Hochschild-Mitchell (co)homology of skew categories and of Galois coverings

Claude Cibils and Eduardo N. Marcos *

Abstract

Let $\mathcal{C}$ be a category over a commutative ring k, its Hochschild-Mitchell homology and cohomology are denoted respectively $HH_*(\mathcal{C})$ and $HH^*(\mathcal{C})$. Let G be a group acting on $\mathcal{C}$, and $\mathcal{C}[G]$ be the skew category. We provide decompositions of the (co)homology of $\mathcal{C}[G]$ along the conjugacy classes of G. For Hochschild homology of a k-algebra, this corresponds to the decomposition obtained by M. Lorenz.

If the coinvariants and invariants functors are exact, we obtain isomorphisms $(HH_*(C))_G \simeq HH_*^{\{1\}}(\mathcal{C}[G])$ and $(HH^*(\mathcal{C}))^G \simeq HH^*_{\{1\}}(\mathcal{C}[G])$, where $\{1\}$ is the trivial conjugacy class of G.

We first obtain these isomorphisms in case the action of G is free on the objects of $\mathcal{C}$. Then we introduce an auxiliary category $M_G(\mathcal{C})$ with an action of G which is free on its objects, related to the infinite matrix algebra considered by J. Cornick. This category enables us to show that the isomorphisms hold in general, and in particular for the Hochschild (co)homology of a k-algebra with an action of G by automorphisms.

We infer that $(HH^*(\mathcal{C}))^G$ is a canonical direct summand of $HH^*(\mathcal{C}[G])$. This provides a frame for monomorphisms obtained previously, and which have been described in low degrees.

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1 Introduction

Let k be a commutative ring. A k-category $\mathcal{C}$ is a small category enhanced over the category of k-modules. In other words the objects of $\mathcal{C}$ are a set denoted $\mathcal{C}_0$, for any pair of objects $x, y \in \mathcal{C}_0$ the set of morphisms from x to y is denoted ${}_y\mathcal{C}_x$ and has a k-module structure, the composition in $\mathcal{C}$ is k-bilinear, and the image of the canonical inclusion $k \hookrightarrow {}_x\mathcal{C}_x$ is central in ${}_x\mathcal{C}_x$ for all $x \in \mathcal{C}_0$. Note that in particular ${}_x\mathcal{C}_x$ is a k-algebra for any $x \in \mathcal{C}_0$.

Mitchell in [2] called those categories "algebras with several objects". Indeed, to a k-algebra Λ we associate a k-category with a single object x_0 with

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${}_{x_0}\mathcal{C}_{x_0} = \Lambda$. Moreover a k-category $\mathcal{C}$ with a finite set of objets provides a k-algebra $a(\mathcal{C}) = \oplus_{x,y\in\mathcal{C}_0} {}_y\mathcal{C}_x$ with product given by the composition of $\mathcal{C}$ combined with the matrix multiplication. However we point out that if $\mathcal{C}$ and $\mathcal{D}$ are finite object k-categories, a k-functor $F : \mathcal{C} \to \mathcal{D}$ provides a multiplicative k-morphism $a(F) : a(\mathcal{C}) \to a(\mathcal{D})$, but in general $a(F)(1) \neq 1$. In particular the Galois coverings that we consider in this work are functors which are not algebra morphisms.

Hochschild-Mitchell homology and cohomology of a k-category $\mathcal{C}$ are theories introduced by Mitchell [2], see also [8, 14, 21, 24]. When the number of objects of $\mathcal{C}$ is finite, they coincide with Hochschild cohomology and homology of $a(\mathcal{C})$, see for instance [8].

Let G be a group. A G-k-category is a k-category with an action of G on $\mathcal{C}$, that is with a group homomorphism $G \to \mathsf{Aut}_k\mathcal{C}$ where $\mathsf{Aut}_k\mathcal{C}$ is the group of k-functors $\mathcal{C} \to \mathcal{C}$ which are isomorphisms. In particular a G-k-algebra Λ is a k-algebra with an action of G by k-automorphisms of Λ.

Let $\mathcal{C}$ be a G-k-category. The skew category $\mathcal{C}[G]$ has been considered in [7]. It has the same set of objects than $\mathcal{C}$ while the morphism between two objects $x, y \in \mathcal{C}_0$ are direct sums, ${}_y\mathcal{C}[G]_x = \oplus_{s\in G}\ {}_y\mathcal{C}_{sx}$, with composition given by $({}_zg_{ty})({}_yf_{sx}) = g \circ tf$. If the G-k-category $\mathcal{C}$ has a finite number of objects and if G is finite, then $a(\mathcal{C})[G] = a(\mathcal{C}[G])$, where $\Lambda[G]$ denotes the usual skew algebra of a G-k-algebra Λ.

The action of G on $\mathcal{C}_0$ is called free if for $x \in \mathcal{C}_0$, the equality $sx = x$ only holds for $s = 1$. In this case the quotient k-category $\mathcal{C}/G$ exists, see for instance [8], and $\mathcal{C} \to \mathcal{C}/G$ is a Galois covering. This construction has several uses in representation theory, see for instance [4, 28, 13, 1, 20].

A main result obtained in [7] is the following: if the action of G is free on the objects of $\mathcal{C}$, then $\mathcal{C}/G$ and $\mathcal{C}[G]$ are equivalent k-categories. Consequently, if the action is not free on the objects, the skew category is a substitute to the quotient category.

In this work we consider a G-k-category $\mathcal{C}$ and we relate the Hochschild-Mitchell (co)homology of $\mathcal{C}$ and of $\mathcal{C}/G$, or of $\mathcal{C}[G]$.

Hochschild-Mitchell (co)homology theories of k-categories can also be defined with bimodules of coefficients. We underline that the comparison results that we obtain do not rely on a change of the bimodules of coefficients. More precisely the (co)homology of k-categories that we consider in this paper are always with coefficients in themselves. In contrast the Cartan-Leray type spectral sequence obtained in [8] makes use of an adequate change in the bimodule of coefficients, see also [18] and [26].

If Λ is a G-graded algebra its Hochschild homology decomposes

$$HH_*(\Lambda) = \bigoplus_{D\in Cl(G)} HH_*^D(\Lambda)$$

where $Cl(G)$ is the set of conjugacy classes of G, see [10, 23, 30]. This decomposition also holds for the Hochschild-Mitchell homology of a G-graded category

$\mathcal{C}$, for instance for $\mathcal{C}[G]$ where $\mathcal{C}$ is a G-k-category. If the action of G is free on C_0, and if the coinvariants functor $(\ )_G$ is exact, we prove that there is an isomorphism

$$HH_*^{\{1\}}(\mathcal{C}[G]) = (HH_*(\mathcal{C}))_G,$$

where $\{1\}$ is the trivial conjugacy class. If $(\ )_G$ is not exact, then the ad-hoc spectral sequence can be settled.

In order to extend the above result to a non free action, we introduce an auxiliary G-k-category $M_G(\mathcal{C})$. It has a natural free action of G on its objects, and there is a G-k-equivalence of categories $M_G(\mathcal{C}) \to \mathcal{C}$. The category $M_G(\mathcal{C})$ is related to the infinite matrix algebra considered by J. Cornick in [10] for a graded algebra, which in turn is linked with Cohen-Montgomery duality in [6], see also [9].

In [17] E. Herscovich proved that equivalent k-categories have isomorphic Hochschild-Mitchell co(homologies), see also [3], note that if the k-categories are isomorphic the result is obvious. Beyond, in [19] it is proven that Hochschild-Mitchell (co)homology is a derived invariant. In case of equivalent k-categories, the quasi-isomorphism is induced by the given equivalence, hence from the G-k-equivalence $M_G(\mathcal{C}) \to \mathcal{C}$ we infer a kG-isomorphism in homology. This enables us to prove that the above isomorphism also holds for Hochschild-Mitchell homology, without assuming freeness of the action on the objects.

If the coinvariants functor $(\ )_G$ is exact, the invariance of the Hochschild-Mitchell homology for equivalence of k-categories implies that for a Galois covering $\mathcal{C} \to \mathcal{C}/G$ the following holds:

$$HH_*^{\{1\}}(\mathcal{C}/G) = (HH_*(\mathcal{C}))_G\,.$$

Hochschild-Mitchell cohomology is defined via a complex of cochains which in each degree is the direct product of k-modules, in contrast with Hochschild-Mitchell homology where the chains are direct sums. Nevertheless we first show that for a G-k-category, the complex is a direct product along the conjugacy classes of G. Moreover the complex of cochains is a differential graded algebra for the cup product. The subcomplex associated to the trivial conjugacy class is a sub differential graded algebra, the corresponding Hochschild-Mitchell cohomology is denoted $HH^*_{\{1\}}(\mathcal{B})$ and is a subalgebra of $HH^*(\mathcal{B})$.

