Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

# OPTIMAL PERIODIC CONTROL FOR SCALAR DYNAMICS UNDER INTEGRAL CONSTRAINT ON THE INPUT

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Abstract. This paper studies a periodic optimal control problem governed by a one-dimensional system, linear with respect to the control $u$, under an integral constraint on $u$. We give conditions for which the value of the cost function at steady state with a constant control $\bar{u}$ can be improved by considering periodic control $u$ with average value equal to $\bar{u}$. This leads to the so-called "over-yielding" met in several applications. With the use of the Pontryagin Maximum Principle, we provide the optimal synthesis of periodic strategies under the integral constraint. The results are illustrated on a single population model in order to study the effect of periodic inputs on the utility of the stock of resource.

Key-Words. Optimal Control, Pontryagin Maximum Principle, Periodic solutions, Over-yielding.

## 1. Introduction.

In many applications, the control of dynamical models allows to drive the state of a system to a relevant operation point, typically a steady state optimizing the performances of the system. However, it may happen that a periodic operation of the system gives a better performance averaged on the period than a steady state (with a periodic control instead of a constant one). This question has already been investigated in the literature. In particular, the so-called $\pi$-criterion characterizes the existence of "best" periods. It consists first in determining an optimal steady state among constant controls, and then in checking on a linear-quadratic approximation if there exists a frequency of a periodic signal near the nominal constant one that could improve the cost (see [8, 7]). For instance, in [2, 3, 21], this method has been applied on the chemostat model, and it has been shown that its productivity can be improved with a periodic control when there is a delay in the dynamics. However, there are relatively few theoretical works about global optimality of periodic controls (apart [23] for the characterization of the value function under quite strong assumptions). Most of the existing works deal with local necessary conditions ([10, 17]), second order conditions ([9, 32, 18]) or approximations techniques ([15, 1, 6]).

In the present work, we consider the problem of determining optimal periodic trajectories under an integral constraint on the control. Our objective is some what different than what has been described above. In [6] for instance, a local analysis is conducted in the context of age-structured system showing how to improve locally the cost function by considering a periodic control versus a constant one (but no integral constraint on the control is considered). First of all, we do not necessarily consider nominal steady states optimizing a criterion. Secondly, we consider admissible trajectories under constraint, as follows. For a given steady state $\bar{x}$ and its associated constant control $\bar{u}$, we consider the set of periodic trajectories with periodic controls having $\bar{u}$ as average (which represents the integral constraint). This question has been motivated by applications for continuous transformation processes for which during each period of time of length $T$, an amount of input matter has to be transformed (which is represented by the integral constraint on the control). The question is to study if the quality of the product, in terms of the averaged concentration $x$ during each time period, could be improved by the way the input matter is delivered during each time period (that is the choice of the control $u$ satisfying the integral constraint). In this context, we say that a *over-yielding* occurs when the average of a $T$-periodic solution $x$ is better than $\bar{x}$. To our knowledge, this problem has not been yet addressed theoretically in the literature. From a mathematical view point, the integral constraint on the input brings two main difficulties:

1. the existence of non-constant periodic trajectories with a control satisfying the integral constraint,
2. the characterization of an optimal control under both constraints of periodicity of the trajectory and the integral constraint on the input,

that we propose to tackle here for scalar dynamics in general framework.

The paper is organized as follows. In Section 2, we formulate the problem and give a precise definition of over-yielding. We then provide assumptions on the dynamics and the cost function that guarantee or prevent over-yielding. In particular, we show that convexity is playing an important role. In section 3, we synthesize optimal periodic controls that improve the cost function compared to steady-state (see Theorem 3.6). In Section 4, we show how to relax the assumptions of Section 2 that are required on an invariant domain $I$ of the dynamics, when these ones are fulfilled only in a neighborhood of $\bar{x}$. This leads us to give a result similar to the one of Section 3 but for restrictive values of the period $T$. Finally, we illustrate the results of Section 3-4 in Section 5 in the context of sustainable resource management (see *e.g.* [13]). Our aim is to consider non-constant periodic inputs (harvesting efforts) but with same average than a constant one over each time period. We study the impact of such inputs on the stock and we determine the input profiles that provide the largest deviations of the average utility of the stock compared to the nominal constant one.

## 2. **Existence of over-yielding.**

Given two functions $f, g : \mathbb{R} \to \mathbb{R}$ of class $C^1$, we consider the control system

$$\dot{x} = f(x) + ug(x), \tag{1}$$

where $u$ is a control variable taking values in $U := [-1, 1]$. We suppose that the system satisfies the following hypotheses:

(H1) There exists $(a, b) \in \mathbb{R}^2$ with $a < b$ such that $g$ is positive on the interval $I := (a, b)$ with

$$f(a) - g(a) = 0 \quad \text{and} \quad f(b) + g(b) = 0.$$

(H2) One has $f - g < 0$ and $f + g > 0$ on $I$.

**Remark 1.** Hypothesis (H1) implies that the interval $I$ is invariant by (1) whereas Hypothesis (H2) is related to controllability properties of (1) (that will be used in the next section for the synthesis of non-constant periodic trajectories). In the rest of the paper, we shall consider initial conditions in $I$ only.

We define for $x \in I$ the function

$$\psi(x) := -\frac{f(x)}{g(x)}.$$

Notice that Hypotheses (H1)-(H2) imply that one has $\psi(I) \subset U$. Therefore, for any $\bar{x} \in I$, we can consider the control value $\bar{u} := \psi(\bar{x})$ in $U$. Note that $\bar{x}$ is an equilibrium of (1) for the constant control $u = \bar{u}$. In the sequel, we shall consider $T$-periodic solutions of (1), where $T \in \mathbb{R}_+^*$, with a $T$-periodic control $u$ that satisfies the integral constraint

$$\frac{1}{T}\int_0^T u(t)\mathrm{d}x = \bar{u}. \tag{2}$$

We then define the set $\mathcal{U}_T$ of admissible controls as

$$\mathcal{U}_T := \{u : [0, +\infty) \to U \text{ s.t. } u \text{ is meas., } \ T\text{-periodic and fullfills (2)}\}. \tag{3}$$

One has the following property.

**Lemma 2.1.** *Under Hypothesis* (H1)*, any $T$-periodic solution $x$ of* (1) *in $I$ with $u \in \mathcal{U}_T$ fulfills the property*

$$\int_0^T (\psi(x(t)) - \psi(\bar{x}))\, \mathrm{d}t = 0. \tag{4}$$

*Proof.* On the interval $I$, the function $g$ is positive and from equation (1), we get

$$\int_0^T \frac{\dot{x}(t)}{g(x(t))}\mathrm{d}t = -\int_0^T \psi(x(t))\mathrm{d}t + \int_0^T u(t)\mathrm{d}t.$$

Define the function

$$h(x) := \int_{\bar{x}}^x \frac{\mathrm{d}\xi}{g(\xi)}, \quad x \in I,$$

together with the function $t \mapsto y(t) := h(x(t))$ for $t \in [0, T]$. For any control function $u$ that fulfills the constraint (2), one then has

$$y(T) - y(0) = -\int_0^T (\psi(x(t)) - \bar{u})\, \mathrm{d}t,$$

where $\bar{u} = \psi(\bar{x})$. For any $T$-periodic solution $x$ in $I$, $y$ is also $T$-periodic and one obtains the property (4). □

We now assume that the equilibrium $\bar{x}$ is asymptotically stable for the dynamics (1) in $I$ with the constant control $\bar{u}$, requiring the following hypothesis.

($\bar{\mathrm{H}}$) The function $\psi$ satisfies the property.

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in I \setminus \{\bar{x}\}.$$

For convenience, we denote by $t \mapsto x(t, u, x_0)$ the solution of (1) with $u \in \mathcal{U}_T$ and taking the value $x_0 \in I$ at time 0. In the following, we shall consider $T$-periodic solutions with the initial condition $x(0) = \bar{x}$ (*i.e.* that are such that $x(T, u, \bar{x}) = \bar{x}$ for $u \in \mathcal{U}_T$). We first show that Hypothesis ($\bar{\mathrm{H}}$) guarantees the existence of non-constant such solutions.

**Lemma 2.2.** *Under Hypotheses* (H1)-($\bar{\mathrm{H}}$), *there exist non-constant $T$-periodic solutions of* (1) *with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$, for any $T > 0$.*

*Proof.* Consider the constant control $u = \bar{u}$ and its associated dynamics in $I$

$$\dot{x} = \bar{f}(x) := g(x)(\bar{u} - \psi(x)) = g(x)(\psi(\bar{x}) - \psi(x)).$$

As the function $g$ is positive on $I$, Hypothesis ($\bar{\mathrm{H}}$) implies that one has $\bar{f} < 0$ on $(\bar{x}, b)$, and $\bar{f} > 0$ on $(a, \bar{x})$. Therefore, one has the properties

$$\begin{array}{lcl} x_0 \in (\bar{x}, \bar{b}) & \Rightarrow & x(T, \bar{u}, x_0) < x_0, \\ x_0 \in (a, \bar{x}) & \Rightarrow & x(T, \bar{u}, x_0) > x_0. \end{array} \tag{5}$$

Consider now any bounded $T$-periodic measurable function $v : [0, +\infty) \to U$ satisfying

$$\int_0^T v(t)\mathrm{d}t = 0,$$

and the control function

$$u_\varepsilon(t) := \bar{u} + \varepsilon v(t),$$

where $\varepsilon \in \mathbb{R}$. Clearly, $u_\varepsilon$ satisfies the constraint (2) and for $\varepsilon$ small enough, one has $u_\varepsilon(t) \in U$ for any $t \geq 0$. Define then the function

$$\theta(x_0, \varepsilon) := x(T, u_\varepsilon, x_0) - x_0,$$

for $(x_0, \varepsilon) \in I \times \mathbb{R}$. By the Theorem of continuous dependency of the solutions of ordinary differential equations w.r.t. initial conditions and parameters (see for instance [28]), $\theta$ is a continuous function. From (5), we deduce that

$$\begin{array}{lcl} x_0 \in (\bar{x}, b) & \Rightarrow & \theta(x_0, 0) < 0, \\ x_0 \in (a, \bar{x}) & \Rightarrow & \theta(x_0, 0) > 0, \end{array}$$

and by continuity of $\theta$, there exists $\varepsilon \neq 0$, $x_0^+ \in (\bar{x}, b)$ and $x_0^- \in (a, \bar{x})$ such that $\theta(x_0^+, \varepsilon) < 0$ and $\theta(x_0^-, \varepsilon) > 0$. By the Mean Value Theorem, we deduce the existence of $x_0 \in (x_0^-, x_0^+)$ such that $\theta(x_0, \varepsilon) = 0$, that is, the existence of a $T$-periodic solution $x$ of (1) with a non-constant control $u$ that satisfies the constraint (2). From Lemma 2.1, such solution satisfies

$$\int_0^T (\psi(x(t)) - \psi(\bar{x}))\, \mathrm{d}t = 0,$$

which implies that the map $t \mapsto \psi(x(t)) - \psi(\bar{x})$ cannot be of constant sign on $[0, T]$. Hypothesis ($\bar{\mathrm{H}}$) implies that $x(t) - \bar{x}$ has to change its sign. Therefore there exists $\bar{t} \in (0, T)$ with $x(\bar{t}) = \bar{x}$ in such a way that the control function $\tilde{u}$ defined by $t \mapsto \tilde{u}(t) := u(t + \bar{t})$ guarantees to have $x(T, \tilde{u}, \bar{x}) = \bar{x}$. $\square$

Now, let $\ell : \mathbb{R} \to \mathbb{R}$ be a function of class $C^1$ and consider the cost function

$$J_T(u) := \frac{1}{T} \int_0^T \ell(x_u(t))\mathrm{d}t, \tag{6}$$

where $x_u$ is the unique solution of (1) such that $x_u(0) = \bar{x}$, associated to a control $u \in \mathcal{U}_T$. Our aim in this work is to address the question of finding a periodic trajectory with $x(0) = \bar{x}$ that has a better cost than the constant $\bar{x}$, with a ($T$-periodic) control of mean value $\bar{u}$. For this purpose, we introduce the following terminology.

**Definition 2.3.** Given $T > 0$, we say that (1) exhibits an *over-yielding* for the cost (6) if there exists a $T$-periodic solution $x$ of (1) with $x(0) = \bar{x}$ associated to a control $u \in \mathcal{U}_T$ such that $J_T(u) < \ell(\bar{x})$.

Moreover, we aim to characterize in the next section the strategies realizing the minimum of the criterion (6) among such controls. The possibility of having an over-yielding relies on specific assumptions on the cost function and the dynamics, that we now introduce.

(H3) The function $\ell : I \to \mathbb{R}$ is increasing and the function $\gamma := \psi \circ \ell^{-1}$ is strictly convex increasing over $\ell(I)$.

