.6] there exists a birational map $\alpha : \mathcal{A}_2(1,7)^{lev} \dashrightarrow \mathrm{VSP}(F_4,6)$ mapping a general $(A,\psi) \in \mathcal{A}_2(1,7)^{lev}$ to the set $\{L_{1,A},\dots,L_{6,A}\} \in A$ of the odd 2-torsion points of A, that are naturally mapped to $\mathbb{P}^2$ by the Theta-Null map. Each of the 6 odd 2-torsion points correspond to a choice of an odd theta characteristic via $Th^-_{(1,7)}$. Now, consider a general point (A,ψ,L) of $\mathcal{A}_2(1,7)^-_{sym}$ over $(A,\psi) \in \mathcal{A}_2(1,7)^{lev}$. We may define a rational map $\beta : \mathcal{A}_2(1,7)^-_{sym} \dashrightarrow \mathrm{VSP}_6(F_4,6)$ sending (A,ψ,L) to the linear form in $\chi_6^{-1}(\{L_{1,A},\dots,L_{6,A}\})$ that corresponds to $Th^-_{(1,7)}(A,\psi,L) \in \mathbb{P}^2$, where χ_6 is the map in Remark 3.6. To conclude it is enough to observe that since α is birational the map $\beta : \mathcal{A}_2(1,7)^-_{sym} \dashrightarrow \mathrm{VSP}_6(F_4,6)$ is birational as well.

In particular, since the generic abelian surface is Jacobian, odd theta characteristics correspond to the Weierstrass points of the corresponding curve. It is a classical fact that even theta characteristics correspond to partitions of the Weierstrass points into two 3-elements sets, see for example [DO88, Chapter 8]. This directly implies that $\mathcal{A}_2(1,7)^+_{sym}$ is birational to $\mathrm{VSP}^6(F_4,6)$.

Finally, recall that a $(2,2)$-level structure for a Jacobian abelian surface corresponds to a complete ordering of the Weierstrass points [DO88, Chapter 8]. This in turn implies that the moduli space $\mathcal{A}_2(1,7;2,2)$ is birational to $\mathrm{VSP}_{ord}(F_4,6)$. □

We summarize the situation in the following diagram, where the superscripts on the arrows indicate the degrees of the respective maps.



Theorem 4.2. *The moduli space $\mathcal{A}_2(1,7)^{-}_{sym}$ is unirational, and hence its Kodaira dimension is $-\infty$. Furthermore, the moduli space $\mathcal{A}_2(1,7;2,2)$ admits a rational fibration over $\mathbb{P}^2$ whose general fiber is a curve of general type.*

Proof. Since by Proposition 3.12 $\mathrm{VSP}_6(F_4,6)$ is unirational the first claim follows from Proposition 4.1. The fibers of the restriction to $\mathrm{VSP}_{ord}(F_4,6)$ of the first projection $\pi_{|\mathrm{VSP}_{ord}(F_4,6)} : \mathrm{VSP}_{ord}(F_4,6) \to \mathbb{P}^2$ are mapped by $f_3 : \mathrm{VSP}_{ord}(F_4,6) \to \mathcal{P}_3$ onto the fibers of the morphism $g_3 : \mathcal{P}_3 \to \mathbb{P}^2$. By Proposition 3.12 the general fiber of g_3 is a curve of general type. Hence the general fiber of $\pi_{|\mathrm{VSP}_{ord}(F_4,6)}$ is of general type as well, and by Proposition 4.1 $\pi_{|\mathrm{VSP}_{ord}(F_4,6)}$ induces a rational fibration of $\mathcal{A}_2(1,7;2,2)$ over $\mathbb{P}^2$ whose general fiber is a curve of general type. □

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Michele Bolognesi, IMAG - Université de Montpellier, Place Eugène Bataillon, 34095 Montpellier Cedex 5, France
E-mail address: `michele.bolognesi@umontpellier.fr`

Alex Massarenti, Universidade Federal Fluminense, Rua Mário Santos Braga, 24020-140, Niterói, Rio de Janeiro, Brazil
E-mail address: `alexmassarenti@id.uff.br`