For a G-k-category, the group G acts on the complex of cochains $\mathcal{C}^*(\mathcal{C})$ by automorphisms of the differential graded algebra, hence $HH^*(\mathcal{C})$ is a G-algebra.

In case the action of G on C_0 is free and the invariants functor $(\ )^G$ is exact, we show that there is an isomorphism of algebras

$$(HH^*(\mathcal{C}))^G \simeq HH^*_{\{1\}}(\mathcal{C}[G]).$$

Using the isomorphism on Hochschild-Mitchell cohomology between equivalent k-categories we infer

$$(HH(\mathcal{C}))^G \simeq HH^*_{\{1\}}(\mathcal{C}/G),$$

hence $(HH^*(\mathcal{C}))^G$ is a canonical direct summand of $HH^*(\mathcal{C}/G)$. This way we recover the monomorphism obtained in [25] which is made explicit in low degrees in [16].

Through the same mentioned auxiliary category, we extend the result in case the action of G on $\mathcal{C}_0$ is not necessarily free. We establish that if the invariants functor is exact, the above isomorphism of algebras holds in general. If the invariants functor is not exact a spectral sequence can be considered instead.

In particular, for Λ a G-k-algebra, if the invariants functor is exact, there is an algebra isomorphism

$$(HH(\Lambda))^G \simeq HH^*_{\{1\}}(\Lambda[G]).$$

In other words $(HH(\Lambda))^G$ is a subalgebra of $HH^*(\Lambda[G])$, and there is a canonical two-sided ideal complementing it, provided by the non trivial conjugacy classes of G. This result is related with explicit computations made for instance in [12, 15, 29, 27] for Hochschild (co)homology of specific skew group algebras, in particular for the symmetric algebra over a finite dimensional vector space V over a field k, with G a finite subgroup of $GL(V)$ which order is invertible in k.

2 Graded and skew categories

Let k be a commutative ring. A k-category is a small category $\mathcal{C}$ enriched over the category of k-modules, its set of objects is denoted $\mathcal{C}_0$. For $x, y \in \mathcal{C}_0$, we usually write ${}_yf_x$ for an element f of the k-module of morphisms ${}_y\mathcal{C}_x$ from x to y.

Let G be a group. A G-k-*category* is a k-category $\mathcal{C}$ with an action of G by k-isomorphisms of $\mathcal{C}$. That is firstly G acts on C_0 and secondly for $s \in G$ and ${}_yf_x \in {}_yC_x$ there is ${}_{sy}(sf)_{sx} \in {}_{sy}\mathcal{C}_{sx}$ such that the map given by ${}_y\mathcal{C}_x \to_{sy} \mathcal{C}_{sx}$, $f \mapsto sf$ is a k-module morphism, and $t(sf) = (ts)f$ for $s, t \in G$, as well as $s({}_y1_x) = {}_{sy}1_{sx}$ for $s \in G$ and $x \in \mathcal{C}_0$.

Observe that if $\mathcal{C}$ is a single object G-k-category, then its k-algebra of endomorphisms has an action of G by automorphisms of the algebra.

Definition 2.1 *[7] Let $\mathcal{C}$ be a G-k-category. The* skew category *$\mathcal{C}[G]$ has the same set of objects than $\mathcal{C}$. Let ${}_y\mathcal{C}[G]^s_x = {}_y\mathcal{C}_{sx}$. The morphisms of $\mathcal{C}[G]$ from x to y are*

$${}_y\mathcal{C}[G]_x = \bigoplus_{s\in G} {}_y\mathcal{C}[G]^s_x. \tag{2.1}$$

The composition is defined through adjusting the first morphism in order to make possible to compose it in $\mathcal{C}$ with the second one, as follows. If

$${}_yf_{sx} \in {}_y\mathcal{C}_{sx} \subseteq \ {}_y\mathcal{C}[G]_x \text{ and } {}_zg_{ty} \in {}_z\mathcal{C}_{ty} \subseteq \ {}_z\mathcal{C}[G]_y, \text{ then}$$

$$({}_zg_{ty})({}_yf_{sx}) = {}_z(g \ \circ \ tf)_{tsx} \in {}_z\mathcal{C}[G]_x,$$

where $\circ$ denotes the composition of $\mathcal{C}$.

Remark 2.2

1. *By definition, the direct summands of (2.1) are in one to one correspondence with elements of G.*

2. *If $\mathcal{C}$ is a single object G-k-category with endomorphism algebra Λ, it is shown in [7] that the single object k-category $\mathcal{C}[G]$ has endomorphism algebra the usual skew group algebra,* i.e. $\Lambda[G] = \Lambda \otimes kG$ *where Λ and kG are subalgebras, and with product determined by the intertwining $kG \otimes \Lambda \to \Lambda \otimes kG$ given by $s \otimes a \mapsto sa \otimes s$.*

3. *Observe that*
$$_z\mathcal{C}[G]^t_y \; _y\mathcal{C}[G]^s_x \subset \; _z\mathcal{C}[G]^{ts}_x. \tag{2.2}$$

Definition 2.3 *Let G be a group. A k-category $\mathcal{B}$ is* G-graded *if for all $x, y \in \mathcal{B}_0$ there is a direct sum decomposition by means of k-modules*

$$_y\mathcal{B}_x = \bigoplus_{s \in G} {_y\mathcal{B}^s_x}$$

such that $_z\mathcal{B}^t_y \; _y\mathcal{B}^s_x \subset \; _z\mathcal{B}^{ts}_x$ for all objects $x, y, z \in \mathcal{B}_0$ and for every $s, t \in G$. A morphism $f \in {_y\mathcal{B}^s_x}$ is called homogeneous of degree s *from x to y, we often write it $_yf^s_x$ instead of f.*

A morphism $_yf_{sx} \in {_y\mathcal{C}_{sx}} = {_y\mathcal{C}[G]^s_x}$ is denoted by $_y[f]^s_x$. Recall that in $\mathcal{C}[G]$

$$_z[g]^t_y \; _y[f]^s_x = {_z[(g) \circ (tf)]^{ts}_x}, \tag{2.3}$$

that is the k-category $\mathcal{C}[G]$ is G-graded, as observed in (2.2).

3 Homology for free actions and for Galois coverings

We recall the definition of Hochschild-Mitchell homology, see for instance [2, 21, 24, 8].

Definition 3.1 *Let $\mathcal{C}$ be a k-category and let $C_\bullet(\mathcal{C})$ be the chain complex given by:*

$$\mathcal{C}_0(\mathcal{C}) = \bigoplus_{x \in \mathcal{C}_0} {_x\mathcal{C}_x},$$

$$C_n(\mathcal{C}) = \bigoplus_{x_0, x_1, \ldots, x_n \in \mathcal{C}_0} {_{x_0}\mathcal{C}_{x_n}} \otimes \cdots \otimes \; {_{x_2}\mathcal{C}_{x_1}} \otimes \; {_{x_1}\mathcal{C}_{x_0}},$$

with boundary map d given by the usual formulas used to compute the Hochschild homology of an algebra, see for instance [22, 32, 5].

The Hochschild-Mitchell homology *of $\mathcal{C}$ is the homology of the complex above, that is*

$$HH_*(\mathcal{C}) = H_*(C_\bullet(\mathcal{C})).$$

Next we decompose the above chain complex for a G-graded k-category, as Lorenz [23] have done for the complex which computes the Hochschild homology of a G-graded k-algebra (see also [31]).

Proposition 3.2 *Let $\mathcal{B}$ be a G-graded k-category. Let D be a conjugacy class of G. Then*

$$C_n^D(\mathcal{B}) = \bigoplus_{\substack{s_n \dots s_1 s_0 \in D \\ x_n, \dots, x_1, x_0 \in \mathcal{B}_0}} {}_{x_0}\mathcal{B}_{x_n}^{s_n} \otimes \cdots \otimes {}_{x_2}\mathcal{B}_{x_1}^{s_1} \otimes {}_{x_1}\mathcal{B}_{x_0}^{s_0}$$

is a subcomplex of $C_\bullet(\mathcal{B})$, and there is a decomposition

$$C_\bullet(\mathcal{B}) = \bigoplus_{D \in Cl_G} C_\bullet^D(\mathcal{B})$$

where Cl_G denotes the set of conjugacy classes of G.

Proof. The following is a verification for $n = 2$ which provides the way for proving the result for any n. It has the advantage of avoiding long and useless technical computations. For subsequent proofs we will often maintain this approach of providing the computations for small values of n.