**Remark 2.** Hypothesis (H3) implies Hypothesis ($\bar{\mathrm{H}}$). Therefore, by Lemma 2.2, there exist $T$-periodic solutions $x$ of (1) with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$, that are different of the constant solution $\bar{x}$, when (H1)-(H2)-(H3) are fulfilled. Hypothesis (H3) also implies that $\psi$ is increasing.

**Proposition 2.1.** *If* (H1)-(H3) *hold true, any non-constant $T$-periodic solution $x$ of* (1) *with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$ satisfies $J_T(u) < \ell(\bar{x})$.*

*Proof.* Consider a $T$-periodic solution $x$ with $x(0) = \bar{x}$ associated to a control in $\mathcal{U}_T$. From Lemma 2.1, equality (4) is satisfied and we deduce

$$\int_0^T (\gamma(\ell(x(t))) - \gamma(\ell(\bar{x})))\,\mathrm{d}t = 0.$$

For a non-constant solution, we find by Jensen's inequality

$$\gamma\left(\frac{1}{T}\int_0^T \ell(x(t))\mathrm{d}t\right) < \frac{1}{T}\int_0^T \gamma(\ell(x(t))) = \gamma(\ell(\bar{x})).$$

Since $\gamma$ is increasing over $\ell(I)$, we obtain

$$J_T(u) = \frac{1}{T}\int_0^T \ell(x(t))\mathrm{d}t < \ell(\bar{x}).$$

□

**Remark 3.** (i) The result of Proposition 2.1 applies in the simple case where $\ell(x) = x$ and $\psi$ is strictly convex and increasing over $I$.
(ii) If $\psi$ is strictly convex and increasing over $I$ and $\ell$ is strictly concave increasing over $I$, the result of Proposition 2.1 also holds true (by a similar reasoning).

We now provide sufficient conditions for preventing any over-yielding.

(H4) There exists a continuous function $\bar{\psi}$ such that
 (i) $\bar{\psi} \geq \psi$ on $I$ with $\bar{\psi}(\bar{x}) = \psi(\bar{x})$,
 (ii) the function $\bar{\gamma} := \bar{\psi} \circ \ell^{-1}$ is concave increasing on $\ell(I)$.

**Proposition 2.2.** *If* (H1)-(H4) *hold true then no over-yielding is possible.*

*Proof.* We suppose by contradiction that there exists a periodic solution $x$ associated to a control $u \in \mathcal{U}_T$ such that

$$J_T(u) = \frac{1}{T}\int_0^T \ell(x(t))\mathrm{d}t < \ell(\bar{x}),$$

The function $\bar{\gamma}$ being increasing on $\ell(I)$, we have

$$\bar{\gamma}\Big(\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t\Big) < \bar{\gamma}(\ell(\bar{x})) = \bar{\psi}(\bar{x}) = \psi(\bar{x}). \tag{7}$$

Using Jensen's inequality for $\bar{\gamma}$, we can write

$$\bar{\gamma}\Big(\frac{1}{T}\int_0^T \ell(x(t))\,\mathrm{d}t\Big) \geq \frac{1}{T}\int_0^T \bar{\gamma}(\ell(x(t)))\,\mathrm{d}t. \tag{8}$$

As one has $\bar{\psi} = \bar{\gamma}\circ\ell \geq \psi$ over $I$, we get

$$\frac{1}{T}\int_0^T \bar{\gamma}(\ell(x(t)))\,\mathrm{d}t \geq \frac{1}{T}\int_0^T \psi(x(t))\,\mathrm{d}t. \tag{9}$$

Combining inequalities (7), (8), (9), we obtain

$$\psi(\bar{x}) > \frac{1}{T}\int_0^T \psi(x(t))\,\mathrm{d}t,$$

which is a contradiction with the equality (4) given by Lemma 2.1. □

**Remark 4.** Thanks to the previous proposition, if $\ell(x) = x$ for $x \in \mathbb{R}$ and $\psi$ is strictly concave, then no over-yielding is possible. In the same way, if $\ell$ is increasing on $I$ and $\gamma$ strictly concave increasing over $\ell(I)$, then the same conclusion follows.

3. **Determination of optimal periodic solutions.** In this Section, we assume that Hypotheses (H1)-(H2)-(H3) hold true, so that we know that over-yielding is possible, according to Proposition 2.1. For a given $T > 0$, we shall say that a solution $x$ of (1) is $T$-*admissible* if it is $T$-periodic with $x(0) = \bar{x}$ and $u \in \mathcal{U}_T$. We reformulate the control constraint (2) by considering the augmented dynamics

$$\left\{\begin{array}{ll} \dot{x} & = f(x) + ug(x), \\ \dot{y} & = u, \end{array}\right. \tag{10}$$

together with the boundary conditions:

$$(x(0), y(0)) = (\bar{x}, 0) \quad \text{and} \quad (x(T), y(T)) = (\bar{x}, \bar{u}T). \tag{11}$$

The optimal control problem can be then stated as follows

$$\inf_{u\in\mathcal{U}} \int_0^T \ell(x(t))\mathrm{d}t \quad \text{s.t. } (x, y) \text{ satisfies } (10) - (11), \tag{12}$$

where $\mathcal{U}$ denotes the set of measurable control functions $u$ over $[0, T]$ taking values in $U$.

3.1. **Application of the Pontryagin Maximum Principle.** We derive necessary optimality conditions using the Pontryagin Maximum Principle [29]. Let $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian associated to (12) :

$$H = H(x, y, \lambda_x, \lambda_y, \lambda_0, u) = \lambda_0\ell(x) + \lambda_x f(x) + u(\lambda_x g(x) + \lambda_y),$$

where $\lambda := (\lambda_x, \lambda_y)$ denotes the adjoint vector. Let $u \in \mathcal{U}$ be an optimal control and $(x, y)$ a solution of (10)-(11) associated to $u$. Then, there exists a scalar $\lambda_0 \leq 0$ and an absolutely continuous map $\lambda : [0, T] \to \mathbb{R}^2$ satisfying the adjoint equation

$$\left\{\begin{array}{l} \dot{\lambda}_x = -\lambda_0\ell'(x(t)) - \lambda_x(f'(x(t)) + u(t)g'(x(t))), \\ \dot{\lambda}_y = 0, \end{array}\right. \tag{13}$$

for a.e. $t \in [0,T]$. Moreover, $(\lambda_0, \lambda) \neq 0$ and the Hamiltonian condition writes

$$u(t) \in \arg\max_{\omega \in U} H(x(t), \lambda(t), \lambda_0, \omega) \quad \text{a.e. } t \in [0,T]. \tag{14}$$

The switching function $t \mapsto \phi(t) := \lambda_x(t) g(x(t)) + \lambda_y(t)$ provides the following properties satisfied by the control $u$:

$$\begin{cases} \phi(t) > 0 & \Rightarrow \quad u(t) = 1, \\ \phi(t) < 0 & \Rightarrow \quad u(t) = -1, \\ \phi(t) = 0 & \Rightarrow \quad u(t) \in [-1,1]. \end{cases}$$

Moreover, if we differentiate $\phi$ w.r.t $t$, we find that for $t \in [0,T]$

$$\dot{\phi}(t) = \lambda_x(t)[f(x(t))g'(x(t)) - f'(x(t))g(x(t))] - \lambda_0 \ell'(x(t))g(x(t)).$$

An extremal trajectory is a quadruple $(x, \lambda, \lambda_0, u)$ where $(x, \lambda)$ satisfies the state-adjoint equations and $u$ the Hamiltonian condition (14). We recall that a singular arc occurs if $\phi$ vanishes on some time interval $[t_1, t_2]$ with $t_1 < t_2$, and a switching time $t_s \in (0,T)$ is such that an extremal control $u$ is non-constant in any neighborhood of $t_s$ (which implies that $\phi(t_s) = 0$).

**Lemma 3.1.** *Under Hypotheses* (H1)-(H2)-(H3)*, there is no abnormal extremal trajectory, i.e.* $\lambda_0 \neq 0$.

*Proof.* If $\lambda_0 = 0$, then $\lambda_x$ cannot vanish from the adjoint equation. Otherwise $\lambda_x$ would be zero over $[0,T]$ and the switching function would be constant equal to $\lambda_y$. Since $\lambda_y$ cannot be simultaneously equal to 0, $\phi$ would be of constant sign over $[0,T]$ implying that $u = 1$ or $u = 0$ over $[0,T]$ and a contradiction with the periodicity of $x(\cdot)$ (recall that $f + g > 0$ and $f - g < 0$ over $I$). As a consequence, $\lambda_x$ is of constant sign. Now, since $\lambda_0 = 0$, one has

$$\dot{\phi}(t) = \lambda_x(t) g(x(t))^2 \psi'(x(t)), \qquad t \in [0,T].$$

We deduce that $\dot{\phi}$ is of constant sign (recall that $\psi' > 0$), hence $\phi$ is monotone. Consequently, the extremal trajectory has at most one switching point. Thus, one has $x(t) > \bar{x}$ for any time $t \in (0,T)$ implying a contradiction with (4). If $x(t) < \bar{x}$ for any time $t \in (0,T)$, we conclude in the same way. $\square$

Without any loss of generality, we may assume that $\lambda_0 = -1$.

**Remark 5.** Considering $T$-periodic optimal solutions in $I$ without requiring the initial condition $x(0) = \bar{x}$, but only $x(T) = x(0)$ provides the transversality condition $\lambda_x(T) = \lambda_x(0)$. However, Lemma 2.1 and Hypothesis (H3) (or simply ($\bar{\text{H}}$)) imply that any $T$-periodic optimal solution $x(\cdot)$ in $I$ has to pass by $\bar{x}$. Therefore, we can impose $x(0) = \bar{x}$ without any loss of generality, and deduce that $\lambda_x(\cdot)$ is necessarily $T$-periodic (even though we shall not use this property in the following).

3.2. **Properties of switching times.** Let us denote by $x_m$ and $x_M$ the minimum and maximum on $[0,T]$ of a $T$-admissible solution $x$. Note that for any time $t \in (0,T)$ such that $x(t) \in \{x_m, x_M\}$, then one has $\phi(t) = 0$ (otherwise $x(\cdot)$ would be monotone in a neighborhood of $t$ implying a contradiction).

**Proposition 3.1.** *Under Hypotheses* (H1)-(H2)-(H3)*, any extremal satisfies the following properties.*

1. *At any switching time* $t_s \in (0,T)$*, one has* $x(t_s) \in \{x_m, x_M\}$.
2. *It has no singular arc.*

*Proof.* Let $t_1$, $t_2$ in $[0,T]$ be such that $x(t_1) = x_m$ and $x(t_2) = x_M$ with $x_m$, $x_M$ in $I$. We deduce that $\lambda_x(t_1)g(x_m) = \lambda_x(t_2)g(x_M) = -\lambda_y$. Now, since $H$ is conserved along any extremal trajectory (see for instance [12]), one has

$$H = -\ell(x_M) - \lambda_y \frac{f(x_M)}{g(x_M)} = -\ell(x_m) - \lambda_y \frac{f(x_m)}{g(x_m)},$$

implying that (recall that $\gamma = \psi \circ \ell^{-1}$)

$$\frac{1}{\lambda_y} = \frac{\psi(x_M) - \psi(x_m)}{\ell(x_M) - \ell(x_m)} = \frac{\gamma(\ell(x_M)) - \gamma(\ell(x_m))}{\ell(x_M) - \ell(x_m)}. \tag{15}$$

Suppose now that $t_s$ is a switching time such that $x(t_s) \in (x_m, x_M)$. Using a similar computation as above, we find that

$$\frac{1}{\lambda_y} = \frac{\psi(x_M) - \psi(x(t_s))}{\ell(x_M) - \ell(x(t_s))} = \frac{\gamma(\ell(x_M)) - \gamma(\ell(x(t_s)))}{\ell(x_M) - \ell(x(t_s))}. \tag{16}$$

Since $\gamma$ and $\ell$ are respectively strictly convex and increasing on $[x_m, x_M]$, (15) and (16) imply a contradiction, thus $x(t_s) \in \{x_m, x_M\}$ as was to be proved.

Suppose now by a contradiction that there exists a time interval $[t_1, t_2]$ such that $\phi(t) = \dot{\phi}(t) = 0$ for $t \in [t_1, t_2]$. It is then easy to see that the trajectory is necessarily constant over $[t_1, t_2]$ (indeed, combining $\phi = \dot{\phi} = 0$ over $[t_1, t_2]$ one finds that $\lambda_y \psi'(x(t)) - \ell'(x(t)) = 0$ for $t \in [t_1, t_2]$ implying that $x$ must be constant over $[t_1, t_2]$). Thus there exists $x_s \in [x_m, x_M]$ such that $x(t) = x_s$ for any time $t \in [t_1, t_2]$. Now, since the extremities of the singular arc $t_1$ and $t_2$ must be switching times, one must have $x_s \in \{x_m, x_M\}$. Suppose for instance that $x_s = x_M$. From the expression of $\phi$ and $\phi'$, we deduce that

$$\frac{1}{\lambda_y} = \frac{\psi'(x_M)}{\ell'(x_M)} = \gamma'(\ell(x_M)),$$

which is a contradiction with (15) (since $\gamma$ is strictly convex). We have a similar contradiction if $x_s = x_m$, which ends the proof. □

At this stage, we have thus proved that optimal trajectories are of bang-bang type (*i.e.* they are concatenations of arcs with $u = \pm 1$) such that at each switching time $t_s$ one has $x(t_s) \in \{x_m, x_M\}$. By a similar reasoning as in the proof of Proposition 3.1, one can show that the number of switching times is finite. Moreover, the number of switching times is necessarily even (otherwise a switch will have to occur at $x(T) = \bar{x}$ in contradiction with point 1 of Proposition 3.1). We focus now on extremal trajectories with two switches.

### 3.3. **Trajectories with two switches.**

For a given $T > 0$, we consider trajectories $t \mapsto x(t)$ solutions of (1) on $[0, T]$ with $x(0) = \bar{x}$ and associated to a control $u$ defined by two switching times $t_1$, $t_2$ with $0 \leq t_1 < t_2 \leq T$:

$$u(t) = \left| \begin{array}{rl} 1, & t \in [0, t_1), \\ -1, & t \in [t_1, t_2), \\ 1, & t \in [t_2, T). \end{array} \right. \tag{17}$$

These trajectories, that we shall call $B_+B_-B_+$ *trajectories*, will play an important role in the following. Note that under Hypotheses (H1)-(H2) a $B_+B_-B_+$ trajectory is characterized uniquely by its maximal and minimal values $x_M = x(t_1)$ and $x_m = x(t_2)$ in $I$. For convenience, we define on the interval $I$ the function

$$\eta(x) := \frac{1}{f(x)+g(x)} - \frac{1}{f(x)-g(x)}.$$

From Hypothesis (H2), note that $\eta$ is a positive function on $I$.

**Lemma 3.2.** *Under Hypotheses* (H1)-(H2)*, if a $B_+B_-B_+$ trajectory is $T$-periodic, then the pair $(x_m, x_M)$ satisfies*

$$\int_{x_m}^{x_M} \eta(x)\mathrm{d}x = T. \tag{18}$$

*Moreover, if the corresponding control satisfies* (2) *then the pair $(x_m, x_M)$ satisfies*

$$\int_{x_m}^{x_M} \eta(x)\psi(x)\mathrm{d}x = \bar{u}T. \tag{19}$$

*Proof.* For $t \in [0, t_1) \cup [t_2, T)$, one has $\dot{x} = f(x) + g(x) > 0$ and one can write

$$t_1 = \int_{\bar{x}}^{x_M} \frac{\mathrm{d}x}{f(x)+g(x)}, \quad T - t_2 = \int_{x_m}^{x(T)} \frac{\mathrm{d}x}{f(x)+g(x)}.$$

Similarly for $t \in [t_1, t_2)$, one has $\dot{x} = f(x) - g(x) < 0$ and

$$t_2 - t_1 = -\int_{x_m}^{x_M} \frac{\mathrm{d}x}{f(x)-g(x)}.$$

One then obtains

$$T = \int_{x_m}^{x(T)} \frac{\mathrm{d}x}{f(x)+g(x)} - \int_{x_m}^{x_M} \frac{\mathrm{d}x}{f(x)-g(x)} + \int_{\bar{x}}^{x_M} \frac{\mathrm{d}x}{f(x)+g(x)},$$

and for a $T$-periodic solution, $x(T) = \bar{x}$ gives exactly the property (18). Proceeding with the same decomposition of the interval $[0, T]$, one can write

$$\int_0^T u(t)\mathrm{d}t = \int_{\bar{x}}^{x_M} \frac{\mathrm{d}x}{f(x)+g(x)} - \int_{x_M}^{x_m} \frac{\mathrm{d}x}{f(x)-g(x)} + \int_{x_m}^{\bar{x}} \frac{\mathrm{d}x}{f(x)+g(x)},$$

which gives the quality

$$\int_{x_m}^{x_M} \left( \frac{1}{f(x)+g(x)} + \frac{1}{f(x)-g(x)} \right) \mathrm{d}x = \bar{u}T,$$

when $u$ fulfills (2). Finally, notice that one has

$$\frac{1}{f(x)+g(x)} + \frac{1}{f(x)-g(x)} = \eta(x)\psi(x),$$

for $x \in I$, and thus property (19) is satisfied. □

We first analyze the possibilities of satisfying the integral condition (18).

**Lemma 3.3.** *Under Hypotheses* (H1)-(H2)*, for any $T > 0$ there exists an unique function $\beta_T : [a, b] \mapsto [a, b]$ that satisfies $\beta_T(\alpha) > \alpha$ for any $\alpha \in I$ and*

$$\int_{\alpha}^{\beta_T(\alpha)} \eta(x)\mathrm{d}x = T, \quad \alpha \in I.$$

*Moreover $\beta_T$ is of class $C^1$, increasing and bijective from $[a, b]$ to $[a, b]$.*

*Proof.* The function $f+g$ is of class $C^1$ and positive on $I=(a,b)$ with $(f+g)(b)=0$. It follows that one has the inequality $(f+g)(x)\leq K_+(b-x)$ for any $x\in I$, where $K_+:=-\min_{x\in\bar{I}}(f+g)'(x)>0$. As the function $\eta$ satisfies

$$\eta(x)>\frac{1}{f(x)+g(x)}\geq\frac{1}{K_+(b-x)}>0,\quad x\in I,$$

one deduces that the map

$$\chi:(\xi_-,\xi_+)\mapsto\chi(\xi_-,\xi_+):=\int_{\xi_-}^{\xi_+}\eta(x)\mathrm{d}x,$$

is such that for any $\alpha\in I$, $\chi(\alpha,\cdot)$ is increasing with $\chi(\alpha,\alpha)=0$ and $\chi(\alpha,b)=+\infty$. By the Implicit Function Theorem, there exists an unique map $\beta_T:I\mapsto I$ of class $C^1$, such that $\chi(\alpha,\beta_T(\alpha))=T$ for any $\alpha\in I$. Moreover, one has

$$\beta_T'(\alpha)=\frac{\eta(\alpha)}{\eta(\beta_T(\alpha))}>0,\quad\alpha\in I.$$

The function $\beta_T$ is thus increasing, and then admits limits at the points $a_+$ and $b_-$. Therefore one has $\beta_T(a_+):=\lim_{a\to a_+}\beta_T(a)\geq a$ and $\beta_T(b_-):=\lim_{b\to b_-}\beta_T(b)\leq b$ that verify $\chi(a,\beta_T(a_+))=T$ and $\chi(b,\beta_T(b_-))=T$. As previously, $f-g<0$ on $I$ with $(f-g)(a)=0$ implies that one has $(f-g)(x)\geq K_-(x-a)$ for any $x\in I$ with $K_-:=-\min_{x\in\bar{I}}(f-g)'(x)>0$. Thus one has

$$\eta(x)>-\frac{1}{f(x)-g(x)}\geq\frac{1}{K_-(x-a)}>0,\quad x\in I.$$

If $\beta_T(a_+)>a$, one should then have $\chi(a,\beta_T(a_+))=+\infty$ which is not possible since one has $\chi(\alpha,\beta_T(\alpha))=T$ for any $\alpha\in I$. So, one has $\beta_T(a_+)=a$. As the function $\eta$ is positive on $I$, one also has $\beta_T(\alpha)>\alpha$ for any $\alpha\in I$, and we deduce that $\beta_T(b_-)=b$. This proves that $\beta_T$ can be extended to a one-to-one mapping from $[a,b]$ to $[a,b]$. □

We are now ready to show that there exists an unique $B_+B_-B_+$ trajectory that satisfies both integral conditions (18) and (19).

**Proposition 3.2.** *Under Hypotheses* (H1)-(H2)-($\bar{\text{H}}$)*, there exists an unique pair* $(x_m,x_M)\in I^2$ *satisfying* (18)-(19)*, and one has* $x_m<\bar{x}<x_M$*.*

*Proof.* From Lemma 3.3, condition (18) implies to have $x_M=\beta_T(x_m)$. We thus have simply to show the uniqueness of $x_m$ for the condition (19) to be fulfilled. Consider the function $F:[a,b]\to\mathbb{R}$ defined by

$$F(\alpha):=\int_{\alpha}^{\beta_T(\alpha)}\eta(x)(\psi(x)-\psi(\bar{x})),\tag{20}$$

and notice that conditions (18) and (19) are both satisfied exactly when $F(x_m)=0$. From Hypothesis ($\bar{\text{H}}$) and the properties satisfied by the function $\beta_T$ (see Lemma 3.3), one has $F(\alpha)>0$ for any $\alpha\in[\bar{x},b)$, and $F(\alpha)<0$ for any $\alpha\in(a,\beta_T^{-1}(\bar{x})]$. By the Mean Value Theorem, there exists $x_m\in(\beta_T^{-1}(\bar{x}),\bar{x})$ such that $F(x_m)=0$. Moreover, one has

$$F'(\alpha)=\eta(\beta_T(\alpha))\left(\psi(\beta_T(\alpha))-\psi(\bar{x})\right)\beta_T'(\alpha)-\eta(\alpha)\left(\psi(\alpha))-\psi(\bar{x})\right).$$

As $\beta_T$ is increasing and $\psi$ satisfies ($\bar{\text{H}}$), we obtain $F'(\alpha)>0$ for any $\alpha<\bar{x}$ with $\beta_T(\alpha)>\bar{x}$, that is exactly for $\alpha\in(\beta_T^{-1}(\bar{x}),\bar{x})$, and we conclude about the existence and uniqueness of $x_m$, $x_M$ in $I$, with $x_m<\bar{x}$ and $x_M>\bar{x}$. □

It is worth to mention that $x_m$ and $x_M$ depend on the period $T$. In the next Lemma, we provide properties of $x_m$ and $x_M$ as functions of $T$.

**Lemma 3.4.** *Under Hypotheses* (H1)-(H2)-($\bar{\mathrm{H}}$), *the functions* $T \mapsto x_m(T)$ *and* $T \mapsto x_M(T)$ *are continuously differentiable, and respectively decreasing and increasing. Moreover, one has*

$$\lim_{T\to+\infty} x_m(T) = a \quad \text{and} \lim_{T\to+\infty} x_M(T) = b. \tag{21}$$

*Proof.* For each $T > 0$, we know from Proposition 3.2 that there exists an unique pair $(x_m(T), x_M(T)) \in I^2$ satisfying (18)-(19). By the Implicit Function Theorem, $x_m$ and $x_M$ are are continuously differentiable w.r.t. $T$. Let us denote by ${x_m}'$, ${x_M}'$ the derivatives of $x_m$ and $x_M$ w.r.t. $T$. Differentiating (18)-(19) w.r.t. $T$ then yields

$$\underbrace{\begin{bmatrix} \eta(x_M(T)) & -\eta(x_m(T)) \\ \\ \eta(x_M(T))\psi(x_M(T)) & -\eta(x_m(T))\psi(x_m(T)) \end{bmatrix}}_{X(T)} \begin{bmatrix} {x_M}'(T) \\ \\ {x_m}'(T) \end{bmatrix} = \begin{bmatrix} 1 \\ \\ \psi(\bar{x}) \end{bmatrix},$$

where $\det(X(T)) := \eta(x_M(T))\eta(x_m(T))\left(\psi(x_M(T)) - \psi(x_m(T))\right) > 0$. Then ${x_M}'(T)$, ${x_m}'(T)$ are given by the expressions

$$\begin{cases} {x_M}'(T) & = & \dfrac{\eta(x_m(T))\left(\psi(\bar{x}) - \psi(x_m(T))\right)}{\det(X(T))} > 0, \\[2ex] {x_m}'(T) & = & \dfrac{\eta(x_m(T))\left(\psi(\bar{x}) - \psi(x_m(T))\right)}{\det(X(T))} < 0. \end{cases}$$

From (18) and (19), one has

$$\frac{T}{2}(\bar{u}+1) = \int_{x_m(T)}^{x_M(T)} \frac{\mathrm{d}x}{f(x)+g(x)} < \int_a^{x_M(T)} \frac{\mathrm{d}x}{f(x)+g(x)}.$$

Taking the limit when $T$ tends to $+\infty$ in both side of this inequality, one obtains $\lim_{T\to+\infty} x_M(T) = b$. Similarly one can prove that $\lim_{T\to+\infty} x_m(T) = a$. $\square$

3.4. **Optimal solutions.** According to Proposition 3.2, for any $T > 0$ there exists an unique $B_+B_-B_+$ trajectory $\hat{x}_T(\cdot)$ that is $T$-admissible, generated by a control that we shall denote $\hat{u}_T$. Moreover, there exists a unique $\bar{t} \in (0, T)$ such that $\hat{x}_T(\bar{t}) = \bar{x}$. Therefore, there are exactly two $T$-admissible solutions $\hat{x}_T(\cdot)$, $\check{x}_T(\cdot)$ with two switches, given by $\hat{u}_T$ and $\check{u}_T$ with

$$\check{u}_T(t) := \hat{u}_T(t+\bar{t}), \quad t \geq 0,$$

which have the same cost. Similarly, we denote by $B_-B_+B_-$ the trajectory $\check{x}_T$. We now study the monotonicity of the cost $J_T(\hat{u}_T)$ with respect to $T$. This property is crucial for the optimal synthesis (Theorem 3.6) and relies on the convexity assumptions on the data.