Let ${}_{x_0}f_2^{s_2}{}_{x_2} \otimes {}_{x_2}f_1^{s_1}{}_{x_1} \otimes {}_{x_1}f_0^{s_0}{}_{x_0} \in C_2^D(\mathcal{B})$, with $s_2 s_1 s_0 \in D$. We have

$$\begin{array}{ccccc} \multicolumn{5}{l}{d(f_2 \otimes f_1 \otimes f_0) =} \\ f_2 f_1 \otimes f_0 & - & f_2 \otimes f_1 f_0 & + & f_0 f_2 \otimes f_1 \in \\ \left({}_{x_0}\mathcal{B}_{x_1}^{s_2 s_1} \otimes {}_{x_1}\mathcal{B}_{x_0}^{s_0}\right) & \oplus & \left({}_{x_0}\mathcal{B}_{x_2}^{s_2} \otimes {}_{x_2}\mathcal{B}_{x_0}^{s_1 s_0}\right) & \oplus & \left({}_{x_1}\mathcal{B}_{x_2}^{s_0 s_2} \otimes {}_{x_2}\mathcal{B}_{x_1}^{s_1}\right). \end{array}$$

Note that for the last summand $s_0 s_2 s_1 = s_0(s_2 s_1 s_0)s_0^{-1} \in D$. $\diamond$

Corollary 3.3 *Let $\mathcal{B}$ be a G-graded k-category and let $HH_*^D(\mathcal{B}) = H_*(C_\bullet^D(\mathcal{B}))$. There is a decomposition*

$$HH_*(\mathcal{B}) = \bigoplus_{D \in Cl_G} HH_*^D(\mathcal{B}).$$

For a G-k-category $\mathcal{C}$ our aim is to compare $HH_*(\mathcal{C})$ and $HH_*(\mathcal{C}[G])$. We first observe the following fact.

Proposition 3.4 *Let $\mathcal{C}$ be a G-k-category. The group G acts on the chain complex $C_\bullet(\mathcal{C})$ by automorphims.*

Proof. For $s \in G$ and $(f_n \otimes \cdots \otimes f_1 \otimes f_0) \in C_n(\mathcal{C})$ we define

$$sf = (sf_n \otimes \cdots \otimes sf_1 \otimes sf_0).$$

For $n=2$ we have

$$d(sf) = {}_{sx_0}(sf_2\ sf_1)_{sx_1} \otimes {}_{sx_1}(sf_0)_{sx_0} - {}_{sx_0}(sf_2)_{sx_2} \otimes {}_{sx_2}(sf_1\ sf_0)_{sx_0} + {}_{sx_1}(sf_0\ sf_2)_{sx_2} \otimes {}_{sx_2}(sf_1)_{sx_1}$$

and

$$sd(f) = {}_{sx_0}\left(s(f_2 f_1)\right)_{sx_1} \otimes {}_{sx_1}(sf_0)_{sx_0} - {}_{sx_0}(sf_2)_{sx_2} \otimes {}_{sx_2}\left(s(f_1 f_0)\right)_{sx_0} + {}_{sx_1}\left(s(f_0 f_2)\right)_{sx_2} \otimes {}_{sx_2} f_{1x_1}.$$

◇

Corollary 3.5 *Let $\mathcal{C}$ be a G-k-category. The Hochschild-Mitchell homology $HH_*(\mathcal{C})$ is a kG-module.*

Recall that if M is a kG-module, its kG-module of coinvariants is $M_G = M/ < sm - m >$, where the denominator is the sub kG-module of M generated by $\{sm - m \mid m \in M, s \in G\}$. The module of coinvariants is the largest quotient of M with trivial action of G. Considering M as a kG-bimodule with trivial action on the right, we have $M_G = kG \otimes_{kG\otimes(kG)^{op}} M$.

Remark 3.6 *If k is a field and if G is of finite order invertible in k, the coinvariants functor is exact. Moreover, in this case M_G is canonically isomorphic to the invariants $M^G = \{m \in M \mid sm = m$ for all $s \in G\}$ through the morphism $m \mapsto \frac{1}{|G|}\sum_{s\in G} sm$.*

The following comparison result assumes that the action of the group G is free on the set of objets of a G-k-category $\mathcal{C}$. In this situation the quotient category $\mathcal{C}/G$ exists and is equivalent to $\mathcal{C}[G]$, see [7]. In the next Section we will show that the isomorphim of the next theorem also holds when the action is not free. However the general case is based on the following.

Theorem 3.7 *Let $\mathcal{C}$ be a G-k-category and suppose that the action of G on $\mathcal{C}_0$ is free. Let $\mathcal{C}[G]$ be the skew category, considered with its G-grading (see 2.3). Let $\{1\}$ be the trivial conjugacy class of G. There is an isomorphism*

$$HH_*^{\{1\}}(\mathcal{C}[G]) \simeq H_*\left((\mathcal{C}_\bullet(\mathcal{C})_G\right).$$

If the coinvariant functor is exact then

$$HH_*^{\{1\}}(\mathcal{C}[G]) \simeq \left(H_*(\mathcal{C})\right)_G.$$

Remark 3.8 *If the covariants functor is not exact, it follows that there is a spectral sequence for computing $HH_*^{\{1\}}(\mathcal{C}[G])$.*

Before proving the Theorem we provide some properties of the skew category that we will need.

Lemma 3.9 *Let $\mathcal{C}$ be a $k-G$-category. Let x and y be objects in the same orbit of the action of G. They are isomorphic in $\mathcal{C}[G]$.*

Proof. Let $t \in G$ such that $y = tx$. Recall that ${}_{tx}\mathcal{C}[G]_x = \bigoplus_{s\in G} {}_{tx}\mathcal{C}_{sx}$. Let

$$a = {}_{tx}1_{tx} \in {}_{tx}\mathcal{C}[G]^t_x = {}_{tx}\mathcal{C}_{tx} \text{ and } b = {}_x1_x \in {}_x\mathcal{C}[G]^{t^{-1}}_{tx} = {}_x\mathcal{C}_x.$$

Using the composition defined in $\mathcal{C}[G]$, we obtain that a and b are mutual inverses in the skew category.

Definition 3.10 *Let G be a group acting on a set E. A* transversal T of the action *is a subset of E obtained by choosing precisely one element in each orbit of the action.*

Observe that equivalently $T \subset E$ is a transversal if for each $x \in E$ there exists a unique $u(x) \in T$ such that $x = su(x)$ for some $s \in G$. Note that the action is free if for each $x \in E$, there is only one $s \in G$ such that $x = su(x)$.

Lemma 3.11 *Let $\mathcal{C}$ be a G-k-category, let $T \subseteq \mathcal{C}_0$ be a transversal of the action of G on $\mathcal{C}_0$, and let $\mathcal{C}_T[G]$ be the full subcategory of $\mathcal{C}[G]$ with set of objects T. For each conjugacy class D of G we have*

$$HH^D_*(\mathcal{C}_T[G]) = H^D_*(\mathcal{C}[G]).$$

Proof. In [17, 19] (see also [3]) it is proven that if $\mathcal{C}$ and $\mathcal{D}$ are k-categories and if $F : \mathcal{C} \to \mathcal{D}$ is a k-equivalence, then F induces a quasi-isomorphism $C_\bullet(\mathcal{C}) \to C_\bullet(\mathcal{D})$. Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are G-graded and F is homogeneous, then the induced quasi-isomorphism clearly preserves the decomposition along conjugacy classes of G. Since T is a transversal, the above Lemma 3.9 shows that the inclusion functor $\mathcal{C}_T[G] \subset \mathcal{C}[G]$ is dense. Moreover it corresponds to a full subcategory, hence it is full and faithful, in addition of being homogeneous. The induced quasi-isomorphism provides then the result. $\diamond$

Proof of Theorem 3.7. Let T be a transversal of the free action of G on $\mathcal{C}_0$. In order to define an isomorphism of chain complexes $A : C_\bullet(\mathcal{C})_G \to C_\bullet(\mathcal{C}^{\{1\}}_T[G])$, let ${}_{x_0}f_{2x_2} \otimes {}_{x_2}f_{1x_1} \otimes {}_{x_1}f_{0x_0}$ be a chain of $C_2(\mathcal{C})$. Up to the action, that is in $(C_2(\mathcal{C}))_G$, we begin by modifying the chain in order that the starting (and hence the ending) object x_0 belongs to T. More precisely, there exists a unique $s \in G$ such that $sx_0 = u_0 \in T$. Then

$$f_2 \otimes f_1 \otimes f_0 \equiv s(f_2 \otimes f_1 \otimes f_0) = {}_{u_0}(sf_2)_{sx_2} \otimes {}_{sx_2}(sf_1)_{sx_1} \otimes {}_{sx_1}(sf_0)_{u_0}.$$

In other words we can assume that the chain is of the form

$${}_{u_0}(f_2)_{x_2} \otimes {}_{x_2}(f_1)_{x_1} \otimes {}_{x_1}(f_0)_{u_0} \text{ for } u_0 \in T.$$