**Lemma 3.5.** *Under Hypotheses* (H1)-(H2)-(H3), *one has*

$$S > T > 0 \Rightarrow J_S(\hat{u}_S) < J_T(\hat{u}_T).$$

*Proof.* Following (17), we denote by $t_1$ and $t_2$ the two successive instants of $(0, T)$ for which one has $\hat{u}_T = +1$ over $[0, t_1) \cup [t_2, T)$ and $\hat{u}_T = -1$ over $[t_1, t_2)$. In the same way, we define $s_1$, $s_2$ as the two successive instants of $(0, S)$ such that one has $\hat{u}_S = +1$ over $[0, s_1) \cup [s_2, T)$ and $\hat{u}_S = -1$ over $[s_1, s_2)$. Let us also denote by $x$, $y$

the solutions of (1) corresponding to $\hat{u}_T$ and $\hat{u}_S$ respectively and set $x_M := x(t_1)$, $x_m := x(t_2)$, $y_M := y(s_1)$, $y_m := y(s_2)$.

From Lemma 3.4, one has $x_M < y_M$, $x_m > y_m$, $t_1 < s_1$, and $t_2 < s_2$. So, we introduce a $E$ defined by

$$E := \{s \in [0, S]\ ;\ y(s) > x_M \text{ or } y(s) < x_m\},$$

together with a function $\varphi : [0, T] \to [0, S]\backslash E$ by

$$\varphi(t) := \begin{cases} t & \text{if} \quad t \in [0, t_1), \\ t + \delta_1 & \text{if} \quad t \in [t_1, t_2), \\ t + \delta_2 & \text{if} \quad t \in [t_2, T], \end{cases}$$

where $\delta_1$, resp. $\delta_2$ is the time spent by $y$ over $x$, resp. below $x$. By construction one has $x(t) = y(\varphi(t))$ for $t \in [0, T]$ and $\varphi$ is bijective, thus $\text{meas}(E) = S - T$. Moreover, for any monotonic function $\rho : I \to \mathbb{R}$ one has

$$\int_0^T \rho(x(t))\mathrm{d}t = \int_0^T \rho(y(\varphi(t)))\mathrm{d}t = \int_{[0,S]\backslash E} \rho(y(s))\mathrm{d}s,$$

by considering the change of variable $s = \varphi(t)$. We then get

$$\int_0^T \ell(x(t))\mathrm{d}t = \int_{[0,S]\backslash E} \ell(y(s))\mathrm{d}s,$$

and

$$\int_0^T \gamma(\ell(x(t)))\mathrm{d}t = \int_{[0,S]\backslash E} \gamma(\ell(y(s)))\mathrm{d}s. \tag{22}$$

As both controls $\hat{u}_T$ and $\hat{u}_S$ satisfy the constraint (4), one has

$$\frac{1}{T}\int_0^T \gamma(\ell(x(t)))\mathrm{d}t = \frac{1}{S}\int_0^S \gamma(\ell(y(s)))\mathrm{d}s = \bar{u},$$

which implies

$$\frac{1}{S-T}\int_E \gamma(\ell(y(s)))\mathrm{d}s = \bar{u}.$$

Let us now consider a function $\hat{\gamma} : [\ell(y_m), \ell(y_M)] \to \mathbb{R}$ defined by

$$\hat{\gamma}(\xi) := \left| \begin{array}{ll} \ell(x_m) + \dfrac{\gamma(\ell(x_M)) - \gamma(\ell(x_m))}{\ell(x_M) - \ell(x_m)}(\xi - \ell(x_m)) & \text{for } \xi \in [\ell(x_m), \ell(x_M)], \\ \gamma(\xi) & \text{otherwise,} \end{array} \right.$$

(see Fig. 1). First, note that $\hat{\gamma}$ is convex increasing and satisfies

$$\hat{\gamma}(\xi) > \gamma(\xi) \text{ for } \xi \in (\ell(x_m), \ell(x_M)). \tag{23}$$

As one has $\gamma = \hat{\gamma}$ in $[\ell(y_m), \ell(y_M)]\backslash[\ell(x_m), \ell(x_M)]$, we also have

$$\frac{1}{S-T}\int_E \hat{\gamma}(\ell(y(s)))\mathrm{d}s = \bar{u}.$$

By Jensen's inequality, we obtain

$$\frac{1}{S-T}\int_E \ell(y(s))\mathrm{d}s \leq \hat{\gamma}^{-1}(\bar{u}). \tag{24}$$

Now, since $\hat{\gamma}$ is affine over $[\ell(x_m), \ell(x_M)]$, one obtains

$$\hat{\gamma}\left(\frac{1}{T}\int_0^T \ell(x(t))\mathrm{d}t\right) = \frac{1}{T}\int_0^T \hat{\gamma}(\ell(x(t)))\mathrm{d}t > \frac{1}{T}\int_0^T \gamma(\ell(x(t)))\mathrm{d}t = \bar{u},$$

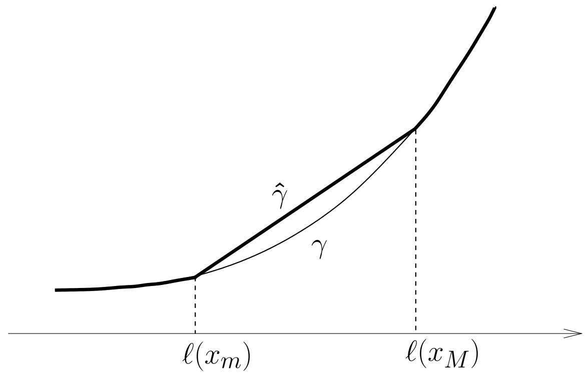



FIGURE 1. Functions $\gamma = \psi \circ \ell^{-1}$ and $\hat{\gamma}$ defined above.

using the fact that $x(t) \in [x_m, x_M]$ for $t \in [0,T]$, (23) and (4). Therefore, one has

$$\frac{1}{T}\int_0^T \ell(x(t))\mathrm{d}t > \hat{\gamma}^{-1}(\bar{u}). \tag{25}$$

We get by (22), (24) and (25)

$$\begin{array}{rcl}\displaystyle\frac{1}{S}\int_0^S \ell(y(s))\mathrm{d}s & = & \displaystyle\frac{1}{S}\int_E \ell(y(s))\mathrm{d}s + \frac{1}{S}\int_{[0,S]\setminus E} \ell(y(s))\mathrm{d}s \\ & \leq & \displaystyle\frac{S-T}{S}\hat{\gamma}^{-1}(\bar{u}) + \frac{1}{S}\int_0^T \ell(x(t))\mathrm{d}t \\ & < & \displaystyle\frac{S-T}{S}\frac{1}{T}\int_0^T \ell(x(t))\mathrm{d}t + \frac{T}{S}\frac{1}{T}\int_0^T \ell(x(t))\mathrm{d}t \\ & = & \displaystyle\frac{1}{T}\int_0^T \ell(x(t))\mathrm{d}t,\end{array}$$

which concludes the proof. □

We now give our main result.

**Theorem 3.6.** *Assume that Hypotheses* (H1)-(H2)-(H3) *are fulfilled. Then, for any $T > 0$, there are two optimal solutions of* (12) *given by the controls $\hat{u}_T$ and $\check{u}_T$.*

*Proof.* Proposition 3.1 shows that an optimal trajectory consists in $2n$ (with $n \geq 1$) switches, that occur exactly at its maximal and minimal values. As (H3) implies ($\bar{\mathrm{H}}$), Proposition 3.2 gives the uniqueness of a $T$-admissible $B_+B_-B_+$ trajectory, which amounts to state that there are exactly two extremals with two switches (corresponding to $n = 1$), given by the controls $\hat{u}_T(\cdot)$ and $\check{u}_T(\cdot)$.

If $n > 1$, an optimal trajectory is $\frac{T}{n}$-periodic (as $n$ switches have to occur for the same value $x_m$ of $x$ and the other $n$ ones at the same value $x_M$) with exactly two switches on $[0, T/n]$. Therefore its cost is equal to $J(\hat{u}_{T/n})$. By Lemma 3.5, one has $J(\hat{u}_{T/n}) > J(\hat{u}_T)$, which proves that only the case $n = 1$ can be optimal. □

An interesting consequence of Lemma 3.5 is the monotonicity property of the cost function evaluated at the optimal solution as a function of $T$.

**Corollary 3.1.** *The optimal criterion $T \mapsto J_T(\hat{u}_T)$ is decreasing w.r.t. $T$.*

4. **Relaxing the assumptions for local over-yielding.** The previous sections have shown the crucial role played by the monotonicity property of the function $\psi$ and the convexity of the function $\gamma$ on the interval $I$ (see Hypotheses ($\bar{\mathrm{H}}$) and (H3)). In the present section, we consider situations for which these conditions are not fulfilled on the whole interval $I$ but only in a neighborhood of $\bar{x}$. Typically, there could exist other values of $\bar{x}$ satisfying $\psi(\bar{x}) = \bar{u}$ (Hypothesis ($\bar{\mathrm{H}}$) is thus not fulfilled on $I$) or $\gamma$ could be only locally convex in a neighborhood of $\bar{x}$ (Hypothesis (H3) is thus not fulfilled on $I$). The idea is then to restrict the values of the period $T$ for characterizing (periodic) optimal solutions remaining in a neighborhood of $\bar{x}$ (and presenting over-yielding).

We first revisit Proposition 3.2 as follows.

**Proposition 4.1.** *Assume that Hypotheses* (H1)-(H2) *are fulfilled with* $\psi'(\bar{x}) > 0$. *Then there exists* $T_{max} > 0$ *such that for any* $T \in (0, T_{max})$*, there exists a unique* $(x_m, x_M) \in (a, b)^2$ *that verify* (18) *and* (19) *with*

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in [x_m, x_M] \setminus \{\bar{x}\}. \tag{26}$$

*Proof.* Consider a sub-interval $J = (\tilde{a}, \tilde{b}) \subset I$ with $\tilde{a} < \bar{x} < \tilde{b}$ such that the property

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in J \setminus \{\bar{x}\}. \tag{27}$$

is fulfilled (as $\psi'$ is strictly positive at $\bar{x}$, we know that such an interval exists). Let us then consider the function $\tilde{f}$ defined on the interval $[a, b]$ by

$$\tilde{f}(x) := \left| \begin{array}{ll} f(x) & \text{if } x \in J, \\ & \\ -g(x)\left(\dfrac{\psi(\tilde{a}) + 1}{\tilde{a} - a}(x - a) - 1\right) & \text{if } x \in [a, \tilde{a}], \\ & \\ -g(x)\left(\dfrac{1 - \psi(\tilde{b})}{b - \tilde{b}}(b - x) + 1\right) & \text{if } x \in [\tilde{b}, b]. \end{array} \right.$$

Clearly, the pair $(\tilde{f}, g)$ satisfies Hypotheses (H1)-(H2)-($\bar{\mathrm{H}}$). The function $\tilde{f}$ is not $C^1$ but Lipschitz continuous, but one can easily check that Lemma 3.3, Proposition 3.2 and Lemma 3.4 are still valid with $f$ merely Lipschitz continuous. This gives the existence and uniqueness of $x_m$ and $x_M$ that verify (18) and (19) for the pair $(\tilde{f}, g)$ and any $T > 0$. As $T \mapsto x_m(T)$ and $T \mapsto x_M(T)$ are respectively decreasing and increasing w.r.t. $T$ (recall Lemma 3.4), there exists $\tilde{T} > 0$ such that $x_m(\tilde{T}) = \tilde{a}$ or $x_M(\tilde{T}) = \tilde{b}$. As $f$ coincide with $\tilde{f}$ on $\mathrm{cl}(J)$, we conclude that $x_m$, $x_M$ are the unique numbers that verify (18) and (19) on $\mathrm{cl}(J)$ for the pair $(f, g)$ and any $T \leq \tilde{T}$. This can be done for any sub-interval $J$ that verifies condition (27). We then consider $T_{max}$ as the supremum of $\tilde{T}$ for all such sub-intervals $J$. □

Given $T < T_{max}$, one may wonder if is enough to require Hypothesis (H3) to be fulfilled on $[x_m, x_M]$ (instead of $I$) to obtain the optimality of the controls $\hat{u}_T$, $\check{u}_T$ as in Theorem 3.6. However, there could exist extremal trajectories taking values outside the interval $[x_m, x_M]$, without requiring additional assumption on the function $\psi$ outside this set. For this purpose, we consider $x_T^- \in \mathbb{R}$, $x_T^+ \in \mathbb{R}$ that are uniquely defined by the times $0 < t^- < t^+ < T$ such that

$$\left\{ \begin{array}{lllll} x_T^- & := & x(t^-, -1, \bar{x}) & = & x(t^- - T, 1, \bar{x}), \\ x_T^+ & := & x(t^+, 1, \bar{x}) & = & x(t^+ - T, -1, \bar{x}). \end{array} \right. \tag{28}$$

One can straightforwardly check that under Hypotheses (H1)-(H2), any $T$-periodic solution $x$ of (1) with $x(0)=\bar{x}$ and control $u$ taking values in $U$ verifies

$$x(t) \in [x_T^-, x_T^+], \quad \forall t \in [0,T]. \tag{29}$$

Clearly, one has $x_T^- < \bar{x} < x_T^+$ and $(x_T^-, x_T^+) \to (\bar{x}, \bar{x})$ when $T \to 0$.

We give now a result requiring the condition (26) to be fulfilled on $[x_T^-, x_T^+]$, which guarantees that any optimal solution is in the interval $[x_m, x_M]$.

**Proposition 4.2.** *Assume that Hypotheses* (H1)-(H2) *are fulfilled with $\psi'(\bar{x}) > 0$. Take $T \in (0, T_{max})$ such that*

$$(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0, \quad \forall x \in [x_T^-, x_T^+] \setminus \{\bar{x}\} \tag{30}$$

*where $x_T^-$, $x_T^+$ are defined in* (28)*. Then there exists unique $x_m$, $x_M$ in $[x_T^-, x_T^+]$ satisfying* (18) *and* (19)*. If $\psi$ is increasing on $[x_m, x_M]$, then any $T$-admissible solution $x(\cdot)$ verifies*

$$\hat{x} := \max_{t \in [0,T]} x(t) \le x_M \quad \textit{and} \quad \check{x} := \min_{t \in [0,T]} x(t) \ge x_m.$$

*Proof.* Fix $T \in (0, T_{max})$ that fulfills condition (30). Note that this is possible since $\psi$ is increasing in a neighborhood of $\bar{x}$ and $(x_T^-, x_T^+) \to (\bar{x}, \bar{x})$ when $T \to 0$.

According to Proposition 4.1, there exists unique $x_m$, $x_M$ that verify (18) and (19). Since there exists a $T$-admissible trajectory taking the values $x_m$ and $x_M$, one has necessarily

$$x_T^- < x_m < \bar{x} < x_M < x_T^+.$$

Consider now any $T$-admissible solution $x$. From the property (29), one has $\hat{x} \le x_T^+$ and $\check{x} \ge x_T^-$. Moreover, from condition (30) and Lemma 2.1, one has $\hat{x} > \bar{x} > \check{x}$. Let $\hat{t} \in ]0, T[$ be such that $x(\hat{t}) = \hat{x}$ and suppose that one has $\hat{x} > x_M$. We can assume, without loss of generality, that $x(t) \ge \bar{x}$ is satisfied for any $t \in [0, \hat{t}]$ (if not, consider $t_0 := \sup\{t < \hat{t} \; ; \; x(t) < \bar{x}\}$ and replace $x(\cdot)$ by $x(\cdot + t_0)$). Let $(A, B) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ be defined by

$$A := \int_{\bar{x}}^{\hat{x}} \frac{\mathrm{d}x}{f(x) + g(x)} \quad \text{and} \quad B := -\int_{\bar{x}}^{\hat{x}} \frac{\mathrm{d}x}{f(x) - g(x)}.$$

It can be observed that $A$ and $B$ are the fastest times for a solution of (1) to reach, respectively, $\hat{x}$ from $\bar{x}$ (with the constant control $u = 1$) and $\bar{x}$ from $\hat{x}$ (with the constant control $u = -1$). Clearly, one has $\hat{t} \ge A$ and $T - \hat{t} > B$.

We construct now a $T$-periodic solution $\tilde{x}$ of (1) such that $\tilde{x}(0) = \bar{x}$ and associated to a control $\tilde{u}$ defined as follows

$$\tilde{u}(t) = \left| \begin{array}{lll} \bar{u} & \text{if} & t \in [0, \hat{t} - A[, \\ 1 & \text{if} & t \in [\hat{t} - A, \hat{t}[ \cup [t^\dagger, T], \\ -1 & \text{if} & t \in [\hat{t}, t^\dagger[, \end{array} \right. \tag{31}$$

where $t^\dagger$ is given by

$$t^\dagger = T - \int_{x^\dagger}^{\bar{x}} \frac{\mathrm{d}x}{f(x) + g(x)},$$

and $x^\dagger$ is a solution of $\kappa(x^\dagger) = T - \hat{t}$, the map $\kappa(\cdot)$ being defined by

$$\kappa(\xi) := \int_{\xi}^{\bar{x}} \frac{\mathrm{d}x}{f(x) + g(x)} - \int_{\xi}^{\hat{x}} \frac{\mathrm{d}x}{f(x) - g(x)}, \quad \xi \in (a, b).$$

By Hypothesis (H2), the function $\kappa$ is decreasing and one has

$$\kappa(x_m) = \int_{x_m}^{\hat{x}} \eta(x)\mathrm{d}x - A > \int_{x_m}^{x_M} \eta(x)\mathrm{d}x - \hat{t} = T - \hat{t},$$

and $\kappa(\bar{x}) = B < T - \hat{t}$. Therefore $x^\dagger$ is uniquely defined with $x^\dagger \in (x_m, \bar{x})$. Moreover, one has

$$t^\dagger = \hat{t} - \int_{x^\dagger}^{\hat{x}} \frac{\mathrm{d}x}{f(x) - g(x)} \in ]\hat{t}, T[.$$

Expression (31) is thus well defined. The solution $\tilde{x}$ is depicted on Fig. 2.

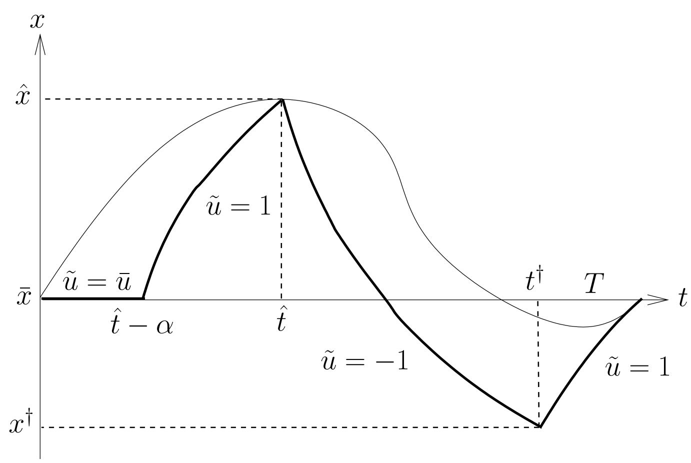



FIGURE 2. The solution $\tilde{x}$ in thick line, $x$ in thin line.

Clearly $\tilde{x}$ reaches $\hat{x}$ at time $\hat{t}$ and it is below the function $x$ on the interval $[0, \hat{t}]$. On the interval $[\hat{t}, t^\dagger]$, $\tilde{x}$ has the fastest descent and therefore stays also below $x$ on this interval. At time $t = t^\dagger$, one has $\tilde{x}(t^\dagger) = x^\dagger$. Finally, the constant control $u = 1$ is the only one that allows to connect $x^\dagger$ at time $t^\dagger$ to $\bar{x}$ at time $T$. So, any periodic solution has to be above $\tilde{x}$ on $[t^\dagger, T]$. We conclude that one has $x(t) \geq \tilde{x}(t)$ for any $t \in [0, T]$. As $\psi(x) > \psi(\bar{x})$ for $x \in [x_M, \hat{x}]$ and $\psi$ is increasing on $[x_m, x_M]$, and as we have shown that $x(t) > x_m$ for any $t \in [0, T]$, one can write

$$\begin{aligned}
\int_0^T (\psi(x(t)) - \psi(\bar{x}))\mathrm{d}t &> \int_{\{t \in [0,T] | x(t) \leq x_M\}} (\psi(x(t)) - \psi(\bar{x}))\mathrm{d}t \\
&\geq \int_{\{t \in [0,T] | x(t) \leq x_M\}} (\psi(\tilde{x}(t)) - \psi(\bar{x}))\mathrm{d}t \\
&= \int_{x^\dagger}^{x_M} (\psi(x) - \psi(\bar{x}))\eta(x)\mathrm{d}x.
\end{aligned}$$

To conclude, since one has $x^\dagger > x_m$ and $\eta > 0$ on $I$, one obtains

$$\int_0^T (\psi(x(t)) - \psi(\bar{x}))\mathrm{d}t > \int_{x_m}^{x_M} (\psi(x) - \psi(\bar{x}))\eta(x)\mathrm{d}x = 0,$$

which is not possible according to Lemma 2.1. We then conclude that the inequality $\hat{x} \leq x_M$ is satisfied. In a similar manner, one can prove the other inequality $\check{x} \geq x_m$. □

For periods $T > 0$ that fulfill conditions of Proposition 4.2, we know that optimal solutions remain in the set $[x_m, x_M]$. We then obtain the same conclusion as Theorem 3.6 when Hypothesis (H3) is fulfilled on the interval $[x_m, x_M]$ only, as stated by the following Theorem.

**Theorem 4.1.** *Assume that Hypotheses* (H1)-(H2) *are fulfilled and consider* $T > 0$ *such that*

*i)* $(\psi(x) - \psi(\bar{x}))(x - \bar{x}) > 0$ *for any* $x \in [x_T^-, x_T^+] \setminus \{\bar{x}\}$*, where* $x_T^-$*,* $x_T^+$ *are defined in* (28)*,*

*ii)* $\ell$ *is increasing on* $[x_m, x_M]$ *and* $\gamma = \psi \circ \ell^{-1}$ *is strictly convex increasing on* $[\ell(x_m), \ell(x_M)]$*, where* $x_m$ *and* $x_M$ *are given by Proposition 4.1.*

*Then, there are two optimal solutions of* (12)*, given by the controls* $\hat{u}_T$ *and* $\check{u}_T$*.*

*Proof.* First, assumption ii) implies that $\psi$ is increasing on the interval $[x_m, x_M]$. Thanks to i), we know from Proposition 4.2 that any extremal is such that $x$ takes values within the interval $[x_m, x_M]$. With assumption ii) instead of Hypothesis (H3), the reader can easily check that the arguments of Theorem 3.6 apply in the same manner on $[x_m, x_M]$ (instead of the whole interval $I$), to prove that only the extremals $\hat{x}$ and $\check{x}$ are optimal. □

5. **Periodic versus constant strategies in a single population model.** We consider an exploited stock of a renewable resource (fish, forest..) represented by its density $x(t)$ which follows a dynamics

$$\dot{x} = f_0(x) - E(t)x, \tag{32}$$

where the growth function $f_0 : \mathbb{R}_+ \to \mathbb{R}$ is of class $C^1$ and satisfies $f_0(0) = 0$. The harvesting effort $E$, which is considered as a measurable control, takes values within an interval $[0, E_{max}]$ (with $E_{max} > 0$). Such models have been extensively studied in the bio-economics literature (see for instance [13] and the references cited herein). Typically an optimal steady state $\bar{x}$ associated to a constant control $\bar{E}$ is determined as maximizing a bio-economic profit of the harvesting over a discounted infinite horizon. However, it is not always possible or desirable to apply the theoretical value $\bar{E}$ of the harvesting effort in a constant manner (because of labor laws, seasonality...), but its average value is usually guaranteed on a period $T$. In this context, our objective is to study the impacts on the stock of applying a periodic harvesting effort instead of a constant one. There are several ways of measuring the impacts on a stock, in terms of a function $\ell(x)$ which measures the *well-being* of the stock or its *utility* (such as recreative activities). In the simplest case, $\ell(x)$ is just equal to the stock density $x$ but more generally one can consider that $\ell : \mathbb{R}_+ \to \mathbb{R}$ is a $C^1$ concave increasing function.

Given a constant control $\bar{E} \in (0, E_{max})$, we then consider an associated steady-state $\bar{x}$ of (32) such that

$$h(\bar{x}) = \bar{E}, \tag{33}$$

where $h : \mathbb{R}_+ \to \mathbb{R}$ is the function defined as

$$h(x) := \left| \begin{array}{lll} \dfrac{f_0(x)}{x} & \text{if} & x > 0, \\ f_0'(x) & \text{if} & x = 0. \end{array} \right.$$

Note that equation (33) may have several solutions. We consider one of them which leads to a stable equilibrium (one can easily check that this amounts to have

$\bar{E} > f_0'(\bar{x})$). Our aim is to study if the average criterion

$$J_T(E) := \frac{1}{T}\int_0^T \ell(x(t))\mathrm{d}t, \tag{34}$$

can be improved by considering $T$-periodic inputs $E(\cdot)$ satisfying

$$\frac{1}{T}\int_0^T E(t)\mathrm{d}t = \bar{E}, \tag{35}$$

and $T$-periodic solutions of (32) associated to $E(\cdot)$ with

$$x(0) = x(T) = \bar{x}. \tag{36}$$

In order to use the previous setting, we consider the following change of variables:

$$u := 1 - \frac{2E}{E_{max}} \ ; \ f(x) := f_0(x) - \frac{E_{max}}{2}x \ ; \ g(x) := \frac{E_{max}}{2}x,$$

and the function $\psi$ becomes

$$\psi(x) = -\frac{f(x)}{g(x)} = 1 - \frac{2}{E_{max}}h(x).$$

So, (32) has exactly the form (1) with $u \in [-1, 1]$. Let $\bar{u} \in (-1, 1)$ be the constant control associated to $\bar{E}$ such that $\bar{u} = \psi(\bar{x})$. We now study the effects of $T$-periodic inputs for two growth functions $f_0$: the classical logistic function, and the modified one with a depensation term (that will highlight Section 4).