For $i = 1, 2$, let $u_i = u(x_i)$ be the unique element of T which is in the orbit of x_i. Moreover let s_i be the unique element of G such that $x_i = s_i u_i$. We define

$$A\left({}_{u_0}(f_2)_{s_2u_2} \ \otimes \ {}_{s_2u_2}(f_1)_{s_1u_1} \ \otimes \ {}_{s_1u_1}(f_0)_{u_0}\right) =$$
$${}_{u_0}(f_2)_{s_2u_2} \ \otimes \ {}_{u_2}(s_2^{-1}f_1)_{s_2^{-1}s_1u_1} \ \otimes \ {}_{u_1}(s_1^{-1}f_0)_{s_1^{-1}u_0} =$$
$${}_{u_0}[f_2]^{s_2}_{u_2} \ \otimes {}_{u_2}[s_2^{-1}f_1]^{s_2^{-1}s_1}_{u_1} \ \otimes {}_{u_1}[s_1^{-1}f_0]^{s_1^{-1}}_{u_0}.$$

This chain belongs to $C_2^{\{1\}}(\mathcal{C}_T[G])$ since $s_2(s_2^{-1}s_1)s_1^{-1} = 1$. For a 3-chain the formula defining A is

$$A({}_{u_0}f_{3s_3u_3} \otimes {}_{s_3u_3}f_{2s_2u_2} \otimes {}_{s_2u_2}f_{1s_1u_1} \otimes {}_{s_1u_1}f_{0u_0}) =$$
$${}_{u_0}[f_3]^{s_3}_{u_3} \ \otimes \ {}_{u_3}[s_3^{-1}(f_2)]^{s_3^{-1}s_2}_{u_2} \ \otimes \ {}_{u_2}[(s_2^{-1}f_1)]^{s_2^{-1}s_1}_{u_1} \ \otimes \ {}_{u_1}[s_1^{-1}f_0]^{s_1^{-1}}_{u_0}.$$

Next we verify that A is a chain map.

$$\begin{aligned}
dA(f_2 \otimes f_1 \otimes f_0) &= [f_2]^{s_2}[s_2^{-1}f_1]^{s_2^{-1}s_1} \otimes [s_1^{-1}f_0]^{s_1^{-1}} - \\
&\quad [f_2]^{s_2} \otimes [s_2^{-1}f_1]^{s_2^{-1}s_1}[s_1^{-1}f_0]^{s_1^{-1}} + \\
&\quad [s_1^{-1}f_0]^{s_1^{-1}}[f_2]^{s_2} \otimes [s_2^{-1}f_1]^{s_2^{-1}s_1} \\
&= [f_2f_1]^{s_1} \otimes [s_1^{-1}f_0]^{s_1^{-1}} - \\
&\quad [f_2]^{s_2} \otimes [(s_2^{-1}f_1)(s_2^{-1}f_0)]^{s_2^{-1}} + \\
&\quad [(s_1^{-1}f_0)(s_1^{-1}f_2)]^{s_1^{-1}s_2} \otimes [s_2^{-1}f_1]^{s_2^{-1}s_1}.
\end{aligned}$$

Recall that

$$\begin{aligned}
d(f_2 \otimes f_1 \otimes f_0) = \ & {}_{u_0}(f_2f_1)_{s_1u_1} \otimes {}_{s_1u_1}(f_0)_{u_0} - \\
& {}_{u_0}(f_2)_{s_2u_2} \otimes {}_{s_2u_2}(f_1f_0)_{u_0} + \\
& {}_{s_1u_1}(f_0f_2)_{s_2u_2} \otimes {}_{s_2u_2}(f_1)_{s_1u_1}.
\end{aligned}$$

In order to compute Ad, notice that up to the action, that is in the coinvariants, the last term of the previous sum can be rewrited:

$${}_{s_1u_1}(f_0f_2)_{s_2u_2} \otimes {}_{s_2u_2}(f_1)_{s_1u_1} \equiv \ {}_{u_1}(s^{-1}(f_0f_2))_{s_1^{-1}s_2u_2} \otimes {}_{s_1^{-1}s_2u_2}(s_1^{-1}f_1)_{u_1}.$$

This way the last summand of $d(f_2 \otimes f_1 \otimes f_0)$ starts and ends at $u_1 \in T$, which is required in order to apply A. Hence

$$\begin{aligned}
Ad(f_2 \otimes f_1 \otimes f_0) &= A\big({}_{u_0}(f_2f_1)_{s_1u_1} \otimes {}_{s_1u_1}(f_0)_{u_0} - {}_{u_0}(f_2)_{s_2u_2} \otimes {}_{s_2u_2}(f_1f_0)_{u_0} + \\
&\quad {}_{u_1}(s_1^{-1}(f_0f_2))_{s_1^{-1}s_2u_2} \otimes \ {}_{s_1^{-1}s_2u_2}(s_1^{-1}f_1)_{u_1}\big) \\
&= [(f_2f_1)]^{s_1} \otimes [s_1^{-1}f_0]^{s_1^{-1}} - [f_2]^{s_2} \otimes [s_2^{-1}(f_1f_0)]^{s_2^{-1}} + \\
&\quad [s_1^{-1}(f_0f_2)]^{s_1^{-1}s_2} \otimes [(s_1^{-1}s_2)^{-1}s_1^{-1}f_1]^{(s_1^{-1}s_2)^{-1}}
\end{aligned}$$

and this shows $Ad = dA$.

Let $g_2 \otimes g_1 \otimes g_0 \in C_2^{\{1\}}(\mathcal{C}_T[G])$, that is

$$g_2 \otimes g_1 \otimes g_0 = {}_{u_0}[g_2]^{s_2}_{u_2} \otimes {}_{u_2}[g_1]^{s_1}_{u_1} \otimes {}_{u_1}[g_0]^{s_0}_{u_0} = {}_{u_0}g_{2s_2u_2} \otimes {}_{u_2}g_{1s_1u_1} \otimes {}_{u_1}g_{0s_0u_0}$$

with $s_2s_1s_0 = 1$. Let

$$B : C_\bullet^{\{1\}}(\mathcal{C}_T[G]) \to (C_\bullet(\mathcal{C}))_G$$

be defined by

$$B(g_2 \otimes g_1 \otimes g_0) = {}_{u_0}(g_2)_{s_2u_2} \otimes {}_{s_2u_2}(s_2g_1)_{s_2s_1u_1} \otimes {}_{s_2s_1u_1}(s_2s_1g_0)_{s_2s_1s_0u_0}.$$

We observe that since $s_2s_1s_0 = 1$, we have that $s_2s_1s_0u_0 = u_0$. Next we will show that A and B are mutual inverses. This will imply that B is a chain map, since A is a chain map.

$$\begin{array}{l} AB(g_2 \otimes g_1 \otimes g_0) = \\ A({}_{u_0}g_{2s_2u_2} \otimes {}_{s_2u_2}(s_2g_1)_{s_2s_1u_1} \otimes {}_{s_2s_1u_1}(s_2s_1g_0)_{u_0}) = \\ {}_{u_0}g_{2s_2u_2} \otimes {}_{u_2}(s_2^{-1}s_2g_1)_{s_1u_1} \otimes {}_{u_1}((s_2s_1)^{-1}(s_2s_1)g_0)_{(s_2s_1)^{-1}u_0}. \end{array}$$

Since $s_2s_1s_0 = 1$, we obtain

$${}_{u_0}g_{2s_2u_2} \otimes {}_{u_2}g_{1s_1u_1} \otimes {}_{u_1}g_{0s_0u_0} = {}_{u_0}[g_2]^{s_2}_{u_2} \otimes {}_{u_2}[g_1]^{s_1}_{u_1} \otimes {}_{u_1}[g_0]^{s_0}_{u_0}.$$

$$\begin{array}{l} BA\left({}_{u_0}f_{2s_2u_2} \otimes {}_{s_2u_2}f_{1s_1u_1} \otimes {}_{s_1u_1}f_{0u_0}\right) = \\ B\left({}_{u_0}[f_2]^{s_2}_{u_2} \otimes {}_{u_2}[s_2^{-1}f_1]^{s_2^{-1}s_1}_{u_1} \otimes {}_{u_1}[s_1^{-1}f_0)]^{s_1^{-1}}_{u_0}\right) = \\ {}_{u_0}f_{2s_2u_2} \otimes {}_{s_2u_2}(s_2s_2^{-1}f_1)_{s_1u_1} \otimes {}_{s_2s_2^{-1}s_1u_1}((s_2s_2^{-1}s_1)s_1^{-1}f_0)_{(s_2s_2^{-1}s_1)s_1^{-1}u_0} = \\ {}_{u_0}f_{2s_2u_2} \otimes {}_{s_2u_2}f_{1s_1u_1} \otimes {}_{s_1u_1}f_{0u_0}. \end{array}$$

$\diamond$

We end this section by restating the Theorem above in terms of Galois coverings. We recall first the definition of a quotient category.

Definition 3.12 *(see [4, 28]) Let $\mathcal{C}$ be a G-k-category such that the action of G on $\mathcal{C}_0$ is free. The* quotient category *$\mathcal{C}/G$ has set of objects the set of orbits $\mathcal{C}_0/G$. Let α and β be orbits. The k-module of morphisms from α to β is*

$${}_\beta(\mathcal{C}/G)_\alpha = \left(\bigoplus_{\substack{x\in\alpha \\ y\in\beta}} {}_y\mathcal{C}_x\right)_G .$$

Let γ, β, α be orbits. Let

$$g \in {}_z\mathcal{C}_{y'} \text{ and } f \in {}_y\mathcal{C}_x,$$

where $z \in \gamma$, y and $y' \in \beta$, and $x \in \alpha$. Let s be the unique element of G such that $sy = y'$, then $f \equiv sf$ in the coinvariants. The composition gf in $\mathcal{C}/G$ is

$$gf = {}_z g_{y'} \ {}_{sy} sf_{sx} \in {}_\gamma(\mathcal{C}/G)_\alpha.$$

There is no difficulty in verifying that this is a well defined associative composition.

Definition 3.13 *A Galois covering of k-categories is a functor $\mathcal{C} \to \mathcal{C}/G$, where $\mathcal{C}$ is a G-k-category with a free action of G on the objects and where the functor is the canonical projection functor.*

Let $\mathcal{C} \to \mathcal{C}/G$ be a Galois covering and let T be a transversal of the action of G on $\mathcal{C}_0$. For each orbit α, let $u_\alpha \in T$ be the unique element of T which belongs to α.

It is shown in Lemma 2.2 of [7] that through a canonical identification we have

$$_\beta(\mathcal{C}/G)_\alpha = \bigoplus_{s\in G} {}_{u_\beta}\mathcal{C}_{su_\alpha}.$$

This provides that $\mathcal{C}/G$ is graded by G. Indeed let $_\beta(\mathcal{C}/G)^s_\alpha = {}_{u_\beta}\mathcal{C}_{su_\alpha}$, and notice that

$$({}_{u_\gamma}g_{tu_\beta})({}_{u_\beta}f_{su_\alpha}) = \left({}_{u_\gamma}g_{tu_\beta}\right)\left({}_{tu_\beta}tf_{tsu_\alpha}\right) = {}_{u_\gamma}(g(tf))_{tsu_\alpha} \in {}_\gamma(\mathcal{C}/G)^{ts}_\alpha.$$

The following result can be deduced from [7]. We provide a proof for completeness.

Proposition 3.14 *Let $\mathcal{C} \to \mathcal{C}/G$ be a Galois covering. Let $T \subset \mathcal{C}_0$ be a transversal of the free action of G, and consider the G-grading of $\mathcal{C}/G$ determined by T.*

The graded categories $\mathcal{C}/G$ and $\mathcal{C}_T[G]$ are isomorphic by an homogeneous functor, and the graded categories $\mathcal{C}/G$ and $\mathcal{C}[G]$ are equivalent by an homogeneous functor.

Proof. There is a bijection between the objects of $\mathcal{C}/G$ and those of $\mathcal{C}_T[G]$. The previous considerations shows that the morphisms of both categories are subsequently identified in an homogeneous manner. Moreover we have already used that the inclusion $C_T[G] \subset \mathcal{C}[G]$ provides an equivalence of categories, and this equivalence is homogeneous. $\diamond$

Corollary 3.15 *Let $\mathcal{C} \to \mathcal{C}/G$ be a Galois covering.*

$$HH^{\{1\}}_*(\mathcal{C}/G) = H_*(C_\bullet(\mathcal{C})_G).$$

If the coinvariant functors is exact

$$HH^{\{1\}}_*(\mathcal{C}/G) = (HH_*(\mathcal{C}))_G\,.$$

4 Homology for skew categories

In this section we construct an auxiliary G-k-category affording a free action of G on its objects, in order to prove Theorem 3.7 without the assumption that the action of G is free on the objects.

Definition 4.1 *(see also [10], [11]) Let $\mathcal{C}$ be a G-k-category. The objects of the k-category $M_G(\mathcal{C})$ are $G \times \mathcal{C}_0$. The k-module of morphisms of $M_G(\mathcal{C})$ from (s,x) to (t,y) is*

$$_{(t,y)}(M_G(\mathcal{C}))_{(s,x)} = {}_y\mathcal{C}_x.$$

The composition is given in the evident way by the composition of $\mathcal{C}$. The action of G on $M_G(\mathcal{C})$ is defined as follows: for $r \in G$, let $r(s,x) = (rs, rx)$, and for

$$f \in {}_{(t,y)}(M_G(\mathcal{C}))_{(s,x)} = {}_y\mathcal{C}_x \text{ let } rf \in {}_{ry}\mathcal{C}_{rx} = {}_{(rt,ty)}(M_G(\mathcal{C}))_{(rs,rx)}.$$

Observe that this action is free on the objects of $M_G(\mathcal{C})$.

Remark 4.2 *Let Λ be a G-k-algebra, considered as a single object G-k-category. The category $M_G(\Lambda)$ is the so-called Λ-complete category over G: its set of objects is G and each k-module of morphisms is a copy of Λ. The composition is given by the product in the algebra. The action of G on the objects is the product of G, while on morphism it is given by the G-action on Λ combined with the action on the objects.*

If G is finite, the k-algebra $a(M_G(\Lambda))$ is a matrix algebra over Λ with columns and rows indexed by G. Observe that the action of $s \in G$ on a given matrix is the action of s on the columns and the rows, combined with the action of s on the entries of the matrix.

We recall next the definition of the tensor product $\mathcal{C} \otimes \mathcal{D}$ of k-categories $\mathcal{C}$ and $\mathcal{D}$. The objects are $\mathcal{C}_0 \times \mathcal{D}_0$, while the morphisms are given by:

$$_{(c',d')}(\mathcal{C} \otimes \mathcal{D})_{(c,d)} = {}_{c'}C_c \otimes {}_{d'}\mathcal{D}_d$$

with the obvious composition. Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are G-k-categories then $\mathcal{C} \otimes \mathcal{D}$ is a G-k-category through the diagonal action of G.

Hence

$$M_G(\mathcal{C}) = M_G(k) \otimes \mathcal{C},$$

where k is the single objet G-k-category, the object has endomorphisms k, and the action of G is trivial.

Lemma 4.3 *Let $\mathcal{C}$ be a G-k-category. There is an equivalence of G-k-categories $L : M_G(\mathcal{C}) \to \mathcal{C}$.*

Proof. Let $L : M_G(\mathcal{C}) \to \mathcal{C}$ be the functor defined on the objects by $L(s,x) = x$, while on morphisms L is given by the suitable identity maps. Hence L is a fully faithful G-functor which is surjective on the objects, so it is an equivalence of G-k-categories. ◇

Proposition 4.4 *Let $\mathcal{C}$ be a G-k-category. The chain complexes $C_\bullet(M_G(\mathcal{C}))$ and $C_\bullet(\mathcal{C})$ are kG-quasi-isomorphic.*

Proof. As before we use the result in [17]. The above equivalence of categories $L : M_G(\mathcal{C}) \to \mathcal{C}$ induces a quasi-isomorphism $\mathcal{C}_\bullet(M_G(\mathcal{C})) \to C_\bullet(\mathcal{C})$. Since L is a G-k-functor, the quasi-isomorphism is a kG-module chain map, which provides a kG-isomophism in homology. ◇

Theorem 4.5 *Let $\mathcal{C}$ be a G-k-category.*

$$HH_*^{\{1\}}(\mathcal{C}[G]) = H_*\left(C_\bullet(\mathcal{C})_G\right).$$

If the coinvariants functor is exact

$$HH_*^{\{1\}}(\mathcal{C}[G]) = \left(HH_*(\mathcal{C})\right)_G.$$

Proof. Due to Theorem 3.7 the result holds for the G-k-category $M_G(\mathcal{C})$. The equivalence of G-k-categories $L : M_G(\mathcal{C}) \to \mathcal{C}$ provides an homogeneous equivalence of G-graded k-categories

$$L[G] : M_G(\mathcal{C})[G] \to \mathcal{C}[G]$$

which gives a quasi-isomorphism

$$C_\bullet(M_G(\mathcal{C})[G]) \to C_\bullet(\mathcal{C}[G])$$

which preserves the decomposition of chain complexes along the conjugacy classes of G. Hence

$$HH_*^{\{1\}}(M_G(\mathcal{C})[G]) = HH_*^{\{1\}}(\mathcal{C}[G]).$$

By the above proposition

$$H_*(C_\bullet(M_G(\mathcal{C}))_G) = H_*(C_\bullet(\mathcal{C})_G).$$

If the coinvariants functor is exact then

$$HH_*\left(M_G(\mathcal{C})\right)_G = \left(HH_*(\mathcal{C})\right)_G.$$

◇

5 Cohomology

Definition 5.1 *Let $\mathcal{C}$ be a k-category. Let $C^\bullet(\mathcal{C})$ be the complex of cochains given by:*

$$\mathcal{C}^0(\mathcal{C}) = \prod_{x\in\mathcal{C}_0} {}_x\mathcal{C}_x,$$

$$\mathcal{C}^n(\mathcal{C}) = \prod_{x_{n+1},\dots,x_1\in\mathcal{C}_0} C_{x_{n+1},\dots,x_1} \text{ for } n>0$$

where

$$C_{x_{n+1},\dots,x_1} = \mathsf{Hom}_k({}_{x_{n+1}}\mathcal{C}_{x_n}\otimes\cdots\otimes {}_{x_2}\mathcal{C}_{x_1},\ {}_{x_{n+1}}\mathcal{C}_{x_1})$$

The coboundary d is given by the formulas below which are the usual ones for computing Hochschild homology, see for instance [5, 32].