5.1. **The logistic growth.** We recall the classical expression of this model

$$f_0(x) := rx\left(1 - \frac{x}{K}\right),$$

where $r > 0$ and $K > 0$. One can easily check that there exists a positive equilibrium $\bar{x}$ of (32) satisfying (33) as soon as $\bar{E} < r$. Moreover, $\bar{x}$ is a stable equilibrium (see [13]). We assume hereafter that one has $\bar{E} < r$. Since one has

$$(f-g)(x) = x\left(r - E_{max} - \frac{r}{K}x\right) \quad ; \quad (f+g)(x) = rx\left(1 - \frac{x}{K}\right),$$

Hypotheses (H1)-(H2) are satisfied for the interval $I := (\lambda(E_{max}), K)$ where

$$\lambda(E_{max}) = \left| \begin{array}{cl} 0 & \text{if} \quad E_{max} > r, \\ h^{-1}(E_{max}) & \text{if} \quad E_{max} < r. \end{array} \right.$$

Note that $\psi$ is an affine function: $\psi(x) = c_1 x + c_0$ with $c_0 = 1 - \frac{2r}{E_{max}}$, $c_1 = \frac{2r}{KE_{max}}$. When $\ell$ is strictly concave, the function $\gamma = \psi \circ \ell^{-1}$ is strictly convex (and increasing). Hypothesis (H3) is thus satisfied. According to Proposition 2.1, there is a systematic over-yielding whatever is $T > 0$ *i.e.* the average criterion $J_T$ is always below $\ell(\bar{x})$. Its lowest value is given by the two strategies $B_+B_-B_+$ or $B_-B_+B_-$ (see Theorem 3.6). Note that when $\ell(x) = x$, the function $\gamma$ is affine and consequently the criterion $J_T$ is always equal to $\bar{x}$ *i.e.* the average of the stock is always equal to $\bar{x}$.

We now illustrate the over-yielding with the function

$$\ell(x) := \frac{4x}{1+x},$$

which is concave increasing. Numerical simulations have been conducted with the parameters values $r = 3$, $K = 7$, $\bar{x} = 3.5$, $E_{max} = 2.5$ and $\bar{E} = 1.5$. Results are depicted on Fig. 3.

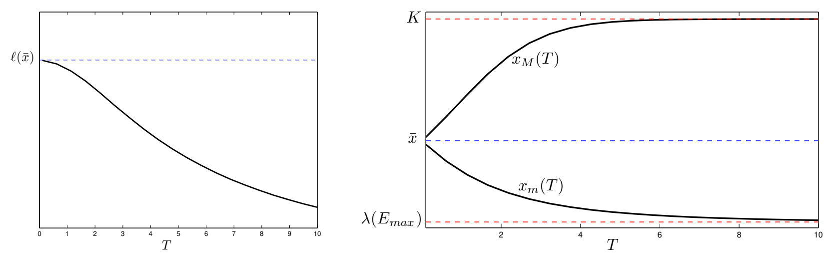



FIGURE 3. Optimal criterion $J_T(\hat{u}_T)$ (left) and $x_m$, $x_M$ (right) as functions of the period $T$ for the logistic growth.

5.2. **The logistic with depensation.** Some populations are known to present a *depensation* in the first part of their growth function [13], which is also called a *weak Allee effect*. This is represented by the following modification of the logistic function

$$f_0(x) := rx^\alpha \left(1 - \frac{x}{K}\right),$$

with $\alpha > 2$. For this function, one has

$$h(x) = rx^{\alpha-1} \left(1 - \frac{x}{K}\right),$$

which is increasing on $[0, x^\star)$ and decreasing on $(x^\star, K]$ with

$$x^\star := \frac{\alpha - 1}{\alpha} K,$$

(see Fig. 4). In presence of depensation in the model, one can also easily check that the function $\psi$ is concave decreasing on $[0, x_c)$, convex decreasing on $(x_c, x^\star)$, and convex increasing on $(x^\star, K]$ with

$$x_c := \frac{\alpha - 2}{\alpha} K < x^\star,$$

(see Fig. 4).

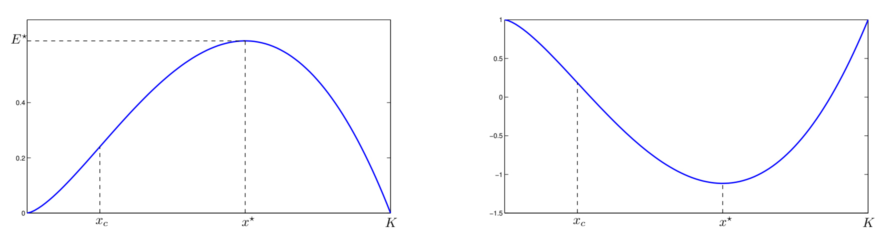



FIGURE 4. Graphs of the functions $h$ (left) and $\psi$ (right) for $r = 0.3$, $K = 5$ and $\alpha = 2.5$.

We shall consider here the function $\ell(x) = x$ (*i.e.* the criterion is simply the level of the stock $x$). Let us define

$$E^\star := h(x^\star).$$

We distinguish now two cases depending if $E_{max}$ is below or above $E^\star$.

5.2.1. *Case 1: $E_{max} < E^\star$.* Note first that there are two solutions $\lambda_1(E_{max})$ and $\lambda_2(E_{max})$ on the interval $(0, K)$ of the equation $h(x) = E_{max}$ such that $\lambda_1(E_{max}) < x^\star < \lambda_2(E_{max})$. One can then check that Hypotheses (H1)-(H2)-(H3) are fulfilled on the interval $I := (\lambda_2(E_{max}), K)$. For any $\bar{E} \in (0, E_{max})$, one can also show, as in the logistic model, that there exists a unique solution $\bar{x} \in I$ of (33) which is moreover a stable steady-state of (32) (see [13]). Proposition 2.1 guarantees then an over-yielding whatever is $T > 0$.

Fig. 5 depicts the optimal cost value $J_T(\hat{u}_T)$ for the following parameter values: $r = 0.3$, $K = 5$, $a = 2.5$, $\bar{x} = 4$, $\bar{E} = 0.48$, and $E_{max} = 0.5893$.

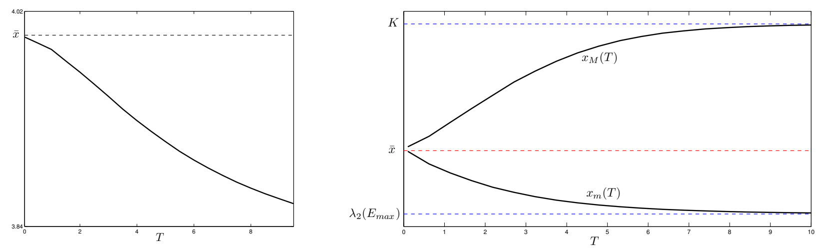



FIGURE 5. Optimal criterion $J_T(\hat{u}_T)$ (left) and $x_m$, $x_M$ (right) as functions of the period $T$ for the depensation model (case 1).

Finally, in presence of depensation in the model with a maximal harvesting effort $E_{max} < E^\star$, our analysis shows that periodic solutions cause a systematic decrease of the mean value of the stock (compared to constant harvesting).

5.2.2. *Case 2: $E_{max} > E^\star$.* One can easily check that Hypotheses (H1)-(H2) are fulfilled on the interval $I := (0, K)$, but not Hypothesis (H3). Since $\bar{x}$ is a stable steady-state of the dynamics, the point $\bar{x}$ belongs to the interval $(x^\star, K)$ (see [13]). Note also that $\psi$ is increasing in a neighborhood of $\bar{x}$. Proposition 4.1 guarantees then the existence of the $T$-periodic trajectory $B_+B_-B_+$ (or $B_-B_+B_-$) that satisfies the integral constraint, for $T$ not too large. Moreover for $T$ small enough, the function $\psi$ is strictly convex on $[x_m(T), x_M(T)]$, and we can conclude about the optimality of these trajectories according to Theorem 4.1.

Using the same parameter values except $E_{max} = 0.8235$, the function $F$ defined in (20) is depicted on Fig. 6 (left) for different values of $T$. We recall (see the proof of Proposition 3.2) that the existence of $x_m$, $x_M$ is equivalent to the existence of a zero of $F$. One can see on this figure that for $T > 6$, the equation $F(\alpha) = 0$ has no roots, and therefore $x_M$ and $x_M$ no longer exist for theses values of the parameter $T$. This means that the $B_+B_-B_+$ (or $B_-B_+B_-$) strategy cannot satisfy the periodicity and the input constraint for $T > 6$. On the contrary, for $T < 6$, the $B_+B_-B_+$ and $B_-B_+B_-$ strategies are admissible, and $x_m$, $x_M$ are plotted as function of $T$ on Fig. 6 (right). Note that equation $h(x) = \bar{E}$ has two solutions $\underline{x} < \bar{x}$ (such that $\psi(\underline{x}) = \psi(\bar{x})$). Finally, on Fig. 7, we present the cost of the $B_+B_-B_+$ (or $B_-B_+B_-$) strategy as a function of $T$ (for $T < 6$).

6. **Application to the chemostat model.** Let us recall that the chemostat apparatus, invented simultaneously by [24] and [26] in the fifteens, is extensively used as an experimental device for studying the growth of micro-organisms. The mathematical model of the chemostat is often considered in the literature as a mathematical

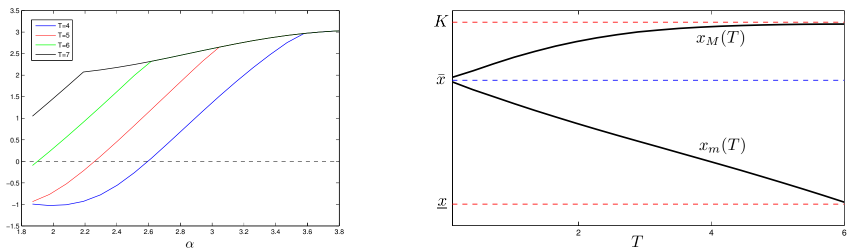



FIGURE 6. Plot of the function $F$ defined by (20) (left), and $x_m$, $x_M$ (right) as functions of the period $T$ for the depensation model (case 2).

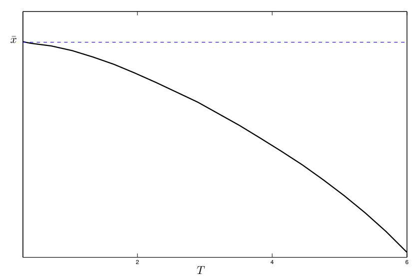



FIGURE 7. Optimal criterion $J_T(\hat{u}_T)$ for the depensation model (case 2)

representation of the micro-organisms growth [31, 20], and not exclusively for the precise experimental chemostat device. It can be found in several real life situations, in natural environments, such as lakes, lagoons... [33], or in industrial applications such as waster-water treatment plants [16]. Originally, the word "chemostat" refers to a steady state operation and therefore many mathematical analyses address the long time behavior of the chemostat model under a constant input flow rate of substrate (which has also to be equal to the output flow for maintaining a constant volume of the water reservoir or tank).

Periodic flow rates in the chemostat model have been investigated in ecological contexts, typically for taking into account seasonality. In particular, it has been shown that periodic solutions could allow the coexistence of different species, while this is not possible in constant environment (see for instance [11, 27]). However, the impact of periodic inputs on the performances of the ecological functions of the ecosystem (resource conversion, biomass growth...) does not seem to have been studied in the ecological literature, even for the single species case. In industrial frameworks, its is known that periodic operations can impact positively or negatively the productivity of continuous cultures [19, 30]. The optimization of the bio-processes productivity has already been investigated with periodic controls, as mentioned in the introduction (see [2, 3, 21]). Our purpose here is different as we impose an integral constraint on the input flow rate, fixing the amount of nutrient that has to be delivered per period. Our objective is to compare temporal profiles that deliver the same amount of nutrient in terms of average biomass or nutrient concentrations in the chemostat model. Our study is two fold:

1. From an ecological view point, we investigate if a non constant flow rate can impact positively or negatively the average density of the consumers,
2. From a bio-process view point, we determine the best periodic flow rate maximizing the average water quality.

We consider then the classical chemostat model:

$$\left\{\begin{array}{lcl} \dot{b} & = & \mu(s,b)b - Db \\ \dot{s} & = & -\dfrac{1}{Y}\mu(s,b)b + D(s_{in}-s) \end{array}\right. \tag{37}$$

where $b$ and $s$ denote the concentrations, respectively of consumers (biomass) and resource (substrate). The parameters $s_{in} > 0$ and $Y > 0$ are the input concentration of nutrient and the biomass yield factor, respectively. The dilution rate $D$ is the input variable, taking values within an interval $[D_m, D_M]$ with $0 \leq D_m < D_M$. The function $\mu$ is the specific growth function, which is $C^1$, non-negative and verifies $\mu(0,b) = 0$ for any $b$. According to the literature, we distinguish two classes of growth functions:

i) $\mu$ does not depend on $b$. We assume that $\mu$ is either increasing on $[0,+\infty)$ or increasing on $[0,\hat{s})$ and decreasing on $(\hat{s},+\infty)$, with $\hat{s} > 0$. Typical instances of monotonic functions are given by the Hill expression [25]

$$\mu(s) := \mu_{\max}\frac{s^n}{K_s^n + s^n}$$

where $\mu_{max} > 0$, $K_s > 0$ and $n > 1$ are parameters. The well-known Monod function [24] corresponds to the particular value $k = 1$ in this expression:

$$\mu(s) := \mu_{\max}\frac{s}{K_s + s}$$

A usual instance of non-monotonic growth functions is the Haldane expression [4]:

$$\mu(s) := \bar{\mu}\frac{s}{K_s + s + s^2/K_i}$$

where $\bar{\mu}$, $K_s$ and $K_i$ are positive parameters. For this function, one has $\hat{s} = \sqrt{K_s K_i}$.

ii) $\mu$ is *density-dependent* i.e. it depends explicitly on density $b$ of consumers. In that case, $\mu$ is decreasing with respect to $b$, representing the crowding effect of consumers in competition for the common resource $s$. A prototype of such functions is the Contois kinetics [14]:

$$\mu(s,b) = \mu_{max}\frac{s}{s + Kb}$$

(where $\mu_{max}$ and $K$ are positive parameters, which is increasing w.r.t. $s$.

Notice that without any loss of generality one can choose $Y = 1$ (re-scaling the biomass concentration $b$). Considering then the total density of matter $m := b + s$, one has

$$\dot{m} = D(t)(s_{in} - m)$$

Then, for any $\bar{D} \in (D_m, D_M)$ and $T$-periodic $D(\cdot)$ (where $T$ is a positive number) such that

$$\frac{1}{T}\int_0^T D(t)dt = \bar{D} > 0 \tag{38}$$

a $T$-periodic solution of (37) has to fulfill $m(t) = s_{in}$ for any $t$. Therefore, we shall consider the dynamics on the $m = b + s = s_{in}$ invariant domain:

$$\dot{s} = (s_{in} - s)(-\nu(s) + D), \tag{39}$$

with

$$\nu(s) = \mu(s, s_{in} - s)$$

We define the *break-even concentration* as

$$\lambda(D) := \inf\{s < s_{in} \text{ s.t. } \nu(s) < D\}$$

We fix a reference value $\bar{D} \in (D_m, D_M)$ such that $\bar{s} := \lambda(\bar{D}) < s_{in}$ and $\bar{s} < \hat{s}$ for the non-monotonic case (we implicitly assume that the interval $(D_m, D_M)$ allows this). $s = \bar{s}$ is clearly an equilibrium of (39) for the constant control $D = \bar{D}$. Moreover, for all the instances of the function $\mu$ we have described previously (Monod, Haldane, Hill and Contois), the function $\nu$ is increasing about $s = \bar{s}$, which implies that $s = \bar{s}$ is a locally asymptotically stable equilibrium.

We consider now the criterion

$$J_T(D(\cdot)) = \frac{1}{T}\int_0^T s(t)dt$$

for periodic solutions of (39) with $s(0) = \bar{s}$ and inputs $D(\cdot)$ that fulfill the integral constraint (38) with $D(t) \in [D_m, D_M]$ for any $t \in [0, T]$. We first study if it is possible to have $J_T(D(\cdot)) < \bar{s}$. Notice that is exactly equivalent to have

$$\frac{1}{T}\int_0^T b(t)dt > \bar{b} = s_{in} - \bar{s}$$

(as one has $m(t) = s(t) + b(t) = s_{in}$ for any $t$). From an ecological view point, this amounts to study if a periodic environment allows to maintain a higher population of consumers $b$ in average. In industrial waste-water treatments, the water quality is usually defined by the substrate $s$ (considered as a pollutant) after separation of the biomass from the liquid, that has to be as small as possible. The question of interest is to determine the operating conditions that give the lowest concentration $s$ averaged on a given period $T$. Here, we consider that one can play with the temporal distribution of the input flow rate of a water loaded with a high concentration of substrate $s_{in}$.

Let us begin by showing that Assumptions of the previous sections are fulfilled. We stick with the notation $s$ (instaed of $x$) for the state varible as it is often used with the chemostat model. We posit

$$\begin{aligned} f(s) &:= \left(-\nu(s) + \frac{D_M + D_m}{2}\right)(s_{in} - s) \\ g(s) &:= \frac{D_M - D_m}{2}(s_{in} - s) \end{aligned}$$

and

$$u := \frac{2}{D_M - D_m}D - \frac{D_M + D_m}{D_M - D_m}$$

so that dynamics (39) has exactly the form (1) with $u \in U$. We take $a = \lambda(D_m)$ and $b = \lambda(D_M)$ which gives $f(a) - g(a) = f(b) + g(b) = 0$. As $b \leq s_{in}$, $g$ is clearly positive on $(a, b)$. Thus Hypothesis (H1) is fulfilled. Remark that for all the growth functions listed above, the function $\nu$ is increasing on $(a, b)$, which implies that Hypothesis (H2) is also satisfied. The function $\psi : I \to \mathbb{R}$ is given by the expression

$$\psi(s) = -\frac{f(s)}{g(s)} = \frac{2}{D_M - D_m}\nu(s) - \frac{D_M + D_m}{D_M - D_m}.$$

Let $\bar{u}$ be the control associated to the value $\bar{D}$, which also satisfied $\bar{u} = \psi(\bar{x})$. Finally, our criterion amounts to choose the function $l$ to be the identity function.

One can straightforwardly check the following Lemma concerning the Hill functions.

**Lemma 6.1.** *The Hill functions are increasing on $\mathbb{R}_+$, strictly convex on $[0, s_c]$ and concave on $[s_c, +\infty]$ where*

$$s_c := K_s \left(\frac{n-1}{n+1}\right)^{\frac{1}{n}}$$

6.1. **Study of the possibility of over-yielding.** We start by giving conditions for which an over-yielding is not possible.

**Proposition 6.1.** *An over-yielding is not possible in the following cases.*

*i) The function $\mu$ is of Monod type.*
*ii) The function $\mu$ is of Haldane type.*
*iii) The function $\mu$ is of Hill type with $\lambda(D_m) \geq s_c$ or $\bar{s} > K_s(n-1)^{1/n}$.*
*iv) The function $\mu$ is of Contois type with $K \leq 1$.*

*Proof.* Remark first that one has $\gamma = \psi$ (as $l$ is identity) and the function $\psi$ has the same monotony and convexity characteristics than the function $\nu$.

For the Monod kinetics, Hypothesis (H4) is fulfilled with $\bar{\psi} = \psi$, as $\mu$ is concave increasing on $\mathbb{R}_+$ (and thus $\psi$ also). By Proposition 2.2 (and Remark 4), we conclude that an over-yielding is not possible.

For the Haldane function, $\mu$ is concave on $(0, \hat{s})$ and we can choose

$$\bar{\psi}(s) = \left| \begin{array}{ll} \psi(s), & s \leq \hat{s} \\ \psi(\hat{s}) & s > \hat{s} \end{array} \right. \tag{40}$$

so that Hypothesis (H4) is fulfilled. As for the Monod kinetics, an over-yielding is not possible.

According to Lemma 6.1, the Hill function $\mu$ is concave and increasing on $I$ when $a \geq s_c$, and as previously we conclude that an over-yielding is not possible. When $a < s_c$, we define the number $s^\dagger$ as the smallest $s \in \mathbb{R}_+$ such that the tangent to the graph of $\mu$ at $s$ is above the graph of $\mu$ on $\mathbb{R}_+$. One can easily check that it exactly corresponds to the abscissa $s^\dagger$ such that the tangent to the graph of $\mu$ at $s^\dagger$ passes by the origin, that is such that

$$\mu(s^\dagger) - \mu'(s^\dagger)s^\dagger = 0$$

which gives the expression $s^\dagger = K_s(n-1)^{1/n}$. We then consider the function (see Figure 8)

$$\bar\psi(s) = \left| \begin{array}{lll} \mu'(s^\dagger)s & \text{for} & s \leq s^\dagger, \\ \psi(s) & \text{for} & s > s^\dagger \end{array} \right.$$

which is concave increasing on $\mathbb{R}_+$ and above the function $\psi$. When $\bar s > s^\dagger$, one has $\bar\psi(\bar s) = \psi(s)$ and Hypothesis (H4) is then verified. As previously, we conclude that no over-yielding is possible.

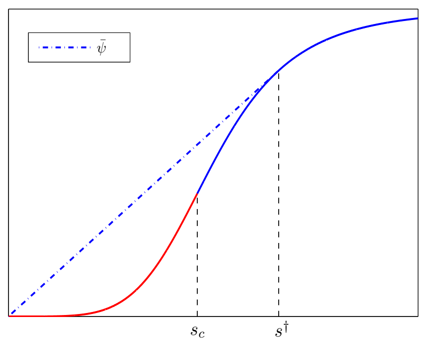



FIGURE 8. Example of graphs of functions $\psi$ and $\bar\psi$ for the Hill case with $\mu_{max} = 2$, $K_s = 2$ and $n = 5$.

For the Contois kinetics, one has

$$\nu(s) = \mu(s, s_{in} - s) = \mu_{max}\frac{s}{s + K(s_{in} - s)}, \quad s \in (0, s_{in})$$

from which one computes the expressions

$$\begin{array}{rcl} \nu'(s) & = & \mu_{max}\dfrac{s_{in}K}{(s + K(s_{in} - s))^2} \\ \nu''(s) & = & \mu_{max}\dfrac{2s_{in}K(K-1)}{(s + K(s_{in} - s))^3} \end{array} \tag{41}$$

The function $\nu$ is thus concave increasing on any interval $I \subset (0, s_{in})$ when $K \leq 1$, preventing then any possibility of over-yielding. $\square$

We focus now on situations for which over-yielding exists.

**Proposition 6.2.** *Over-yielding exists in the following cases.*

*i) For the Hill functions with $\lambda(D_M) \leq s_c$ or the Contois function with $K > 1$, any non constant $T$-periodic solution verifies $J_T(D(\cdot)) < \bar s$, whatever is $T > 0$.*
*ii) For the Hill function $\bar s < s_c$, any non constant $T$-periodic solution verifies $J_T(D(\cdot)) < \bar s$ for $T > 0$ small enough.*

*Proof.* Remind first that that the function $\gamma$ has the same monotony and convexity properties than the function $\nu$.

When the function $\mu$ is of Hill type, we know by Lemma 6.1 that the function $\nu$ (which is identical to $\mu$) is strictly convex and increasing on $[0, s_c)$ and concave on $[K_c, +\infty)$. So, when the interval $I$ is included in $(0, s_c)$, Hypothesis (H3) is fulfilled and by Proposition 2.1, we obtain that any non-constant periodic solution has a better cost than the constant solution $s = \bar{s}$.

For the Contois function, the expressions of the two first derivatives of $\nu$ given in (41) show that the function $\nu$ is strictly convex and increasing on $\mathbb{R}_+$ when $K > 1$ and Proposition 2.1 applies again.

For the Hill function with $\bar{s} < s_c$, the function $\nu$ is strictly convex only locally about $\bar{s}$. However, $\nu$ is always increasing on $\mathbb{R}_+$ and Proposition (4.1) ensures the existence and uniqueness of $s_m(T)$ and $s_M(T)$ for any $T > 0$. As already noticed in Section 4, one has $(s_T^-, s_T^+) \to (\bar{s}, \bar{s})$ when $T \to 0$ and thus one has also $(s_m(T), s_M(T)) \to (\bar{s}, \bar{s})$. Therefore, there exists $\bar{T} > 0$ such that for $T \in (0, \bar{T})$ one has $[s_m(T), s_M(T)] \subset [0, s_c]$ and we conclude by Theorem (4.1) that the $BB$ trajectories exhibit an over-yielding. □

### 6.2. **Numerical illustrations of over-yielding.**

6.2.1. *The Contois case.* From Proposition 6.2, we know that the $BB$ trajectories are always optimal when $K > 1$. Figure 9 depicts these optimal trajectories for the optimal controls $\hat{u}$ and $\check{u}$ with the parameters values $\mu_{max} = 1$, $K = 2.5$, $s_{in} = 3$ and the reference value $\bar{D} \simeq 0.3874$ which corresponds to $\bar{s} \simeq 1.8377$.