Let φ be a cochain of degree n, that is a family of k morphisms $\varphi = \{\varphi_{(x_{n+1},\dots,x_1)}\}$. Its coboundary $d\varphi$ is the family $\{(d\varphi)_{(x_{n+2},\dots,x_1)}\}$ given by

$$\begin{array}{l}(d\varphi)_{(x_{n+2},\dots,x_1)}\left({}_{x_{n+2}}(f_{n+1})_{x_{n+1}}\otimes\cdots\otimes {}_{x_2}(f_1)_{x_1}\right) = \\ (-1)^{n+1}f_{n+1}\varphi_{(x_{n+1},\dots,x_1)}(f_n\otimes\cdots\otimes f_1)+ \\ \sum_{i=1}^n(-1)^{i+1}\varphi_{(x_{n+1},\dots,x_{i+2},x_i,\dots x_1)}(f_{n+1}\otimes\cdots\otimes f_{i+1}f_i\otimes\cdots\otimes f_1)+ \\ \varphi_{(x_{n+2},\dots x_2)}(f_{n+1}\otimes\cdots\otimes f_2)f_1.\end{array} \tag{5.1}$$

Note that d is well defined since the cochains are direct products. The Hochschild-Mitchell cohomology *of $\mathcal{C}$ is $HH^*(\mathcal{C}) = H^*(C^\bullet(\mathcal{C}))$.*

In degree zero we set

$$HH^0(\mathcal{C}) = \{({}_xf_x)_{x\in\mathcal{C}_0} \mid {}_yg_x\ {}_xf_x = {}_yf_y\ {}_yg_x \text{ for all } {}_yg_x\in {}_y\mathcal{C}_x\}.$$

As for Hochschild cohomology of algebras, the cup product is defined at the cochain level as follows. Let $\varphi\in\mathcal{C}_{x_{n+1},\dots,x_1}$ and $\psi\in\mathcal{C}_{y_{m+1},\dots,y_1}$. If $x_{n+1}\neq y_1$ the cup product $\psi\smile\varphi$ is zero. Otherwise the cup product $\psi\smile\varphi\in\mathcal{C}_{y_{m+1},\dots,y_1,x_n,\dots,x_1}$ is

$$(\psi\smile\varphi)(f_{n+m}\otimes\cdots\otimes f_1) = \psi(f_{n+m}\otimes\cdots\otimes f_{n+1})\varphi(f_n\otimes\cdots\otimes f_1).$$

The cup product verifies the graded Leibniz rule, and it provides a graded commutative k-algebra structure on $HH^*(\mathcal{C})$. In particular $HH^0(\mathcal{C})$ is a commutative k-algebra which is the center of the category.

Proposition 5.2 *Let $\mathcal{B}$ be a G-graded category, and let $Cl(G)$ be the set of conjugacy classes of G. There is a decomposition*

$$HH^*(\mathcal{B}) = \prod_{D\in Cl(G)} HH^*_D(\mathcal{B})$$

where $HH^*_{\{1\}}(\mathcal{B})$ is a subalgebra.

Proof. For $D \in Cl(G)$ we provide a subcomplex of cochains $C_D^\bullet(\mathcal{B})$ of $\mathcal{C}^\bullet(\mathcal{B})$ as follows. Let φ be a cochain of degree n. We say that φ is *homogeneous of type* $(s_n, \dots, s_1, s_0)$ if:

1. Each component of φ has its image contained in the homogeneous morphisms of degree s_0.
2. For $(s'_n, \dots, s'_1) \neq (s_n, \dots, s_1)$, each component of φ restricted to tensors of homogeneous morphisms degree $(s'_n, \dots, s'_1)$ is zero.

The formula (5.1) which defines the coboundary d has $n+2$ summands. Let d_{n+1} be the first one, let d_0 be the last one, and let d_i denotes the in between summands indexed according to the appearance of the composition "$f_{i+1}f_i$" for $i = n, \dots, 1$.

Let φ be homogeneous of type $(s_n, \cdots, s_1, s_0)$. We observe the following:

- $d_{n+1}\varphi$ is a sum of homogeneous cochains of types $(s, s_n, \dots, s_1, ss_0)$ for $s \in G$.
- $d_i\varphi$ is a sum of homogeneous cochains of types

$$(s_n, \dots, s_{i+1}, s'', s', s_{i-1}, \dots, s_1, s_0)$$

  for $s'', s' \in G$ with $s''s' = s_i$.
- $d_0\varphi$ is a sum of homogeneous cochains of types $(s_n, \dots, s_1, s, s_0 s)$ for $s \in G$.

Let the *class* of $(s_n, \dots, s_1, s_0)$ be the product $s_n \dots s_1 s_0^{-1} \in G$. The above considerations show that if φ is homogeneous of type $(s_n, \dots, s_1, s_0)$, hence of class $c = s_n \dots s_1 s_0^{-1}$, then $d\varphi$ is a sum of homogeneuos cochains, possibly of different types but whose classes are conjugated to c.

Let D be a conjugacy class and let $C_D^\bullet(\mathcal{C})$ be the homogeneous cochains which classes of types are in D. We have showed that $C_D^\bullet(\mathcal{C})$ is a cochains subcomplex of $C^\bullet(\mathcal{C})$. Moreover

$$C^\bullet(\mathcal{C}) = \prod_{D \in Cl(G)} C_D^\bullet(\mathcal{C}).$$

Clearly, if φ and ψ are homogeneous cochains which classes of types are both 1, then $\psi \smile \varphi$ is also of class type 1.

Let $\mathcal{C}$ be a G-k-category. We assert that $C^n(\mathcal{C})$ is a kG-module as follows. Let $\varphi = \{\varphi_{(x_{n+1},\dots,x_1)}\} \in C^n(\mathcal{C})$ be a cochain, where

$$\varphi_{(x_{n+1},\dots,x_1)} : {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1} \to {}_{x_{n+1}}\mathcal{C}_{x_1}.$$

Let $s \in G$ and let

$$s.[\varphi_{(x_{n+1},\dots,x_1)}] : {}_{sx_{n+1}}\mathcal{C}_{sx_n} \otimes \cdots \otimes {}_{sx_2}\mathcal{C}_{sx_1} \to {}_{sx_{n+1}}\mathcal{C}_{sx_1}$$

be defined by

$$s.[\varphi_{(x_{n+1},\dots,x_1)}](f_n \otimes \cdots \otimes f_1) = s[\varphi_{(x_{n+1},\dots,x_1)}(s^{-1}f_n \otimes \cdots \otimes s^{-1}f_1)].$$

Finally we set $s.\varphi = \{s.[\varphi_{x_{n+1},\dots,x_1}]\}$.

Remark 5.3 *Let $(\ )^G$ be the invariant functor. Then $\varphi \in (C^n(\mathcal{C}))^G$ if and only if for all $s \in G$ and for any sequence of objects $x_{n+1}, \dots, x_1$ we have that*

$$\varphi_{(sx_{n+1},\dots,sx_1)}(sf_n \otimes \cdots \otimes sf_1) = s\left[\varphi_{(x_{n+1},\dots,x_1)}(f_n \otimes \cdots \otimes f_1)\right].$$

Clearly the action of G commutes with the coboundary of $C^\bullet(\mathcal{C})$. Moreover the action of G is by automorphisms of the cup product. In other words $C^\bullet(\mathcal{C})$ is a differential graded algebra with an action of G by automorphisms of its structure.

In particular $(C^\bullet(\mathcal{C}))^G$ is a graded differential algebra. Moreover the inferred action of G on $HH^*(\mathcal{C})$ is by automorphisms of the algebra, in other words $HH^*(\mathcal{C})$ is a G-k-algebra. If the invariants functor is exact, then $(HH^*(\mathcal{C}))^G = H^*(C^\bullet(\mathcal{C})^G)$ as k-algebras.