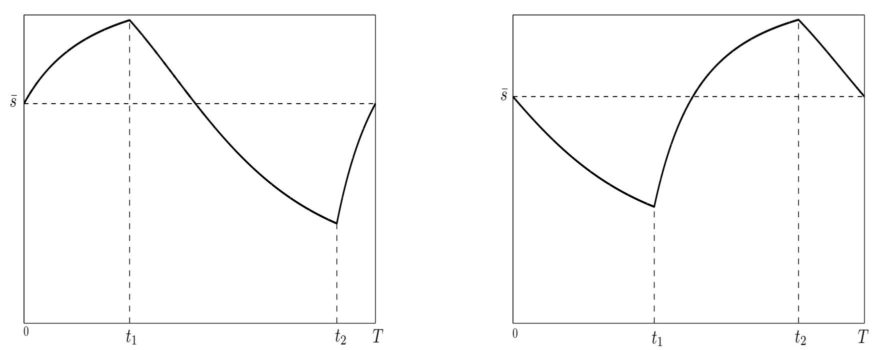



FIGURE 9. Optimal periodic trajectories for $D_m = 0.1$, $D_M = 0.8$ and $T = 10$ in the Contois case ($\hat{u}$ on the left; $\check{u}$ on the right)

Figure 10 gives the evolution of the extreme values $s_m$, $s_M$ of the optimal trajectories and the optimal (average) cost as a function of the period $T$.

6.2.2. *The Hill case.* We have consider a Hill function for the parameters $\mu_{max} = 3$, $K_{s=1.9}$ and $n = 2.5$ with $s_{in} = 3$. For these values, one computes $s_c \simeq 1.3538$. For the reference value $\bar{D} \simeq 0.7221$ that gives $\bar{s} = 1.2$, we are in the conditions of point ii) of Proposition 6.2: the control strategies $\hat{u}$, $\check{u}$ are optimal when the period $T$ is not too large. Figure 11 gives $BB$ trajectories, while Figure 12 presents extremes values and cost as functions of $T$.

One can clearly see on Figure 12 (b) that $BB$ trajectories do not provide over-yielding for large $T$ (because of too large excursions in the interval $I$). However there exists a value $\hat{T}$ of the period that gives the lowest average cost $\hat{J}$ among

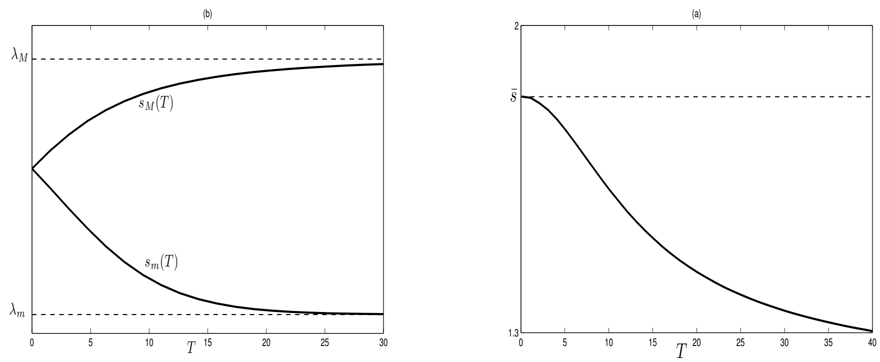



FIGURE 10. (a) Plot of $s_m$ and $s_M$ as function of $T$. (b) Plot of the optimal cost $J_T(\hat{u}) = J_T(\check{u})$ as function of $T$ (in the Contois case for $D_{min} = 0.1$ and $D_{max} = 0.8$)

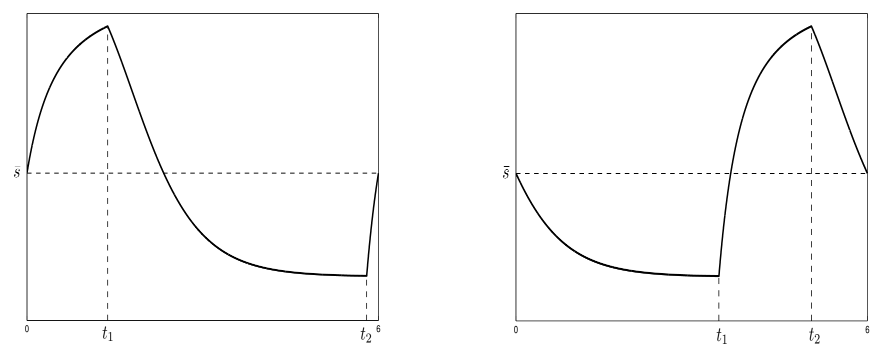



FIGURE 11. $BB$ periodic trajectories for $D_m = 0.2$, $D_M = 2.4$ and $T = 10$ in the Hill case ($\hat{u}$ on the left; $\check{u}$ on the right)

$BB$-trajectories. We conjecture that the optimal cost for periods larger than $\hat{T}$ cannot be lower than $\hat{J}$.

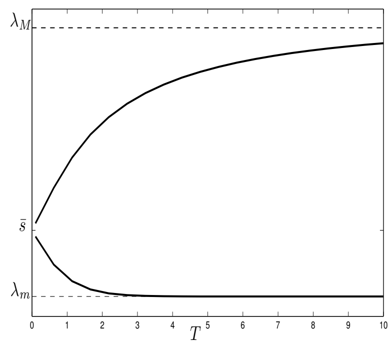
6.2.3. *Operating diagrams.* For a given reference value $\bar{D}$, one can play with two operating parameters:

- the amplitude $\delta$ of the dilution rate $D$ about $\bar{D}$: $D \in [\bar{D} - \delta, \bar{D} + \delta]$ (which amounts to choose $D_m = \bar{D} - \delta$ and $D_M = \bar{D} + \delta$).
- the period $T$.

We consider the *operating diagram* in the $(\delta, T)$ plane that gives the iso-values of the relative gain $G_T$ provided by the $BB$ trajectories compared to the constant control $\bar{D}$, where $G_T$ is defined as

$$G_T = \frac{\bar{s} - J_T(\hat{u})}{\bar{s}}$$

This diagram can serve as a decision support tool for the practitioners to choose which characteristics of periodic operating conditions worth to be applied. Figure 13



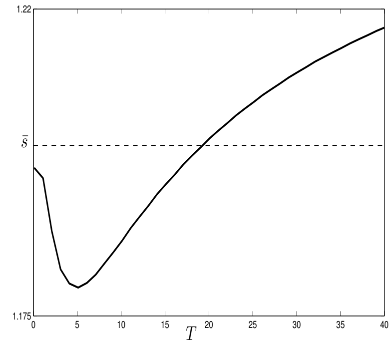
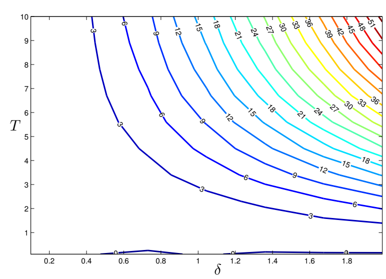




FIGURE 12. (a) Plot $s_m$ and $s_M$ as function of $T$. (b) Plot of the optimal cost $J_T(\hat{u}) = J_T(\check{u})$ as function of $T$ (in the Hill case for $D_{min} = 0.2$ and $D_{max} = 2.4$)

illustrates this diagram for the Contois and Hill functions for the same parameters than for the previous Figures.



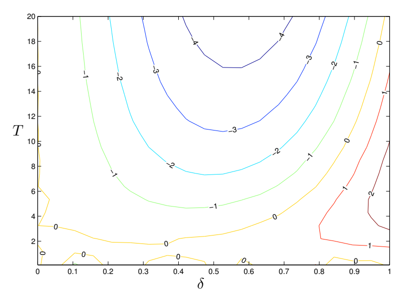




FIGURE 13. Operating diagrams in percentage of the optimal gain $G_T$ with respect to $(\delta, T)$ (Contois on the left, Hill on the right)

6.3. **A procedure to discriminate a Contois kinetics.** It often happens in microbiology that experimenters have to choose between several expressions for the growth function, to be identified on the experimental data. As already noticed in [22], periodic operations in the chemostat is a way to discriminate between models, playing with different frequencies of the dilution rate $D(\cdot)$. A typical situation is the discrimination between a Monod expression, which is often used as a first try, and the density dependent Contois expression which amounts to consider that the affinity coefficient $K_s$ in the Monod expression depends on the biomass concentration: $K_s = Kb$. When the magnitude of $K_s$ can be roughly estimated and $K_s/b$ is larger than one, the following procedure allows to discriminate between these two models.

**Procedure.**

1. Consider the chemostat at a (quasi) steady state for a a nominal value $D_0$ of the dilution rate (with $D_m < D_0 < D_M$) at time 0. Let $s_0$ be the corresponding value of $s$ at steady state.

   *Remark. This ensures the chemostat's state to belong to the invariant set $s + b = s_{in}$.*

2. Choose an arbitrary time $t_1 > 0$ and apply the control
$$D(t) = \left| \begin{array}{ll} D_M & t \in [0, t_1] \\ D_m & t > t_1 \end{array} \right.$$
   until the time $\bar{t} > t_1$ such that $s(\bar{t}) = s_0$ (which exists as $D_m < D_0 < D_M$). Choose another arbitrary time $t_2 > \bar{t}$ and apply the control
$$D(t) = \left| \begin{array}{ll} D_m & t \in [\bar{t}, t_2) \\ D_M & t > t_2 \end{array} \right.$$
   until the time $T > t_2$ such that $s(T) = s_0$ (which exists as $D_m < D_0 < D_M$).

3. Store the measurements history $\{s(t)\}_{t\in[0,T]}$.

4. Let $\bar{D}$ be the mean value of the dilution rate during $[0, T]$ :
$$\bar{D} = \frac{D_M(T + t_1 - t_2) + D_m(t_2 - t_1)}{T}$$
   Apply for $t > T$ the constant value $\bar{D}$ and wait for the (quasi) steady state. Let $\bar{s}$ be the corresponding value of $s$ at steady state.

   *Remark. The trajectory $s(\cdot)$ is a $T$-periodic solution with a control of mean value equal to $\bar{D} = \mu(\bar{s})$. From Lemma 2.1, $\mu(\bar{s})$ is also equal to the mean value $\mu(s(\cdot))$. Therefore $\bar{s}$ belongs to the interval $[s(t_2), s(t_1)]$ and there exists $t_0 > 0$ as the first time such that $s(t_0) = \bar{s}$.*

5. With the data stored on $[0, T]$, determine the average value of $s$ on $[0, T]$:
$$\tilde{s} = \frac{1}{T}\int_0^T s(t)dt$$
   *Remark. The average of the solution from $s(t_0) = \bar{s}$ with the period control on $[t_0, t_0 + T]$ is also equal to $\tilde{s}$.*

6. If $\tilde{s} < \bar{s}$, we invalidate Monod (or Haldane) kinetics. Otherwise, we invalidate Contois (with $K > 1$).

This simple procedure relies on the fact that only the Contois function with $K > 1$ produces an over-yielding, and not the Monod function (see Propositions 6.1, 6.2). Note that this test does not require to test different frequencies.

7. **Conclusion.** In this work, we have shown that under concavity assumptions, the optimal trajectory is the steady-state solution, that is, no over-yielding is possible.

On the contrary, under convexity assumptions, we have proved that there is exactly one optimal trajectory (up to a time translation) which is bang-bang with two switches on a period. This optimality result is global and valid for any period $T$. We have also relaxed the hypotheses to prove the same optimality result globally, but for a limited range of values of the period $T$, when only local convexity is fulfilled.

The determination of the optimal solution for large values of $T$ when neither convexity nor concavity assumptions are fulfilled appears to be much more complex, as the bang-bang solution is no longer admissible.

This analysis was illustrated in the context of a population model subject to a harvesting effort. Depending on the growth model and the criterion, we are able to predict the effect of a periodic harvesting efforts (with the same given mean value) compared to the constant value at steady-state. Such analysis in this context is new to our best knowledge.

Some of the techniques we have proposed here to cope with the integral constraint on the control variable, which is the main characteristic of the problem we have considered, could be deployed for systems in higher dimensions, and will be the matter of a future work.

**Acknowledgments.** The authors thank the AVERROES program between Algeria and France for obtaining the PhD grant of F.-Z. Tani. The first author would like to thank INRA Montpellier and the MISTEA research unit for providing a half year delegation during the academic year 2017-2018. This work was supported by the LabEx NUMEV incorporated into the I-Site MUSE (AAP2017- 2-08) and by the ANR THERMOMIC (ref. ANR-16-CE04-0003). We also would like to thank Pedro Gajardo for helpful discussions.

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