Theorem 5.4 *Let $\mathcal{C}$ be a G-k-category with free action of G on the objects, and let $\mathcal{C}[G]$ be the G-graded skew category. There is an isomorphism of k-algebras*

$$HH^*_{\{1\}}(\mathcal{C}[G]) \simeq H^*\left((C^\bullet(\mathcal{C})^G\right).$$

If the invariant functor $(\ )^G$ is exact, we infer an isomorphism

$$HH^*_{\{1\}}(\mathcal{C}[G]) \simeq (HH^*(C))^G$$

of k-algebras.

Remark 5.5

1. *Based on this Theorem, we will prove that the result also holds if the action of G on $\mathcal{C}_0$ is not free.*
2. *If the invariant functor is not exact, the standard considerations provide a spectral sequence.*

Proof. Let T be a transversal of the action of G on $\mathcal{C}_0$ and let $\mathcal{C}_T[G]$ be the full subcategory of $\mathcal{C}[G]$ with set of objets T. Let D be a conjugacy class of G. We assert that

$$HH^*_D(\mathcal{C}_T[G]) = HH^*_D(\mathcal{C}[G]).$$

Indeed the equivalence of categories given by the inclusion $\mathcal{C}_T[G] \subseteq \mathcal{C}[G]$ induces a quasi-isomorphism of the complexes of cochains which preserves the decomposition along the conjugacy classes of G.

Moreover for the trivial conjugacy class the quasi-isomorphism is a morphism of differential graded algebras. Hence

$$HH^*_{\{1\}}(\mathcal{C}_T[G]) = HH^*_{\{1\}}(\mathcal{C}[G])$$

as k-algebras.

In what follows we will prove that there are morphisms of graded differential algebras

$$(C^\bullet(\mathcal{C}))^G \underset{B}{\overset{A}{\rightleftarrows}} C^\bullet_{\{1\}}(\mathcal{C}_T[G])$$

which are inverses one of each other.

Let $\psi \in \left(C^3(\mathcal{C})\right)^G$ and let $u_4, u_3, u_2, u_1 \in T$. We will define $(A\psi)_{(u_4,u_3,u_2,u_1)}$ on each homogeneous component.

Let

$$f_3 \otimes f_2 \otimes f_1 \in {}_{u_4}\mathcal{C}_T[G]^{s_3}_{u_3} \otimes {}_{u_3}\mathcal{C}_T[G]^{s_2}_{u_2} \otimes {}_{u_2}\mathcal{C}_T[G]^{s_1}_{u_1}.$$

Recall that by the definition of the morphisms of $\mathcal{C}[G]$ we have that $f_i \in {}_{u_{i+1}}\mathcal{C}_{t_i u_i}$ for $i = 1, 2, 3$. Let

$$(A\psi)_{(u_4,u_3,u_2,u_1)}(f_3 \otimes f_2 \otimes f_1) = \psi_{(u_4,s_3u_3,s_3s_2u_2,s_3s_2s_1u_1)}(f_3 \ \otimes s_3f_2 \otimes s_3s_2f_1).$$

We observe that this definition makes sense since

$$f_3 \otimes s_3f_2 \otimes s_3s_2f_1 \in {}_{u_4}\mathcal{C}_{s_3u_3} \otimes {}_{s_3u_3}\mathcal{C}_{s_3s_2u_2} \otimes {}_{s_3s_2u_2}\mathcal{C}_{s_3s_2s_1u_1}.$$

Moreover

$$A\psi(f_3 \otimes f_2 \otimes f_1) \in {}_{u_4}\mathcal{C}_{s_3s_2s_1u_1} = {}_{u_4}\mathcal{C}_T[G]^{s_3s_2s_1}_{u_1},$$

that is we have indeed defined an homogeneous cochain of type $(s_3, s_2, s_1, s_3s_2s_1)$, which is of class $\{1\}$.

The verification that $dA = Ad$ is straightforward, it uses in a crucial way that ψ is an invariant; the formulas defining the composition in $\mathcal{C}[G]$ are required as well . Analogously, it is easy to verify that $A(\psi' \smile \psi) = A(\psi') \smile A(\psi)$.

Let $\varphi \in C^3_{\{1\}}(\mathcal{C}_T[G])$. In order to define $(B\varphi)_{(x_4,x_3,x_2,x_1)}$ we first observe that since the action of G on $\mathcal{C}_0$ is free, there exist $s_4, s_3, s_2, s_1 \in G$ which are unique such that $x_i = s_iu_i$ for $i = 1, 2, 3, 4$.

Let $g_3 \otimes g_2 \otimes g_1 \in {}_{s_4u_4}\mathcal{C}_{s_3u_3} \otimes {}_{s_3u_3}\mathcal{C}_{s_2u_2} \otimes {}_{s_2u_2}\mathcal{C}_{s_1u_1}$. We define $(B\varphi)_{(x_4,x_3,x_2,x_1)}$ as follows:

$$(B\varphi)(g_3 \otimes g_2 \otimes g_1) = s_4\varphi_{(u_4,u_3,u_2,u_1)}(s_4^{-1}g_3 \otimes s_3^{-1}g_2 \otimes s_2^{-1}g_1).$$

In order to verify that this is well defined, note first that

$$\begin{array}{llllll} s_4^{-1}g_3 \otimes s_3^{-1}g_2 \otimes s_2^{-1}g_1 \in & {}_{u_4}\mathcal{C}_{s_4^{-1}s_3u_3} & \otimes & {}_{u_3}\mathcal{C}_{s_3^{-1}s_2u_2} & \otimes & {}_{u_2}\mathcal{C}_{s_2^{-1}s_1u_1} = \\ & {}_{u_4}\mathcal{C}[G]^{s_4^{-1}s_3}_{u_3} & \otimes & {}_{u_3}\mathcal{C}[G]^{s_3^{-1}s_2}_{u_2} & \otimes & {}_{u_2}\mathcal{C}[G]^{s_2^{-1}s_1}_{u_1}. \end{array}$$

Secondly, using that φ is a cochain for the trivial conjugacy class, we obtain

$$\varphi_{(u_4,u_3,u_2,u_1)}(s_4^{-1}g_3, s_3^{-1}g_2, s_2^{-1}g_1) \in {}_{u_4}\mathcal{C}[G]_{u_1}^{s_4^{-1}s_3s_3^{-1}s_2s_2^{-1}s_1} = {}_{u_4}\mathcal{C}[G]_{u_1}^{s_4^{-1}s_1} = {}_{u_4}\mathcal{C}_{s_4^{-1}s_1u_1}.$$

Hence $(B\varphi)(g_3 \otimes g_2 \otimes g_1) \in {}_{su_4}\mathcal{C}_{su_1}$, therefore $B\varphi \in C^3(\mathcal{C})$. Next we check that $B\varphi$ is an invariant cochain. Let $t \in G$, we assert that

$$t(B\varphi)_{(s_4u_4,s_3u_3,s_2u_2,s_1u_1)}(g_3 \otimes g_2 \otimes g_1) = B\varphi_{(ts_4u_4,ts_3u_3,ts_2u_2,ts_1u_1)}(tg_3 \otimes tg_2 \otimes tg_1).$$

Indeed, the second term is by definition

$$ts_4\ \varphi_{(u_4,u_3,u_2,u_1)}\left((ts_4)^{-1}tg_3 \otimes (ts_3)^{-1}tg_2 \otimes (ts_2)^{-1}tg_1\right),$$

which equals the first term.

Let $\psi \in C^3(\mathcal{C})^G$, we assert that $BA\psi = \psi$. Recall that if

$$f_3 \otimes f_2 \otimes f_1 \in {}_{u_4}\mathcal{C}[G]_{u_3}^{t_3} \otimes {}_{u_3}\mathcal{C}[G]_{u_2}^{t_2} \otimes {}_{u_2}\mathcal{C}[G]_{u_1}^{t_1},$$

then

$$(A\psi)_{u_4,u_3,u_2,u_1}(f_3 \otimes f_2 \otimes f_1) = \psi(f_3 \otimes t_3f_2 \otimes t_3t_2f_1).$$

Let $g_3 \otimes g_2 \otimes g_1 \in {}_{s_4u_4}\mathcal{C}_{s_3u_3} \otimes {}_{s_3u_3}\mathcal{C}_{s_2u_2} \otimes {}_{s_2u_2}\mathcal{C}_{s_1u_1}$. Then

$$BA\psi(g_3 \otimes g_2 \otimes g_1) = s_4A\psi(s_4^{-1}g_3 \otimes s_3^{-1}g_2 \otimes s_2^{-1}g_1)$$

where

$$s_4^{-1}g_3 \otimes s_3^{-1}g_2 \otimes s_2^{-1}g_1 \in {}_{u_4}\mathcal{C}[G]_{u_3}^{s_4^{-1}s_3} \otimes {}_{u_3}\mathcal{C}[G]_{u_2}^{s_3^{-1}s_2} \otimes {}_{u_2}\mathcal{C}[G]_{u_1}^{s_2^{-1}s_1}.$$

Hence

$$\begin{aligned}&BA\psi(g_3 \otimes g_2 \otimes g_1) = \\ &s_4\psi(s_4^{-1}g_3 \otimes (s_4^{-1}s_3)s_3^{-1})g_2 \otimes (s_4^{-1}s_3s_3^{-1}s_2)s_2^{-1}g_1) = \\ &s_4\psi(s_4^{-1}g_3 \otimes s_4^{-1}g_2 \otimes s_4^{-1}g_1).\end{aligned}$$

Since ψ is invariant, the later equals $\psi(g_3 \otimes g_2 \otimes g_1)$.

Let $\varphi \in C_{\{1\}}(\mathcal{C}_T[G])$, next we will show $AB\varphi = \varphi$. Consider

$$g_3 \otimes g_2 \otimes g_1 \in {}_{t_4u_4}\mathcal{C}_{t_3u_3} \otimes {}_{t_3u_3}\mathcal{C}_{t_2u_2} \otimes {}_{t_2u_2}\mathcal{C}_{t_1u_1}.$$

We have

$$B\varphi(g_3 \otimes g_2 \otimes g_1) = t_4\varphi(t_4^{-1}g_3 \otimes t_3^{-1}g_2 \otimes t_2^{-1}g_1).$$

Let

$$f_3 \otimes f_2 \otimes f_1 \in {}_{u_4}\mathcal{C}[G]_{u_3}^{s_3} \otimes {}_{u_3}\mathcal{C}[G]_{u_2}^{s_2} \otimes {}_{u_2}\mathcal{C}[G]_{u_1}^{s_1}.$$

Then

$$AB\varphi(f_3 \otimes f_2 \otimes f_1) = (B\varphi)(f_3 \otimes s_3f_2 \otimes s_3s_2f_1)$$

where

$$f_3 \otimes s_3 f_2 \otimes s_3 s_2 f_1 \in {}_{u_4}\mathcal{C}_{s_3 u_3} \otimes {}_{s_3 u_3}\mathcal{C}_{s_3 s_2 u_2} \otimes {}_{s_3 s_2 u_2}\mathcal{C}_{s_3 s_2 s_1 u_1}$$

Hence

$$AB\varphi(f_3 \otimes f_2 \otimes f_1) = \varphi(f_3 \otimes s_3^{-1} s_3 f_2 \otimes (s_3 s_2)^{-1} s_3 s_2 f_1) = \varphi(f_3 \otimes f_2 \otimes f_1).$$

$\diamond$

Our next aim is to show that the isomorphism of Theorem 5.4 remains valid when the action of the group is not necessarily free on the set of objects of the k-category. The following result have been proved in [3, 17], see also [?].

Proposition 5.6 *Let $\mathcal{C}$ and $\mathcal{D}$ be k-categories and let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of k-categories. There is an induced map*

$$C^\bullet F : C^\bullet(\mathcal{D}) \to \mathcal{C}^\bullet(\mathcal{C})$$

which is a quasi-isomorphism.

Remark 5.7 *For future use we make precise the definition of $C^\bullet(F)$. Let*

$$\varphi = \left(\varphi_{y_{n+1},\dots,y_1}\right) \in C^n(\mathcal{D})$$

where

$$\varphi_{y_{n+1},\dots,y_1} \;:\; {}_{y_{n+1}}\mathcal{D}_{y_n} \otimes \cdots \otimes {}_{y_2}\mathcal{D}_{y_1} \longrightarrow {}_{y_{n+1}}\mathcal{D}_{y_1}$$

is a k-morphism. The component $(x_{n+1}, \cdots, x_1)$ of $(C^\bullet F)(\varphi)$ is given as follows. Let

$$f_{n+1} \otimes \cdots \otimes f_1 \in {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}.$$

Then

$$[(C^\bullet F)(\varphi)]_{x_{n+1},\dots,x_1}(f_{n+1} \otimes \cdots \otimes f_1) =$$

$$({}_{x_{n+1}}F_{x_1})^{-1}\left(\varphi_{F(x_{n+1}),\dots,F(x_1)}\left(F(f_{n+1}) \otimes \cdots \otimes F(f_1)\right)\right)$$

where

$${}_{x_{n+1}}F_{x_1} : {}_{x_{n+1}}\mathcal{C}_{x_1} \to {}_{F(x_{n+1})}\mathcal{D}_{F(x_1)}$$

is the k isomophism provided by the equivalence F.

Observe that in [17] the above Proposition is obtained in a more general setting, that is for a $\mathcal{D}$-bimodule of coefficients $\mathcal{N}$, in our case $\mathcal{N} = \mathcal{D}$. The restricted $\mathcal{C}$-bimodule of coefficients is denoted $F\mathcal{N}$ in [17], observe that $F\mathcal{D}$ is isomorphic to $\mathcal{C}$ via F. This later isomorphism explains that in our setting ${}_{x_{n+1}}F_{x_1}^{-1}$ is required in the above formula while in [17] it is not needed since the bimodule of coefficients there is $F\mathcal{D}$, not $\mathcal{C}$.

Theorem 5.8 *Let $\mathcal{C}$ and $\mathcal{D}$ be G-k-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a G-k-equivalence of categories. Then F induces an isomorphism of G-k-algebras*

$$HH^\bullet(\mathcal{D}) \to HH^\bullet(\mathcal{C}).$$

Proof. The explicit description of $C^\bullet F$ given above enables to check easily that it is multiplicative with respect to the cup product. Moreover, $C^\bullet F$ commutes with the actions of G on $C^\bullet\mathcal{C}$ and $C^\bullet\mathcal{D}$, that is $C^\bullet F$ is a kG-morphism. Therefore the induced map in cohomology is an isomorphism of G-k-algebras. $\diamond$

We recall that if $\mathcal{C}$ is a G-k-category, then $M_G(\mathcal{C})$ is a G-k-category where the action of G on the objects of $M_G(\mathcal{C})$ is free, see Definition 4.1. Moreover there is a a G-k-functor $L : M_G(\mathcal{C}) \to \mathcal{C}$ which is an equivalence of categories.

Theorem 5.9 *Let $\mathcal{C}$ be a G-k-category. Let $\mathcal{C}[G]$ be the graded skew category, and let $\{1\}$ be the trivial conjugacy class of G. There is an isomorphism of k-algebras*

$$HH^*_{\{1\}}(\mathcal{C}[G]) \simeq H^*(C^\bullet(\mathcal{C})^G).$$

If the invariant functor $(\ )^G$ is exact, we have an isomorphism of k-algebras

$$HH^*_{\{1\}}(\mathcal{C}[G]) \simeq (HH^*(\mathcal{C}))^G.$$

Proof. Let

$$L[G] : M_G(\mathcal{C})[G] \to \mathcal{C}[G]$$

be the homogeneous equivalence of G-graded k-categories obtained from the G-k-equivalence of categories $L : M_G(\mathcal{C}) \to \mathcal{C}$.

We observe that if $\mathcal{B}$ and $\mathcal{D}$ are G-graded categories and $K : \mathcal{B} \to \mathcal{D}$ is an homogeneous equivalence, then the quasi-isomorphism

$$C^\bullet(K) : C^\bullet(\mathcal{B}) \to C^\bullet(\mathcal{D})$$

described in Remark 5.7 preserves the decomposition along the conjugacy classes of G. Hence $HH^*_{\{1\}}(\mathcal{B})$ and $HH^*_{\{1\}}(\mathcal{D})$ are isomorphic k-algebras.

Remark 5.10 *Let Λ be a G-k-algebra, considered as a single object G-k-category with endomorphism algebra Λ. The action of G on the object is trivial, which is not free unless G is trivial. Let k be a field and G be a finite group whose order is invertible in k, the invariants functor is exact and the previous Theorem provides the isomorphism*

$$HH^*_{\{1\}}(\Lambda[G]) \simeq (HH^*(\Lambda))^G.$$

We review now the proof that we have provided, specified to the above situation. We have considered the matrix G-algebra $M_G(\Lambda)$, where the action of G is on *the indices of the rows and of the columns, and on Λ. Note that the action of G on the set of $|G|$ idempotents of the diagonal is free.*

The track of the previous categorical proof translates into first decomposing the cochains of $M_G(\Lambda)$ through the mentioned diagonal idempotents. Then the freeness of the action on this set enables to show that the invariants of the complex of cochains of $M_G(\Lambda)$ and the homogeneous cochains of class 1 of $M_G(\Lambda)[G]$ are isomorphic. The final step consists in showing that the Hochschild cohomology, as a kG module, remains the same when considering the algebra $M_G(\Lambda)[G]$.

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C.C.:
IMAG, Univ Montpellier, CNRS, Montpellier, France
Institut Montpelliérain Alexander Grothendieck
Claude.Cibils@umontpellier.fr

E.N.M.:
Departamento de Matemática, IME-USP,
Rua do Matão 1010, cx pt 20570, São Paulo, Brasil.
enmarcos@ime.usp.br